NOTES ON COUNTABLE EXTENSIONS
OF $p^{\omega+n}$-PROJECTIVES

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Abstract. We prove that if $G$ is an Abelian $p$-group of length not exceeding $\omega$ and $H$ is its $p^{\omega+n}$-projective subgroup for $n \in \mathbb{N} \cup \{0\}$ such that $G/H$ is countable, then $G$ is also $p^{\omega+n}$-projective. This enlarges results of ours in (Arch. Math. (Brno), 2005, 2006 and 2007) as well as a classical result due to Wallace (J. Algebra, 1971).

Unless we do not specify some else, by the term “group” we mean “an Abelian $p$-group”, written additively as is the custom when dealing with such groups, for some arbitrary but a fixed prime $p$. All unexplained exclusively, but however used, notions and notations are standard and follow essentially those from [7]. For instance, a group is called separable if it does not contain elements of infinite height. As usual, for any group $A$, $A_r$ denotes the reduced part of $A$.

A recurring theme is the relationship between the properties of a given group and its countable extension (see, e.g., [1]). The study in that aspect starts incidentally by Wallace [12] in order to establish a complete set of invariants for a concrete class of mixed Abelian groups. Specifically, his remarkable achievement states as follows.

**Theorem** (Wallace, 1971). Let $G$ be a reduced group with a totally projective subgroup $H$ so that $G/H$ is countable. Then $G$ is totally projective.

Since any reduced group is summable precisely when its socle is a free valued vector space, as application of ([8], Lemma 7) one can derive the following.

**Theorem** (Fuchs, 1977). Let $G$ be a reduced group with a summable isotype subgroup $H$ so that $G/H$ is countable. Then $G$ is summable.

Without knowing then the cited attainment of Fuchs, we have proved in [1] an analogous assertion for summable groups of countable length via the usage of a more direct group-theoretical approach. In [5] was also showed via the construction of a concrete example that when the summable subgroup $H$ is not isotype in $G$, $G$ may not be summable.

Likewise, in [5] (see [1] too) it was obtained the following affirmation.

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Theorem (Danchev and Keef, 2005 and 2008). Let $G$ be a group with a subgroup $H$ so that $G/H$ is countable. If

(a) $H$ is $\sigma$-summable, then $G$ is $\sigma$-summable provided that it is of limit length and $H$ is isotype in $G$ (when $H$ is not isotype in $G$, $G$ may not be $\sigma$-summable);

(b) $H$ is a $\Sigma$-group, then $G$ is a $\Sigma$-group;

(c) $H$ is a $Q$-group, then $G$ is a $Q$-group provided that it is separable;

(d) $H$ is weakly $\omega_1$-separable, then $G$ is weakly $\omega_1$-separable provided that it is separable.

In [1] and [3] we have shown the following statement as well.

Theorem (Danchev, 2005 and 2006). Let $G$ be a group with a $p^{\omega+n}$-projective subgroup $H$ so that $G/H$ is countable and $n \in \mathbb{N} \cup \{0\}$. If

(e) $H$ is pure and nice in $G$, then $G$ is $p^{\omega+n}$-projective;

(f) $H$ is pure in the separable $G$, then $G$ is $p^{\omega+n}$-projective.

Note that in [4] we have established such type results for $\omega$-elongations of totally projective groups by $p^{\omega+n}$-projective groups or summable groups by $p^{\omega+n}$-projective groups, respectively.

The purpose of the present brief work is to discuss some questions as those alluded to above concerning when a given separable group is $p^{\omega+n}$-projective provided that it has a modulo countable proper $p^{\omega+n}$-projective subgroup, but by removing the pureness of the subgroup in the whole group.

We are now in a position to proceed by proving the next extension of point (f) (see [5] as well).

Theorem. Suppose that $G$ is a group of length at most $\omega$ which contains a subgroup $H$ such that $G/H$ is countable. Then $G$ is $p^{\omega+n}$-projective if and only if $H$ is $p^{\omega+n}$-projective, whenever $n \in \mathbb{N} \cup \{0\}$.

Proof. The necessity is immediate because $p^{\omega+n}$-projectives are closed with respect to subgroups (see, for example, [10]). As for the sufficiency, according to the classical Nunke’s criterion for $p^{\omega+n}$-projectivity (see [10]), there exists $P \leq H[p^n]$ with $H/P$ a direct sum of cyclic groups. But observing that $(G/P)_r/((G/P)_r \cap H/P) \cong ((G/P)_r + H/P)/H/P \cong G/P/H/P \cong G/H$ is countable with $(G/P)_r \cap H/P \subseteq H/P$ a direct sum of cyclic groups, we appeal to Wallace’s theorem, quoted above, to infer that $(G/P)_r$ is totally projective. Hence $G/P$ is simply presented. Referring now to [11] (see [7], v. II, too), we deduce that $G/P/(G/P)^1 = G/P/P_G^{-}/P \cong G/P_G^{-}$ is a direct sum of cycles, where $P_G^{-} = \cap_{i<\omega}(P + p^iG)$ is the closure of $P$ in $G$. It is a straightforward argument that $p^nP_G^{-} \subseteq p^nG$. Since $G$ is separable, that is $p^nG = 0$, we derive that $p^nP_G^{-} = 0$, so employing once again the Nunke’s criterion we are finished.

The condition on separability may be avoided if the following strategy is realizable: Since $P$ is bounded, one can write $P = \cup_{m<\omega}P_m$, where $P_m \subseteq P_{m+1} \leq P$ with $p^kP_m = 0$ for each $m < \omega$ and some $k \in \mathbb{N}$. It is readily seen that
$P_G^- = \cup_{m<\omega} K_m$, where $K_m = \cap_{i<\omega} (P_m + p^i G)$. The crucial moment is whether we may choose a nice subgroup $N$ of $G$ such that $N \subseteq P_m$ and such that $P_m \cap p^m G \subseteq N$ for each integer $m \geq 1$; thus $P/N$ is strongly bounded in $G/N$ in terms of $\mathbb{k}$. Consequently, complying with the modular law from [7], we calculate that $K_m \cap p^m G = \cap_{m \leq i<\omega} (P_m + p^i G) \cap p^m G = \cap_{m \leq i<\omega} (P_m \cap p^m G + p^i G) \subseteq \cap_{m \leq i<\omega} (N + p^i G) = N + \cap_{m \leq i<\omega} p^i G = N + p^\omega G \leq P_G^- [p^\omega]$. Furthermore, we elementarily observe that $P_G^-/(N + p^\omega G) = \cup_{m<\omega} [K_m/(N + p^\omega G)]$ where, for each $m < \omega$, we compute with the aid of the modular law in [7] and the foregoing calculations that $(K_m/(N + p^\omega G)) \cap p^m G/(N + p^\omega G)) = [K_m \cap (p^m G + N)]/(N + p^\omega G) = (N + K_m \cap p^m G)/(N + p^\omega G) \subseteq (N + p^\omega G)/(N + p^\omega G) = \{0\}$. Besides, by what we have already shown above, $G/(N + p^\omega G)/P_G^-/(N + p^\omega G) \cong G/P_G^-$ is a direct sum of cyclic groups. Knowing this, we apply the Dieudonné criterion from [6] (see also [2]) to deduce that $G/(N + p^\omega G)$ is, in fact, a direct sum of cycles. Hence and from Nunke’s criterion in [10], we conclude that $G$ is $p^{\omega+n}$-projective, as asserted. This completes our conclusions in all generality.

**Remark.** Actually, $G/P = (G/P)$, since $p^{\omega+n}(G/P) = 0$ by seeing that $(G/P)^1 = \cap_{i<\omega} (p^i G + P)/P \subseteq G[p^n]/P$ with $P \leq G[p^n]$ and $G^1 = 0$. However, our approach in the proof gives a more general strategy even for inseparable groups. Nevertheless, this general case is still in question.

A group $A$ is said to be C-decomposable if $A = B \oplus K$, where $B$ is a direct sum of cycles with $\text{fin } r(B) = \text{fin } r(A)$.

We also pose the following conjecture.

**Conjecture.** Suppose $G$ is a group whose subgroup $H$ is C-decomposable and $G/H$ is countable. Then $G$ is C-decomposable.

In closing, we notice that Hill jointly with Megibben have found in (9), Proposition 1) that if $G$ is a reduced group which possesses a torsion-complete subgroup $H$ such that $G/H \cong \mathbb{Z}(p^\infty)$, then $G$ is torsion-complete.

So, we are ready to state the following.

**Problem.** Suppose $G$ is a group with a subgroup $H$ which belongs to the class $\mathcal{K}$ of Abelian $p$-groups. If $(G/H)[p]$ is finite, then whether or not $G$ also belongs to $\mathcal{K}$?

Investigate with a priority when $\mathcal{K}$ coincides with the class of thick groups, torsion-complete groups, semi-complete groups, quasi-complete groups or pure-complete groups, respectively.

It is worthwhile noticing that according to the main result, stated above, the results from [4] can be improved by dropping off some unnecessary additional limitations.

**References**


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