THE JET PROLONGATIONS OF 2-FIBRED MANIFOLDS
AND THE FLOW OPERATOR

WŁODZIMIERZ M. MIKULSKI

ABSTRACT. Let $r$, $s$, $m$, $n$, $q$ be natural numbers such that $s \geq r$. We prove that any $2\mathcal{FM}_{m,n,q}$-natural operator $A \colon T_{2\text{-proj}} \rightarrow T^{(s,r)}J$ transforming $2$-proposable vector fields $V$ on $(m,n,q)$-dimensional $2$-fibred manifolds $Y \rightarrow X \rightarrow M$ into vector fields $A(V)$ on the $(s,r)$-jet prolongation bundle $J^{(s,r)}Y$ is a constant multiple of the flow operator $J^{(s,r)}$. All manifolds and maps are assumed to be of class $C^\infty$. Manifolds are assumed to be finite dimensional and without boundaries.

The category of all manifolds and maps is denoted by $\mathcal{M}$. The category of all fibred manifolds (surjective submersions $X \rightarrow M$ between manifolds) and fibred maps is denoted by $\mathcal{FM}$. The category of all fibred manifolds with $m$-dimensional bases and $n$-dimensional fibres and their fibred embeddings is denoted by $\mathcal{FM}_{m,n}$. The category of $2$-fibred manifold (pairs of surjective submersions $Y \rightarrow X \rightarrow M$ between manifolds) and their $2$-fibred maps is denoted by $2\mathcal{FM}$. The category of all fibred manifolds $Y \rightarrow X \rightarrow M$ such that $X \rightarrow M$ is an $\mathcal{FM}_{m,n}$-object and their $2$-fibred maps covering $\mathcal{FM}_{m,n}$-maps is denoted by $2\mathcal{FM}_{m,n}$. The category of all fibred manifolds $Y \rightarrow X \rightarrow M$ such that $X \rightarrow M$ is an $\mathcal{FM}_{m,n}$-object and $Y \rightarrow X$ is an $\mathcal{FM}_{m+n,q}$-object and their $2$-fibred embeddings is denoted by $2\mathcal{FM}_{m,n,q}$. The standard $2\mathcal{FM}_{m,n,q}$-object is denoted by $\mathcal{R}^{m,n,q} = (\mathcal{R}^m \times \mathcal{R}^n \times \mathcal{R}^q \rightarrow \mathcal{R}^m \times \mathcal{R}^n \rightarrow \mathcal{R}^m)$. The usual coordinates on $\mathcal{R}^{m,n,q}$ are denoted by $x^1, \ldots, x^m, y^1, \ldots, y^n, z^1, \ldots, z^q$.

Taking into consideration some idea from [1] one can generalize the concept of jets as follows. Let $r$ and $s$ be integers such that $s \geq r$. Let $Y \rightarrow X \rightarrow M$ be a $2\mathcal{FM}_{m,n}$-object. Sections $\sigma_1, \sigma_2 \colon X \rightarrow Y$ of $Y \rightarrow X$ have the same $(s,r)$-jet $j^x_{(s,r)}(\sigma_1) = j^x_{(s,r)}(\sigma_2)$ at $x \in X$ if

$$j^r_x(J^r\sigma_1|_{X_{p_0}}) = j^r_x(J^r\sigma_2|_{X_{p_0}}),$$

where $J^r\sigma_i \colon X \rightarrow J^rY$ is the $r$-jet map $J^r\sigma_i(x) = j^r_x\sigma_i$, $x \in X$, and $X_{p_0}(x)$ is the fibre of $X \rightarrow M$ through $x$. Equivalently $j^x_{(s,r)}(\sigma_1) = j^x_{(s,r)}(\sigma_2)$ if (in some and then in every $2\mathcal{FM}_{m,n}$-coordinates) $D_{(\alpha,\beta)}\sigma_1(x) = D_{(\alpha,\beta)}\sigma_2(x)$ for all $\alpha \in (\mathcal{N} \cup \{0\})^m$ and $\beta \in (\mathcal{N} \cup \{0\})^n$ with $|\alpha| \leq r$ and $|\alpha| + |\beta| \leq s$, where $D_{(\alpha,\beta)}$ denotes the

2000 Mathematics Subject Classification: Primary: 58A20.

Key words and phrases: $(s,r)$-jet, bundle functor, natural operator, flow operator, $2$-fibred manifold, $2$-proposable vector field.

Received March 13, 2007. Editor I. Kolář.
iterated partial derivative corresponding to \((\alpha, \beta)\). Thus we have the so called \((s,r)\)-jets prolongation bundle
\[ J^{(s,r)}Y = \{ j_x^{(s,r)}\sigma \mid \sigma: X \to Y \text{ is a section of } Y \to X, \ x \in X \}. \]

Given a 2-\(\mathcal{F}M_{m,n}\)-map \(f: Y_1 \to Y_2\) of two 2-\(\mathcal{F}M_{m,n}\)-objects covering \(\mathcal{F}M_{m,n}\)-map \(\bar{f}: X_1 \to X_2\) we have the induced map \(J^{(s,r)}f: J^{(s,r)}Y_1 \to J^{(s,r)}Y_2\) given by \(J^{(s,r)}f(j_x^{(s,r)}\sigma) = j_{\bar{f}(x)}^{(s,r)}(f \circ \sigma \circ \bar{f}^{-1}), j_x^{(s,r)}\sigma \in J^{(s,r)}Y_1\). The correspondence \(J^{(s,r)}: 2-\mathcal{F}M_{m,n} \to \mathcal{F}M\) is a (fiber product preserving) bundle functor.

Let \(Y \to X \to M\) be an 2-\(\mathcal{F}M_{m,n,q}\)-object. A vector field \(V\) on \(Y\) is called 2-projectable if there exist (unique) vector fields \(V_1\) on \(X\) and \(V_0\) on \(M\) such that \(V\) is related with \(V_1\) and \(V_1\) is related with \(V_0\) (with respect to the 2-fibred manifold projections). Equivalently, the flow \(\text{Expt}_V\) of \(V\) is formed by (local) 2-\(\mathcal{F}M_{m,n,q}\)-isomorphisms. Thus we can apply functor \(J^{(s,r)}\) to \(\text{Expt}_V\) and obtain new flow \(J^{(s,r)}(\text{Expt}_V)\) on \(J^{(s,r)}Y\). Consequently we obtain vector field \(J^{(s,r)}V\) on \(J^{(s,r)}Y\). The corresponding 2-\(\mathcal{F}M_{m,n,q}\)-natural operator \(J^{(s,r)}: T_{2\text{-proj}} \sim TJ^{(s,r)}\) is called the flow operator (of \(J^{(s,r)}\)).

The main result of the present note is the following classification theorem.

**Theorem 1.** Let \(r, s, m, n, q\) be natural numbers such that \(s \geq r\). Any 2-\(\mathcal{F}M_{m,n,q}\)-natural operator \(A: T_{2\text{-proj}} \sim TJ^{(s,r)}\) is a constant multiple of the flow operator \(J^{(s,r)}\).

Thus Theorem 1 extends the result from [2] on 2-fibred manifolds. More precisely, in [2] it is proved that any \(\mathcal{F}M_{m,n}\)-natural operator \(A\) lifting projectable vector fields \(V\) from fibred manifolds \(Y \to M\) to vector fields \(A(V)\) on \(JY\) is a constant multiple of the flow operator.

In the proof of Theorem 1 we will use the method from [4] (a Weil algebra technique). We start with the proof of the following lemma. Let \(A: T_{2\text{-proj}} \sim TJ^{(s,r)}\) be a natural operator in question.

**Lemma 1.** The natural operator \(A\) is determined by the restriction \(A(\frac{\partial}{\partial x^1}) | (J^{(s,r)}(\mathcal{R}^{m,n,q}))(0,0), \text{ where } (0, 0) \in \mathcal{R}^m \times \mathcal{R}^n.\)

**Proof.** The assertion is an immediate consequence of the naturality and regularity of \(A\) and the fact that any 2-projectable vector field which is not \((Y \to M)\)-vertical is related with \(\frac{\partial}{\partial x^1}\) by an 2-\(\mathcal{F}M_{m,n,q}\)-map. \(\square\)

Now we prove

**Lemma 2.** Let \(A\) be the operator. Let \(\pi: J^{(s,r)}Y \to X\) be the projection. Then there exists the unique real number \(c\) and the unique \(\pi\)-vertical operator \(V: T_{2\text{-proj}} \sim TJ^{(s,r)}\) with \(V(0) = 0\) such that \(A = cJ^{(s,r)} + V\).

**Proof.** Define \(C = T\pi \circ A(\frac{\partial}{\partial x^1}): (J^{(s,r)}(\mathcal{R}^{m,n,q}))(0,0) \to T_{(0,0)}(\mathcal{R}^m \times \mathcal{R}^n).\) Using the invariance of \(A\) with respect to 2-\(\mathcal{F}M_{m,n,q}\)-maps
\[ (x^1, \ldots, x^m, y^1, \ldots, y^n, \tau z^1, \ldots, \tau z^q) \]
for $\tau > 0$ and putting $t \to 0$ we get that $C(j^{(s,r)}_{(0,0)}(\sigma)) = C(j^{(s,r)}_{(0,0)}(0))$, where $0$ is the zero section. Then using the invariance of $A$ with respect to

$$(x^1, \tau x^2, \ldots, \tau x^m, \tau y^1, \ldots, \tau y^n, \tau z^1, \ldots, \tau z^q)$$

for $\tau > 0$ and putting $t \to 0$ we get that $C(j^{(s,r)}_{(0,0)}(0)) = c \frac{\partial}{\partial \tau^i}|_0$ for some $c \in \mathbb{R}$. We put $V = A - cJ^{(s,r)}$. Then $V$ is of vertical type because of Lemma 1. Clearly, $A = cJ^{(s,r)} + V$.

It remains to show that $V(0) = 0$. Clearly, the flow of $V(0)$ is a family of natural automorphisms $J^{(s,r)} \to J^{(s,r)}$. Since the $2\mathcal{F}M_{m,n,q}$-orbit of $j^{(s,r)}_{(0,0)}(0)$ is the whole $(J^{(s,r)}(R^{m,n,q}))_{(0,0)}$ any element $j^{(s,r)}_{(0,0)} \sigma \in (J^{(s,r)}(R^{m,n,q}))_{(0,0)}$ is transformed by $2\mathcal{F}M_{m,n,q}$-map

$$(x, y, z - \sigma(x, y))$$

into $j^{(s,r)}_{(0,0)}(0)$, then any natural automorphism $\mathcal{E}: J^{(s,r)} \to J^{(s,r)}$ is determined by $\mathcal{E}(j^{(s,r)}_{(0,0)}(0))$. Then using the invariance of $\mathcal{E}$ with respect to

$$(\tau x^1, \ldots, \tau x^m, \tau y^1, \ldots, \tau y^n, \tau z^1, \ldots, \tau z^q)$$

for $\tau > 0$ and putting $\tau \to 0$ we get $\mathcal{E}(j^{(s,r)}_{(0,0)}(0)) = j^{(s,r)}_{(0,0)}(0)$. Then $\mathcal{E} = \text{id}$ and then $\mathcal{V}(0) = 0$. $\blacksquare$

Define a bundle functor $F: \mathcal{M}f \to \mathcal{F}M$ by

$$FN = (J^{(s,r)}(R^m \times R^n \times N))_{(0,0)}, \quad Ff = (J^{(s,r)}(\text{id}_{R^m} \times \text{id}_{R^n} \times f))_{(0,0)}.$$ 

**Lemma 3.** The bundle functor $F: \mathcal{M}f \to \mathcal{F}M$ is product preserving.

**Proof.** It is clear. $\blacksquare$

Let $B = FR$ be the Weil algebra corresponding to $F$.

**Lemma 4.** We have $B = D^{s}_{m+n}/\overline{B}$, where $D^{s}_{m+n} = J^{s}_{(0,0)}(R^{m+n}, R)$ and $\overline{B} = \langle j^{s}_{(0,0)}(x^1), \ldots, j^{s}_{(0,0)}(x^m) \rangle^{r+1}$ is the $(r+1)$-power of the ideal $\langle j^{s}_{(0,0)}(x^1), \ldots, j^{s}_{(0,0)}(x^m) \rangle$, generated by the elements as indicate.

**Proof.** It is a simple observation. $\blacksquare$

We have the obvious action $H: G^{s}_{m,n} \times B \to B$,

$$H(j^{s}_{(0,0)} \psi, [j^{s}_{(0,0)} \gamma]) = [j^{s}_{(0,0)} (\gamma \circ \psi^{-1})]$$

for any $\mathcal{F}M_{m,n}$-map $\psi: (R^m \times R^n, (0,0)) \to (R^m \times R^n, (0,0))$ and $\gamma: R^{m+n} \to R$. This action is by algebra automorphisms.

**Lemma 5.** For any derivation $D \in \text{Der}(B)$ we have the implication: if

$$H(j^{s}_{(0,0)}(\tau \text{id})) \circ D \circ H(j^{s}_{(0,0)}(\tau^{-1} \text{id})) \to 0 \quad \text{as} \quad \tau \to 0 \quad \text{then} \quad D = 0.$$

**Proof.** Let $D \in \text{Der}(B)$ be such that

$$H(j^{s}_{(0,0)}(\tau \text{id})) \circ D \circ H(j^{s}_{(0,0)}(\tau^{-1} \text{id})) \to 0 \quad \text{as} \quad \tau \to 0.$$
For $i = 1, \ldots, m$ and $j = 1, \ldots, n$ write $D([j^s_{(0,0)}(x^i)]) = \sum a^i_{\alpha \beta} [j^s_{(0,0)}(x^\alpha y^\beta)]$ and $D([j^s_{(0,0)}(y^j)]) = \sum b^j_{\alpha \beta} [j^s_{(0,0)}(x^\alpha y^\beta)]$ for some (unique) real numbers $a^i_{\alpha \beta}$ and $b^j_{\alpha \beta}$, where the sums are over all $\alpha \in (\mathbb{N} \cup \{0\})^m$ and $\beta \in (\mathbb{N} \cup \{0\})^n$ with $|\alpha| \leq r$ and $|\alpha| + |\beta| \leq s$. We have

$$H(j^s_{(0,0)}(\tau \text{id}))) \circ D \circ H(j^s_{(0,0)}(\tau^{-1} \text{id}))) ([j^s_{(0,0)}(x^i)]) = \frac{1}{\tau^{[|\alpha|+|\beta|]}} [j^s_{(0,0)}(x^\alpha y^\beta)].$$

Then from the assumption on $D$ it follows that $a^i_{\alpha \beta} = 0$ if $(\alpha, \beta) \neq ((0), (0))$. Similarly, $b^j_{\alpha \beta} = 0$ if $(\alpha, \beta) \neq ((0), (0))$. Then $D(([j^s_{(0,0)}(x^i)])) = a^i_{(0)(0)} [j^s_{(0,0)}(1)]$ and $D(([j^s_{(0,0)}(y^j)])) = b^j_{(0)(0)} [j^s_{(0,0)}(1)]$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Then (since $[j^s_{(0,0)}((x^i)^{r+1})] = 0$ and $D$ is a differentiation) we have

$$0 = D(([j^s_{(0,0)}((x^i)^{r+1})])) = (r+1) [j^s_{(0,0)}((x^i)^{r})] D(([j^s_{(0,0)}(x^i)]))$$

$$= (r+1) a^i_{(0)(0)} [j^s_{(0,0)}((x^i)^{r})].$$

Then $a^i_{(0)(0)} = 0$ if $[j^s_{(0,0)}((x^i)^{r})] \neq 0$. Similarly, $b^j_{(0)(0)} = 0$. Then $D = 0$ because the $[j^s_{(0,0)}(x^i)]$ and $[j^s_{(0,0)}(y^j)]$ generate the algebra $B$.

**Proof of Theorem 1.** Operator $V$ from Lemma 2 defines (by the restriction) $M_f$-natural vector fields $\bar{V}_t = V(t \frac{\partial}{\partial x^i})|FN$ on $FN$ for any $t \in \mathbb{R}$. Clearly, $V$ is determined by $\bar{V}_1$. By Lemma 2, $\bar{V}_0 = 0$. By [2], $\bar{V}_t = \text{op}(D_t)$ for some $D_t \in \text{Der}(B)$. Then using the invariance of $V$ with respect to

$$(\tau x^1, \ldots, \tau x^m, \tau y^1, \ldots, \tau y^n, z^1, \ldots, z^q)$$

for $\tau \neq 0$ and putting $\tau \to 0$ we obtain that

$$H(j^s_{(0,0)}(\tau \text{id}))) \circ D \circ H(j^s_{(0,0)}(\tau^{-1} \text{id}))) \to 0 \quad \text{as} \quad \tau \to 0.$$

Then $D_t = 0$ because of Lemma 5. Then $V = 0$, and then $A = cJ^{(s,r)}$ as well.

**Remark 1.** There is another (non-equivalent) generalization of jets. Let $s \geq r$. Let $Y \to X \to M$ be a 2-fibred manifold. By [2], sections $\sigma_1, \sigma_2: X \to Y$ of $Y \to X$ have the same $r,s$-jets $j^r_s \sigma_1 = j^r_s \sigma_2$ at $x \in X$ if $j^r_s \sigma_1 = j^r_s \sigma_2$ and $j^s_x (\sigma_1 | X_{p_0(x)}) = j^s_x (\sigma_2 | X_{p_0(x)})$, where $X_{p_0(x)}$ is the fiber of $X \to M$ through $x$. Consequently we have the corresponding bundle $J^{r,s}Y$ and the corresponding (fiber product preserving) bundle functor $J^{r,s}: 2\mathcal{FM}_{m,n} \to \mathcal{FM}$. In [3], we proved that any $2\mathcal{FM}_{m,n}$-natural operator $A: T_2\text{-proj} \to TJ^{r,s}$ is a constant multiple of the flow operator $J^{r,s}$ corresponding to $J^{r,s}$ (we used quite different method than the one in [4] or in the present note).
References


Institute of Mathematics, Jagellonian University
Reymonta 4, Kraków, Poland
E-mail: mikulski@im.uj.edu.pl