ON THE OSCILLATORY INTEGRATION OF SOME ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Conditions are given for a class of nonlinear ordinary differential equations \( x'' + a(t)w(x) = 0, \ t \geq t_0 \geq 1 \), which includes the linear equation to possess solutions \( x(t) \) with prescribed oblique asymptote that have an oscillatory pseudo-wronskian \( x'(t) - \frac{x(t)}{t} \).

1. Introduction

A certain interest has been shown recently in studying the existence of bounded and positive solutions to a large class of elliptic partial differential equations which can be displayed as

\[
\Delta u + f(x,u) + g(|x|)x \cdot \nabla u = 0, \quad x \in G_R,
\]

where \( G_R = \{ x \in \mathbb{R}^n : |x| > R \} \) for any \( R \geq 0 \) and \( n \geq 2 \). We would like to mention the contributions [3], [1], [8] – [11], [13, 14], [18] and their references in this respect.

It has been established, see [8, 9], that it is sufficient for the functions \( f, g \) to be Hölder continuous, respectively continuously differentiable in order to analyze the asymptotic behavior of the solutions to (1) by the comparison method [15]. In fact, given \( \zeta > 0 \), let us assume that there exist a continuous function \( A : [\mathbb{R}, +\infty) \to [0, +\infty) \) and a nondecreasing, continuously differentiable function \( W : [0, \zeta] \to [0, +\infty) \) such that

\[
0 \leq f(x,u) \leq A(|x|)W(u) \quad \text{for all} \quad x \in G_R, \ u \in [0, \zeta]
\]

and \( W(u) > 0 \) when \( u > 0 \). Then we are interested in the positive solutions \( U = U(|x|) \) of the elliptic partial differential equation

\[
\Delta U + A(|x|)W(U) = 0, \quad x \in G_R,
\]

for the role of super-solutions to (1).

M. Ehrnström [13] noticed that, by imposing the restriction

\[
x \cdot \nabla U(x) \leq 0, \quad x \in G_R,
\]

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upon the super-solutions $U$, an improvement of the conclusions from the literature
is achieved for the special subclass of equations [1] where $g$ takes only nonnegative
values. Further developments of Ehrnström’s idea are given in [3, 11, 14].

Translated into the language of ordinary differential equations, the research
about $U$ reads as follows: given $c_1, c_2 \geq 0$, find (if any) a positive solution $x(t)$ of
the nonlinear differential equation
\begin{equation}
 x'' + a(t)w(x) = 0, \quad t \geq t_0 \geq 1,
\end{equation}
where the coefficient $a: [t_0, +\infty) \to \mathbb{R}$ and the nonlinearity $w: \mathbb{R} \to \mathbb{R}$ are contin-
uous and given by means of $A, W$, such that
\begin{equation}
 x(t) = c_1 t + c_2 + o(1) \quad \text{when} \quad t \to +\infty
\end{equation}
and
\begin{equation}
 W(x, t) = \frac{1}{t} \left| \frac{x'(t)}{x(t)} \right| 1 = x'(t) - \frac{x(t)}{t} < 0, \quad t > t_0.
\end{equation}
The symbol $o(f)$ for a given functional quantity $f$ has here its standard meaning.
In particular, by $o(1)$ we refer to a function of $t$ that decreases to 0 as $t$ increases
to $+\infty$.

The papers [2, 11, 22, 21, 20] present various properties of the functional quantity
$W$, which shall be called pseudo-wronskian in the sequel. Our aim in this note is
to complete their conclusions by giving some sufficient conditions upon $a$ and $w$
which lead to the existence of a solution $x$ to [2] that verifies [3] while having an
oscillatory pseudo-wronskian (this means that there exist the unbounded from
above sequences $(t^+_n)_{n \geq 1}$ and $(t^-_n)_{n \geq 1}$ such that $t^{0}_{2n-1} < t^+_n < t^0_{2n} < t^-_n < t^0_{2n+1}$
and $W(t^+_n) > W(t^-_n) = 0 > W(t^-_n)$ for all $n \geq 1$). We answer thus to a question
raised in [1, p. 371], see also the comment in [2, pp. 46–47].

2. The sign of $W$

Let us start the discussion with a simple condition to settle the sign issue of the
pseudo-wronskian.

**Lemma 1.** Given $x \in C^2([t_0, +\infty), \mathbb{R})$, suppose that $x''(t) \leq 0$ for all $t \geq t_0$.
Then $W(x, \cdot)$ can change from being nonnegative-valued to being negative-valued at
most once in $[t_0, +\infty)$. In fact, its set of zeros is an interval (possibly degenerate).

**Proof.** Notice that
\[
 \frac{d^2}{dt^2} [x(t)] = \frac{1}{t} \cdot \frac{d}{dt} [tW(x, t)], \quad t \geq t_0.
\]

The function $t \mapsto tW(x, t)$ being nonincreasing, it is clear that, if it has zeros, it
has either a unique zero or an interval of zeros. \hfill \Box

The result has an obvious counterpart.

**Lemma 2.** Given $x \in C^2([t_0, +\infty), \mathbb{R})$, suppose that $x''(t) \geq 0$ for all $t \geq t_0$.
Then, $W(x, \cdot)$ can change from being nonpositive-valued to being positive-valued at
most once in $[t_0, +\infty)$. Again, its set of zeros is an interval (possibly reduced to
one point).
Consider that \( x \) is a positive solution of equation (2) in the case where \( a(t) \geq 0 \) in \([t_0, +\infty)\) and \( w(u) > 0 \) for all \( u > 0 \). Then, we have

\[
\frac{dW}{dt} = -\frac{W}{t} - a(t)w(x(t)), \quad t \geq t_0,
\]

which leads to

\[
W(x, t) = \frac{1}{t} \left[ t_0 W_0 - \int_{t_0}^{t} sa(s)w(x(s)) \, ds \right], \quad W_0 = W(x, t_0),
\]

throughout \([t_0, +\infty)\) by means of Lagrange’s variation of constants formula.

The integrand in (5) being nonnegative-valued, we regain the conclusion of Lemma 1. In fact, if \( T \in [t_0, +\infty) \) is a zero of \( W(x, \cdot) \) then it is a solution of the equation

\[
t_0 W_0 = \int_{t_0}^{T} sa(s)w(x(s)) \, ds.
\]

On the other hand, if the pseudo-wronskian of \( x \) is positive-valued throughout \([t_0, +\infty)\) then it is necessary to have

\[
(t_0 W_0 \geq) \int_{t_0}^{+\infty} sa(s)w(x(s)) \, ds < +\infty.
\]

It has become clear at this point that whenever the equation (2) has a positive solution \( x \) such that \( W_0 \leq 0 \), the functional coefficient \( a \) is nonnegative-valued and has at most isolated zeros and \( w(u) > 0 \) for all \( u > 0 \), the pseudo-wronskian \( W \) satisfies the restriction (4). Now, returning to the problem stated in the Introduction, we can evaluate the main difficulty of the investigation: if the positive solution \( x \) has prescribed asymptotic behavior, see formula (3) or a similar development, then we cannot decide upfront whether or not \( W_0 \leq 0 \). The formula (6) shows that there are also certain difficulties to estimate the zeros of the pseudo-wronskian.

### 3. THE BEHAVIOR OF \( W \)

Let us survey in this section some of the recent results regarding the pseudo-wronskian.

It has been established that its presence in the structure of a nonlinear differential equation

\[
x'' + f(t, x, x') = 0, \quad t \geq t_0 \geq 1,
\]

where the nonlinearity \( f: [t_0, +\infty) \times \mathbb{R}^2 \to \mathbb{R} \) is continuous, allows for a remarkable flexibility of the hypotheses when searching for solutions with the asymptotic development (3) (or similar).

**Theorem 1** ([22, p. 177]). Assume that there exist the nonnegative-valued, continuous functions \( a(t) \) and \( g(s) \) such that \( g(s) > 0 \) for all \( s > 0 \) and \( xg(s) \leq g(x^{1-\alpha} s) \), where \( x \geq t_0 \) and \( s \geq 0 \), for a certain \( \alpha \in (0, 1) \). Suppose further that

\[
|f(t, x, x')| \leq a(t)g\left(\frac{x'}{t}\right) \quad \text{and} \quad \int_{t_0}^{+\infty} \frac{a(s)}{s^{\alpha}} \, ds < \int_{c+|W_0|t_0^{1-\alpha}}^{+\infty} \frac{du}{g(u)}.
\]
Then the solution of equation (8) given by (5) exists throughout \([t_0, +\infty)\) and has the asymptotic behavior
\[
x(t) = c \cdot t + o(t), \quad x'(t) = c + o(1) \quad \text{when} \quad t \to +\infty
\]
for some \(c = c(x) \in \mathbb{R}\).

To compare this result with the standard conditions in asymptotic integration theory regarding the development (9), see the papers \[2, 1, 24\] and the monograph \[19\].

Another result is concerned with the presence of the pseudo-wronskian in the function space \(L^1((t_0, +\infty), \mathbb{R})\).

**Theorem 2** (\[1, p. 371\]). Assume that \(f\) does not depend explicitly of \(x'\) and there exists the continuous function \(F : [t_0, +\infty) \times [0, +\infty) \to [0, +\infty)\), which is nondecreasing with respect to the second variable, such that
\[
|f(t, x)| \leq F(t, \frac{|x|}{t}) \quad \text{and} \quad \int_{t_0}^{+\infty} t \left[1 + \ln \left(\frac{t}{t_0}\right)\right] F(t, |c| + \frac{\varepsilon}{t_0}) \, dt < \varepsilon
\]
for certain numbers \(c \neq 0\) and \(\varepsilon > 0\). Then there exists a solution \(x(t)\) of equation (8) defined in \([t_0, +\infty)\) such that
\[
x(t) = c \cdot t + o(1) \quad \text{when} \quad t \to +\infty \quad \text{and} \quad W(x, \cdot) \in L^1.
\]

The effect of perturbations upon the pseudo-wronskian is investigated in the papers \[2, 22, 21\].

**Theorem 3** (\[22, p. 183\]). Consider the nonlinear differential equation
\[
x'' + f(t, x, x') = p(t), \quad t \geq t_0 \geq 1,
\]
where the functions \(f : [t_0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}\) and \(p : [t_0, +\infty) \to \mathbb{R}\) are continuous and verify the hypotheses
\[
|f(t, x, x')| \leq a(t) \left|x - \frac{x}{t}\right|, \quad \int_{t_0}^{+\infty} ta(t) \, dt < +\infty
\]
and
\[
\lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} sp(s) \, ds = C \in \mathbb{R} - \{0\}.
\]
Then, given \(x_0 \in \mathbb{R}\), there exists a solution \(x(t)\) of equation (10) defined in \([t_0, +\infty)\) such that
\[
x(t_0) = x_0 \quad \text{and} \quad \lim_{t \to +\infty} W(x, t) = C.
\]
In particular,
\[
\lim_{t \to +\infty} \frac{x(t)}{t \ln t} = C.
\]
A slight modification of the discussion in \[21, Remark 3\], see \[2, p. 47\], leads to the next result.
Theorem 4. Assume that \( f \) in (10) does not depend explicitly of \( x' \) and there exists the continuous function \( F: [t_0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty) \), which is nondecreasing with respect to the second variable, such that
\[
|f(t, x)| \leq F(t, |x|) \quad \text{and} \quad \int_{t_0}^{+\infty} sF(s, |P(s)| + \sup_{\tau \geq s} q(\tau)) \, ds \leq q(t), \quad t \geq t_0,
\]
for a certain positive-valued, continuous function \( q(t) \) possibly decaying to 0 as \( t \rightarrow +\infty \). Here, \( P \) is the twice continuously differentiable antiderivative of \( p \), that is \( P''(t) = p(t) \) for all \( t \geq t_0 \). Suppose further that
\[
\limsup_{t \rightarrow +\infty} \left[ t \frac{W(P, t)}{q(t)} \right] > 1 \quad \text{and} \quad \liminf_{t \rightarrow +\infty} \left[ t \frac{W(P, t)}{q(t)} \right] < -1.
\]
Then equation (10) has a solution \( x(t) \) throughout \([t_0, +\infty)\) such that
\[
x(t) = P(t) + o(1) \quad \text{when} \quad t \rightarrow +\infty
\]
and \( W(x, \cdot) \) oscillates.

Finally, the presence of the pseudo-wronskian in the structure of a nonlinear differential equation can lead to multiplicity when searching for solutions with the asymptotic development (5).

Theorem 5 ([20] Theorem 1]). Given the numbers \( x_0, x_1, c \in \mathbb{R} \), with \( c \neq 0 \), and \( t_0 \geq 1 \) such that \( t_0 x_1 - x_0 = c \), consider the Cauchy problem
\[
\begin{align*}
x'' &= \frac{1}{t} g(tx' - x), & t \geq t_0 \geq 1, \\
x(t_0) &= x_0, & x'(t_0) = x_1,
\end{align*}
\]
where the function \( g: \mathbb{R} \rightarrow \mathbb{R} \) is continuous, \( g(c) = g(3c) = 0 \) and \( g(u) > 0 \) for all \( u \neq c \). Assume further that
\[
\int_{c+}^{2c} \frac{du}{g(u)} < +\infty \quad \text{and} \quad \int_{2c}^{(3c)-} \frac{du}{g(u)} = +\infty.
\]
Then problem (11) has an infinity of solutions \( x(t) \) defined in \([t_0, +\infty)\) and developable as
\[
x(t) = c_1 t + c_2 + o(1) \quad \text{when} \quad t \rightarrow +\infty
\]
for some \( c_1 = c_1(x) \) and \( c_2 = c_2(x) \in \mathbb{R} \).

The asymptotic analysis of certain functional quantities attached to the solutions of equations (2), (8) and (10), as in our case the pseudo-wronskian, might lead to some surprising consequences. Among the functional quantities that gave the impetus to spectacular developments in the qualitative theory of linear/nonlinear ordinary differential equations we would like to refer to
\[
\mathcal{K}(x)(t) = x(t)x'(t), \quad t \geq t_0,
\]
employed in the theory of Kneser-solutions, see the papers [6, 7] for the linear and respectively the nonlinear case and the monograph [19], and
\[
\mathcal{H}W(x) = \int_{t_0}^{+\infty} x(s)w(x(s)) \, ds.
\]
The latter quantity is the core of the nonlinear version of Hermann Weyl’s limit-point/limit-circle classification designed for equation (2), see the well-documented monograph [5] and the paper [23].

4. The negative values of $\mathcal{W}$

We shall assume in the sequel that the nonlinearity $w$ of equation (2) verifies some of the hypotheses listed below:

\[ |w(x) - w(y)| \leq k|x - y|, \quad \text{where} \quad k > 0, \tag{12} \]

and

\[ w(0) = 0, \quad w(x) > 0 \quad \text{when} \quad x > 0, \quad |w(xy)| \leq w(|x|)w(|y|) \tag{13} \]

for all $x, y \in \mathbb{R}$. We notice that restriction (13) implies the existence of a majorizing function $F$, as in Theorem 2, given by the estimates

\[ |f(t, x)| = |a(t) w(x)| \leq |a(t)| \cdot w(t) w\left(\frac{|x|}{t}\right) = F(t, \frac{|x|}{t}). \]

We can now use the paper [24] to recall the main conclusions of an asymptotic integration of equation (2). It has been established that whenever $\int_{t_0}^{+\infty} tw(t) |a(t)| \, dt < +\infty$, all the solutions of (2) have asymptotes (3) and their first derivatives are developable as

\[ x(t) = c t + o(t^{-1}) \quad \text{when} \quad t \to +\infty. \tag{14} \]

Consequently, $\mathcal{W}(x, t) = -c_2 t^{-1} + o(t^{-1})$ for all large $t$’s. In this case (the functional coefficient $a$ has varying sign), when dealing with the sign of the pseudo-wronskian, of interest would be the subcase where $c_2 = 0$. Here, the asymptotic development does not even ensure that $\mathcal{W}$ is eventually negative. Enlarging the family of coefficients to the ones subjected to the restriction $\int_{t_0}^{+\infty} t^\varepsilon w(t) |a(t)| \, dt < +\infty$, where $\varepsilon \in [0, 1)$, the developments (3), (14) become

\[ x(t) = ct + o(t^{1-\varepsilon}), \quad x'(t) = c + o(t^{-\varepsilon}), \quad c \in \mathbb{R}, \tag{15} \]

yielding the less precise estimate $\mathcal{W}(x, t) = o(t^{-\varepsilon})$ when $t \to +\infty$. We have again a lack of precision in the asymptotic development of $\mathcal{W}(x, \cdot)$ with respect to the sign issue. We also deduce on the basis of (3), (15) that some of the coefficients $a$ in these classes verify (7), a fact that complicates the discussion.

The next result establishes the existence of a positive solution to (2) subjected to (4), (15) for the largest class of functional coefficients: $\varepsilon = 0$. By taking into account Lemmas 1, 2 and the non-oscillatory character of equation (2) when the nonlinearity $w$ verifies (13), we conclude that for an investigation within this class of coefficients $a$ of the solutions with oscillatory pseudo-wronskian it is necessary that $a$ itself oscillates. Also, when $a$ is non-negative valued we recall that the condition

\[ \int_{t_0}^{+\infty} a(t) \, dt < +\infty \]
is necessary for the linear case of equation (2) to be non-oscillatory, see [16], while in the case given by \( w(x) = x^\lambda, \ x \in \mathbb{R}, \) with \( \lambda > 1 \) (such an equation is usually called an *Emden-Fowler equation*, see the monograph [19]) the condition

\[
\int_{t_0}^{+\infty} ta(t) \, dt = +\infty
\]

is necessary and sufficient for oscillation, see [4]. In the case of Emden-Fowler equations with \( \lambda \in (0, 1) \) and a continuously differentiable coefficient \( a \) such that \( a(t) \geq 0 \) and \( a'(t) \leq 0 \) throughout \([t_0, +\infty)\), another result establishes that equation (2) has no oscillatory solutions provided that condition [16] fails, see [17].

Regardless of the oscillation of \( a \), it is known [1, p. 360] that the linear case of equation (2) has bounded and positive solutions with eventually negative pseudo-wronskian.

**Theorem 6.** Assume that the nonlinearity \( w \) verifies hypothesis [13] and is non-decreasing. Given \( c, d > 0 \), suppose that the functional coefficient \( a \) is nonnegative-valued, with eventual isolated zeros, and

\[
\int_{t_0}^{+\infty} w(t) a(t) \, dt \leq \frac{d}{w(c + d)}.
\]

Then, the equation (2) has a solution \( x \) such that \( W_0 = 0 \),

\[
c - d \leq x(t) < \frac{x(t)}{t} \leq c + d \quad \text{for all} \quad t > t_0
\]

and

\[
\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} \frac{x(t)}{t} = c.
\]

**Proof.** We introduce the set \( D \) given by

\[
D = \{ u \in C([t_0, +\infty), \mathbb{R}) : ct \leq u(t) \leq (c + d)t \text{ for every } t \geq t_0 \}.
\]

A partial order on \( D \) is provided by the usual pointwise order “\( \leq \)”, that is, we say that \( v_1 \leq v_2 \) if and only if \( v_1(t) \leq v_2(t) \) for all \( t \geq t_1 \), where \( v_1, v_2 \in D \). It is not hard to see that \((D, \leq)\) is a complete lattice.

For the operator \( V : D \to C([t_0, +\infty), \mathbb{R}) \) with the formula

\[
V(u)(t) = t\left\{ c + \int_{t}^{+\infty} \frac{1}{s^2} \int_{t_0}^{s} \tau a(\tau) w(u(\tau)) \, d\tau \, ds \right\}, \quad u \in D, \ t \geq t_0,
\]
the next estimates hold
\[ \begin{align*}
c &\leq \frac{V(u)}{t} = c + \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^{s} \tau a(\tau) \cdot w(\tau) w\left(\frac{u(\tau)}{\tau}\right) \, d\tau \, ds \\
&\leq c + \sup_{\xi \in [0, c+d]} \left\{ w(\xi) \right\} \cdot \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^{s} \tau w(\tau) a(\tau) \, d\tau \, ds \\
&= c + w(c + d) \left[ \frac{1}{t} \int_t^{t_0} \tau w(\tau) a(\tau) \, d\tau + \int_t^{+\infty} w(\tau) a(\tau) \, d\tau \right] \\
&\leq c + w(c + d) \int_{t_0}^{+\infty} w(\tau) a(\tau) \, d\tau \leq c + d
\end{align*} \]
by means of (13). These imply that \( V(D) \subseteq D \).

Since \( c \cdot t \leq V(c \cdot t) \) for all \( t \geq t_0 \), by applying the Knaster-Tarski fixed point theorem [12, p. 14], we deduce that the operator \( V \) has a fixed point \( u_0 \) in \( D \). This is the pointwise limit of the sequence of functions \( \left\{ (V^n(c \cdot Id))_{n \geq 1} \right\} \), where \( V^1 = V \), \( V^{n+1} = V^n \circ V \) and \( I = [t_0, +\infty) \).

We deduce that
\[ u_0'(t) = \left[ V(u_0) \right]'(t) = \frac{u_0(t)}{t} - \frac{1}{t} \int_{t_0}^{t} \tau a(\tau) \, d\tau < \frac{u_0(t)}{t}, \]
when \( t > t_0 \), and thus (17), (18) hold true.

The proof is complete. \( \square \)

5. The Oscillatory Integration of Equation (2)

Let the continuous functional coefficient \( a \) with varying sign satisfy the restriction
\[ \int_{t_0}^{+\infty} t^2 |a(t)| \, dt < +\infty. \]

We call the problem studied in the sequel an oscillatory (asymptotic) integration of equation (2).

**Theorem 7.** Assume that \( w \) verifies (12), \( w(0) = 0 \) and there exists \( c > 0 \) such that
\[ \begin{align*}
L^c_+ &> 0 > L^c_-, \\
\end{align*} \]
where
\[ \begin{align*}
L^c_+ &= \limsup_{t \to +\infty} t \frac{\int_t^{+\infty} sw(cs) a(s) \, ds}{\int_t^{+\infty} s^2 |a(s)| \, ds}, \\
L^c_- &= \liminf_{t \to +\infty} t \frac{\int_t^{+\infty} sw(cs) a(s) \, ds}{\int_t^{+\infty} s^2 |a(s)| \, ds}.
\end{align*} \]

Then the equation (2) has a solution \( x(t) \) with oscillatory pseudo-wronskian such that
\[ x(t) = c \cdot t + o(1) \quad \text{when} \quad t \to +\infty. \]
Proof. There exist $\eta > 0$ such that $L^c_+ > \eta$, $L^c_- < -\eta$ and two increasing, unbounded from above sequences $(t_n)_{n \geq 1}$, $(t^n)_{n \geq 1}$ of numbers from $(t_0, +\infty)$ such that $t^n \in (t_n, t_{n+1})$ and

\begin{equation}
 t_n \int_{t_n}^{+\infty} s w(cs) a(s) \, ds + k\eta \int_{t_n}^{+\infty} s^2 |a(s)| \, ds < 0
\end{equation}

and

\begin{equation}
 t^n \int_{t^n}^{+\infty} s w(cs) a(s) \, ds - k\eta \int_{t^n}^{+\infty} s^2 |a(s)| \, ds > 0
\end{equation}

for all $n \geq 1$.

Assume further that

\[ \int_{t_0}^{+\infty} \tau^2 |a(\tau)| \, d\tau \leq \frac{\eta}{k(c + \eta)} \]

and introduce the complete metric space $S = (D, \delta)$ given by

\[ D = \{ y \in C([t_0, +\infty), \mathbb{R}) : t|y(t)| \leq \eta \text{ for every } t \geq t_0 \} \]

and

\[ \delta(y_1, y_2) = \sup_{t \geq t_0} \{ \tau \mid y_1(t) - y_2(t) \} \], \quad y_1, y_2 \in D. \]

For the operator $V : D \to C([t_0, +\infty), \mathbb{R})$ with the formula

\[ V(y)(t) = \frac{1}{t} \int_t^{+\infty} sa(s) w\left( s - \int_s^{+\infty} \frac{y(\tau)}{\tau} \, d\tau \right) \, ds, \quad y \in D, t \geq t_0, \]

the next estimates hold (notice that $|w(x)| \leq k|x|$ for all $x \in \mathbb{R}$)

\begin{equation}
 t|V(y)(t)| \leq k \int_t^{+\infty} s^2 |a(s)| \left[ c + \eta \int_s^{+\infty} \frac{d\tau}{\tau^2} \right] \, ds \leq \eta
\end{equation}

and

\begin{align*}
 t|V(y_2)(t) - V(y_1)(t)| &\leq k \int_t^{+\infty} s^2 |a(s)| \left( \int_s^{+\infty} \frac{d\tau}{\tau^2} \right) \, ds \cdot \delta(y_1, y_2) \\
 &\leq \frac{k}{t_0} \int_t^{+\infty} s^2 |a(s)| \, ds \leq \frac{\eta}{c + \eta} \cdot \delta(y_1, y_2).\]
\end{align*}

These imply that $V(D) \subseteq D$ and thus $V : S \to S$ is a contraction.

From the formula of operator $V$ we notice also that

\begin{equation}
 \lim_{t \to +\infty} tV(y)(t) = 0 \quad \text{for all } y \in D.
\end{equation}

Given $y_0 \in D$ the unique fixed point of $V$, one of the solutions to (2) has the formula $x_0(t) = t \left[ c - \int_t^{+\infty} \frac{y_0(s)}{s} \, ds \right]$ for all $t \geq t_0$. Via (24) and L’Hospital’s rule, we
provide also an asymptotic development for this solution, namely
\[
\lim_{t \to +\infty} [x_0(t) - c \cdot t] = - \lim_{t \to +\infty} t \int_{t}^{+\infty} \frac{y_0(s)}{s} \, ds = - \lim_{t \to +\infty} ty_0(t) \\
= - \lim_{t \to +\infty} tV(y_0)(t) = 0.
\]

The estimate
\[
|ty_0(t) - \int_{t}^{+\infty} sw(cs) a(s) \, ds| \leq k \int_{t}^{+\infty} s^2|a(s)| \left[ \int_{t}^{+\infty} \frac{|y_0(\tau)|}{\tau} \, d\tau \right] \, ds \\
\leq k\eta \cdot \frac{1}{t} \int_{t}^{+\infty} s^2|a(s)| \, ds,
\]
accompanied by (21), (22), leads to
\[
y_0(t_n) = W(x_0, t_n) < 0 \quad \text{and} \quad y_0(t^n) = W(x_0, t^n) > 0.
\]

The proof is complete. \(\square\)

**Remark 1.** When Equation (2) is linear, that is \(w(x) = x\) for all \(x \in \mathbb{R}\), the formula (19) can be recast as
\[
L_+ = \limsup_{t \to +\infty} \frac{t \int_{t}^{+\infty} s^2a(s) \, ds}{\int_{t}^{+\infty} s^2|a(s)| \, ds} > 0 > \liminf_{t \to +\infty} \frac{t \int_{t}^{+\infty} s^2a(s) \, ds}{\int_{t}^{+\infty} s^2|a(s)| \, ds} = L_-.
\]

We claim that for all \(c \neq 0\) there exists a solution \(x(t)\) with oscillatory pseudo-wronskian which verifies (20). In fact, replace \(c\) with \(c_0\) in the formulas (21), (22) for a certain \(c_0\) subjected to the inequality \(\min\{L_+, -L_-\} > \frac{\eta}{c_0}\). It is obvious that, when \(L_+ = -L_- = +\infty\), formulas (21), (22) hold for all \(c_0, \eta > 0\). Given \(c \in \mathbb{R} - \{0\}\), there exists \(\lambda \neq 0\) such that \(c = \lambda c_0\). The solution of Equation (2) that we are looking for has the formula \(x = \lambda \cdot x_0\), where \(x_0(t) = t\left[c_0 - \int_{t}^{+\infty} \frac{y_0(s)}{s} \, ds\right]\) for all \(t \geq t_0\) and \(y_0\) is the fixed point of operator \(V\) in \(D\). Its pseudo-wronskian oscillates as a consequence of the obvious identity
\[
\lambda \cdot W(x_0, t) = W(x, t), \quad t \geq t_0.
\]

**Example 1.** An immediate example of functional coefficient \(a\) for the problem of linear oscillatory integration is given by \(a(t) = t^{-2}e^{-t}\cos t\), where \(t \geq 1\).

We have
\[
\int_{t}^{+\infty} s^2a(s) \, ds = \frac{1}{\sqrt{2}} \cos \left(t + \frac{\pi}{4}\right)e^{-t} \quad \text{and} \quad \int_{t}^{+\infty} s^2|a(s)| \, ds \leq e^{-t}
\]
throughout \([1, +\infty)\) which yields \(L_+ = +\infty, L_- = -\infty\).

Sufficient conditions are provided now for an oscillatory pseudo-wronskian to be in \(L^p((t_0, +\infty), \mathbb{R})\), where \(p > 0\). Since \(\lim_{t \to +\infty} W(x, t) = 0\) for any solution \(x(t)\) of equation (2) with the asymptotic development (20), (14), we are interested in the case \(p \in (0, 1)\).
Theorem 8. Assume that, in the hypotheses of Theorem 7, the coefficient \( a \) verifies the condition

\[
\int_{t_0}^{+\infty} \left[ \int_{t_0}^{+\infty} \frac{t}{s^2 |a(s)|} \, ds \right]^{1-p} t^2 |a(t)| \, dt < +\infty \quad \text{for some} \quad p \in (0, 1).
\]

Then the equation (2) has a solution \( x(t) \) with an oscillatory pseudo-wronskian in \( L^p \) and the asymptotic expansion (20).

Proof. Recall that \( y_0 \) is the fixed point of operator \( V \). Then, formula (23) implies that

\[
|y_0(t)| \leq k(c + \eta) \cdot \frac{1}{t} \int_{t}^{+\infty} s^2 |a(s)| \, ds, \quad t \geq t_0.
\]

Via an integration by parts, we have

\[
\frac{1}{[k(c + \eta)]^p} \int_{t}^{T} |y_0(s)|^p \, ds \leq \frac{T^{1-p}}{1-p} \left[ \int_{t}^{+\infty} s^2 |a(s)| \, ds \right]^{p} \left( \frac{p}{1-p} \int_{t}^{T} \left[ \frac{s}{\int_{s}^{+\infty} \tau^2 |a(\tau)| \, d\tau} \right] \right)^{1-p} s^2 |a(s)| \, ds
\]

for all \( T \geq t \geq t_0 \).

The estimates

\[
\frac{T^{1-p}}{1-p} \left[ \int_{t}^{+\infty} s^2 |a(s)| \, ds \right]^{p} = \frac{T^{1-p}}{1-p} \int_{t}^{+\infty} \left[ \frac{1}{\int_{t}^{+\infty} \tau^2 |a(\tau)| \, d\tau} \right]^{1-p} s^2 |a(s)| \, ds
\]

\[
\leq \frac{1}{1-p} \int_{t}^{+\infty} \left[ \frac{s}{\int_{s}^{+\infty} \tau^2 |a(\tau)| \, d\tau} \right] \right)^{1-p} s^2 |a(s)| \, ds
\]

allow us to establish that

\[
\frac{1}{[k(c + \eta)]^p} \int_{t}^{T} |y_0(s)|^p \, ds \leq \frac{1 + p}{1-p} \left[ \int_{t}^{+\infty} \frac{s}{\int_{s}^{+\infty} \tau^2 |a(\tau)| \, d\tau} \right] \right)^{1-p} s^2 |a(s)| \, ds.
\]

The conclusion follows by letting \( T \to +\infty \).

The proof is complete. \( \square \)

Example 2. An example of functional coefficient \( a \) in the linear case that verifies the hypotheses of Theorem 8 is given by the formula

\[
t^2 a(t) = b(t) = \begin{cases} 
    a_k(t - 9k), & t \in [9k, 9k + 1], \\
    a_k(9k + 2 - t), & t \in [9k + 1, 9k + 3], \\
    a_k(t - 9k - 4), & t \in [9k + 3, 9k + 4], \\
    a_k(9k + 4 - t), & t \in [9k + 4, 9k + 5], \\
    a_k(t - 9k - 6), & t \in [9k + 5, 9k + 7], \\
    a_k(9k + 8 - t), & t \in [9k + 7, 9k + 8], \\
    0, & t \in [9k + 8, 9(k + 1)], 
\end{cases}
\]

with an oscillatory pseudo-wronskian in \( L^p \) and the asymptotic expansion (20).
Here, we take \( a_k = k^{-\alpha} - (k + 1)^{-\alpha} \) for a certain integer \( \alpha > \frac{2-p}{p} \).

To help the computations, the \( k \)-th “cell” of the function \( b \) can be visualized next.

It is easy to observe that

\[
\int_{9k+2}^{9k+4} b(t) \, dt = \int_{9k+6}^{9k+8} b(t) \, dt = 0 \quad \text{for all} \quad k \geq 1.
\]

We have

\[
\int_{9k+2}^{+\infty} b(t) \, dt = \int_{9k+6}^{9k+8} b(t) \, dt = -a_k, \quad \int_{9k+6}^{+\infty} b(t) \, dt = \int_{9k+6}^{9k+8} b(t) \, dt = a_k
\]

and respectively

\[
\int_{9k+2}^{+\infty} |b(t)| \, dt = 3a_k + 4 \sum_{m=k+1}^{+\infty} a_m, \quad \int_{9k+6}^{+\infty} |b(t)| \, dt = a_k + 4 \sum_{m=k+1}^{+\infty} a_m.
\]

By noticing that

\[
L_+ = \lim_{k \to +\infty} \frac{(9k + 6) \int_{9k+6}^{+\infty} b(t) \, dt}{\int_{9k+6}^{+\infty} |b(t)| \, dt}, \quad L_- = \lim_{k \to +\infty} \frac{(9k + 2) \int_{9k+2}^{+\infty} b(t) \, dt}{\int_{9k+2}^{+\infty} |b(t)| \, dt},
\]

we obtain \( L_+ = \frac{9\alpha}{4} \) and \( L_- = -\frac{9\alpha}{4} \).

To verify the condition \([26]\), notice first that

\[
I_k = \int_{9k}^{9(k+1)} \left[ \frac{t}{\int_t^{+\infty} |b(s)| \, ds} \right]^{1-p} t^2 |a(t)| \, dt \\
\leq \int_{9k}^{9(k+1)} \left[ \frac{9(k+1)}{\int_{9(k+1)}^{+\infty} |b(s)| \, ds} \right]^{1-p} a_k \, dt, \quad k \geq 1.
\]

The elementary inequality \( a_k \leq (2\alpha - 1)(k + 1)^{-\alpha} \) implies that

\[
I_k \leq \frac{c_\alpha}{(k + 1)^{(1+\alpha)p-1}}, \quad \text{where} \quad c_\alpha = 9 \left( \frac{9}{4} \right)^{1-p} (2\alpha - 1),
\]

and the conclusion follows from the convergence of the series \( \sum_{k \geq 1} (k + 1)^{-1-(1+\alpha)p} \).
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