BOUNDS ON BASS NUMBERS AND THEIR DUAL

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Abstract. Let \((R, \mathfrak{m})\) be a commutative Noetherian local ring. We establish some bounds for the sequence of Bass numbers and their dual for a finitely generated \(R\)-module.

Introduction

Throughout this paper, \((R, \mathfrak{m}, k)\) is a non-trivial commutative Noetherian local ring with unique maximal ideal \(\mathfrak{m}\) and residue field \(k\). Several authors have obtained results on the growth of the sequence of Betti numbers \(\{\beta_n(k)\}\) (e.g., see [9] and [1]). In [10] Ramras gives some bounds for the sequence \(\{\beta_n(M)\}\) when \(M\) is a finitely generated non-free \(R\)-module. In this paper, we seek to give some bounds for the sequence of Bass numbers.

For a finitely generated \(R\)-module \(M\), let

\[0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \]

be a minimal injective resolution of \(M\). Then, \(\mu^i(M)\) denotes the number of indecomposable components of \(E^i\) isomorphic to the injective envelope \(E(k)\) and is called Bass number of \(M\). This is a dual notion of Betti number. For a prime ideal \(p\), \(\mu^i(p, M)\) denotes the number of indecomposable components of \(E^i\) isomorphic to the injective envelope \(E(R/p)\). It is known that \(\mu^i(M)\) is finite and is equal to the dimension of \(\text{Ext}^i_R(R/m, M)\) considered as a vector space over \(R/m\) (note that \(\mu^i(p, M) = \mu^i(M_p)\)). These numbers play important role in understanding the injective resolution of \(M\), and are the subject of further work. For example, the ring \(R\) of dimension \(d\) is Gorenstein if and only if \(R\) is Cohen-Macaulay and the \(d\)th Bass number \(\mu^d(R)\) is 1. This was proved by Bass in [2]. Vasconcelos conjectured that one could delete the hypothesis that \(R\) be Cohen-Macaulay. This was proved by Paul Roberts in [12].

For a finitely generated \(R\)-module \(M\), it turns out that the least \(i\) for which \(\mu^i(M) > 0\) is the depth of \(M\), while the largest \(i\) with \(\mu^i(M) > 0\) is the injective
A homomorphism \( \varphi : F \to M \) with a flat \( R \)-module \( F \) is called a flat precover of the \( R \)-module \( M \) provided \( \text{Hom}_R(G, F) \to \text{Hom}_R(G, M) \to 0 \) is exact for all flat \( R \)-modules \( G \). If in addition any homomorphism \( f : F \to M \) such that \( f \varphi = \varphi \) is an automorphism of \( F \), then \( \varphi : F \to M \) is called a flat cover of \( M \). A minimal flat resolution of \( M \) is an exact sequence \( \cdots \to F_i \to F_{i-1} \to \cdots \to F_0 \to M \to 0 \) such that \( F_i \) is a flat cover of \( \text{Im}(F_{i-1} \to F_{i-2}) \) for all \( i > 0 \).

A module \( C \) is called cotorsion if \( \text{Ext}^n(F, C) = 0 \) for any flat \( R \)-module \( F \). A flat cover of a cotorsion module is cotorsion and flat, and the kernel of a flat cover is cotorsion. In [4], Enochs showed that a flat cotorsion module is cotorsion and flat, and the kernel of a flat cover is cotorsion. In [4], Enochs showed that a flat cotorsion module \( F \) is uniquely a product \( \prod T_p \), where \( T_p \) is the completion of a free \( R_p \)-module, \( p \in \text{Spec } R \). Therefore, for \( i > 0 \) he defined \( \pi_i(p, M) \) to be the cardinality of a basis of a free \( R_p \)-module whose completion is \( T_p \) in the product \( F_i = \prod T_p \). For \( i = 0 \) he defined \( \pi_0(p, M) \) similarly by using the pure injective envelope of \( F_0 \). In some sense these invariants are dual to the Bass numbers. In [6], Enochs and Xu proved that for a cotorsion \( R \)-module \( M \) which possesses a minimal flat resolution, \( \pi_i(p, M) = \dim_k(p) \text{Tor}^R_1(k(p), \text{Hom}_R(R_p, M)) \). Here \( k(p) \) denotes the quotient field of \( R/p \).

Note that in [3] the authors show that every module has a flat cover, see also [13] and [5].

In this paper, we study the sequence of Bass numbers \( \mu^i(p, M) \) and its dual \( \pi_i(p, M) \). Among the other things we establish the following bounds:

1. \( \mu^2(M)/\mu^1(M) = \ell(R) \) and \( \mu^{n+1}(M)/\mu^n(M) = \ell(R) \) for any \( n \geq 2 \),
2. \( \mu^n(M)/\mu^{n+1}(M) < \ell(R)/\ell(\text{Soc}(R)) \) for any \( n \geq 1 \),

where \( \ell(*) \) refers to the length of *.

1. **Main results**

The following lemma is the key to our main result.

**Lemma 1.1.** Let \( p \) be a prime ideal of \( R \) and let \( L \) be an \( R_p \)-module of finite length. Then the following hold:

(a) For any module \( M \) and any non-negative integer \( n \),
\[
\ell(\text{Ext}^n R_p(L, M)) - \ell(\text{Ext}^n R_p(L, M)) \geq \mu^{n+1}(p, M) - \ell(L)\mu^n(p, M).
\]

(b) For any cotorsion \( R \)-module \( M \) and any non-negative integer \( n \),
\[
\ell(\text{Tor}^{R_p}_{n+1}(L, M)) - \ell(\text{Tor}^{R_p}_n(L, M)) \geq \pi_{n+1}(p, M) - \ell(L)\pi_n(p, M).
\]

**Proof.** (a) We proceed by induction on \( s = \ell(L) \). If \( s = 1 \), then \( L \cong k(p) \), and
\[
\ell(\text{Ext}^n R_p(k(p), M)) - \ell(\text{Ext}^n R_p(k(p), M)) = \mu^{n+1}(p, M) - \mu^n(p, M).
\]
Now assume that \( s > 1 \). Then there is a submodule \( K \) of \( L \) with \( \ell(K) = s - 1 \) such that the sequence \( 0 \to k(p) \to L \to K \to 0 \) is exact. The corresponding long
exact sequence for \( \text{Ext}^{n}_{R_p}(-, M) \) gives the exact sequence
\[
\text{Ext}^{n}_{R_p}(K, M) \rightarrow \text{Ext}^{n}_{R_p}(L, M) \rightarrow \text{Ext}^{n}_{R_p}(k(p), M) \\
\rightarrow \text{Ext}^{n+1}_{R_p}(K, M) \rightarrow \text{Ext}^{n+1}_{R_p}(L, M).
\]
It follows that
\[
\ell(\text{Ext}^{n+1}_{R_p}(L, M)) - \ell(\text{Ext}^{n}_{R_p}(L, M)) \geq \ell(\text{Ext}^{n+1}_{R_p}(K, M)) \\
- \ell(\text{Ext}^{n}_{R_p}(K, M)) - \mu^n(p, M) \\
\geq \mu^{n+1}(p, M) - \ell(K)\mu^n(p, M) - \mu^n(p, M) \\
= \mu^{n+1}(p, M) - \ell(L)\mu^n(p, M),
\]
where the first inequality follows from the property of length and the equality \( \text{Ext}^{n}_{R_p}(k(p), M) = \mu^n(p, M) \), also the second inequality follows by the induction hypothesis.

(b) We proceed by induction on \( s = \ell(L) \). If \( s = 1 \), then \( L \cong k(p) \), and we have
\[
\ell(\text{Tor}^{R_{p+1}}(k(p), M)) - \ell(\text{Tor}^{R_p}(k(p), M)) = \pi_{n+1}(p, M) - \ell(L)\pi_n(p, M).
\]
Now assume that \( s > 1 \). Then there is an \( R_p \)-submodule \( K \) of \( L \) with \( \ell(K) = s - 1 \) such that the sequence \( 0 \rightarrow k(p) \rightarrow L \rightarrow K \rightarrow 0 \) is exact. Set \( N = \text{Hom}_R(R_p, M) \). The corresponding long exact sequence for \( \text{Tor}^{R_p}(-, N) \) leads to the exact sequence
\[
\text{Tor}^{R_{p+1}}(L, N) \rightarrow \text{Tor}^{R_p}(K, N) \rightarrow \text{Tor}^{R_p}(k(p), N) \\
\rightarrow \text{Tor}^{R_p}_n(L, N) \rightarrow \text{Tor}^{R_p}_n(K, N).
\]
It follows that
\[
\ell(\text{Tor}^{R_{p+1}}(L, N)) - \ell(\text{Tor}^{R_p}_n(L, N)) \geq \ell(\text{Tor}^{R_p}_n(K, N)) \\
- \ell(\text{Tor}^{R_p}_n(K, N)) - \pi_n(M) \\
\geq \pi_{n+1}(M) - \ell(K)\pi_n(M) - \pi_n(M) \\
= \pi_{n+1}(M) - \ell(L)\pi_n(M),
\]
where the second inequality follows by the induction hypothesis. \( \square \)

**Corollary 1.2.** Let \( R \) be a zero dimensional ring and let \( M \) be an \( R \)-module. For any prime ideal \( p \) and any integer \( n \geq 1 \) the following hold:

(a)
\[
\mu^{n+1}(p, M) \leq \ell(R_p)\mu^n(p, M).
\]

(b) If \( M \) is a cotorsion \( R \)-module, then
\[
\pi_{n+1}(p, M) \leq \ell(R_p)\pi_n(p, M).
\]

**Proof.** (a) Replace the module \( L \) in Lemma 1.1(a) with \( R_p \) and note that \( \text{Ext}^i_{R_p}(R_p, -) = 0 \) for all \( i \geq 1 \).

(b) Replace the module \( L \) in Lemma 1.1(b) with \( R_p \) and note that \( \text{Tor}^i_{R_p}(R_p, -) = 0 \) for any \( i \geq 1 \). \( \square \)
Proposition 1.3. Let $R$ be a zero dimensional ring. Then the following hold:
(a) Let $M$ be an $R$-module. For any integer $n \geq 1$ and prime ideal $p$,
\[ \mu^{n+1}(p, M) \leq \ell(R_p)\mu^n(p, M). \]
(b) Let $M$ be a cotorsion $R$-module. For any $p \in \text{Spec } R$ and any $n \geq 2$,
\[ \pi_{n+1}(p, M) + \ell(\text{Soc } (R))\pi_{n-1}(p, M) \leq \ell(R_p)\pi_n(p, M). \]

Proof. (a) It is clear from Lemma 1.1(a).
(b) Assume that $p \in \text{Spec } R$ and set $I = \text{Soc } (R_p)$, $N = \text{Hom}_R(R_p, M)$. From the exact sequence
\[ 0 \to I \to R_p \to R_p/I \to 0, \]
it follows that for any $n \geq 1$,
\[ \text{Tor}^{n+1}_R(R_p/I, N) \cong \text{Tor}^R_n(I, N) \cong \mathbb{Q}_n(p, pR_p, N), \]
where the numbers of copies in the direct sum is $\ell(I)$. Hence
\[ \ell(\text{Tor}^{n+1}_R(R_p/I, N)) = \ell(I)\mu_n(p, M) \quad \text{for } n \geq 1. \]
Thus, by Lemma 1.1(b), for $n \geq 2$,
\[ \ell(I)\mu_n(p, M) + \ell(I)\pi_{n-1}(p, M) \geq \pi_{n+1}(p, M) - \ell(R_p/I)\pi_n(p, M). \]
Therefore, $\ell(I)\pi_{n-1}(p, M) + \pi_{n+1}(p, M) \leq \ell(R_p)\pi_n(M)$. \hfill \Box

Theorem 1.4. Let $R$ be a zero dimensional local ring. For any finitely generated non-injective $R$-module $M$ the following hold:
1. $\mu^{n+1}(M)/\mu^n(M) < \ell(R)$ for any $n \geq 2$,
2. $\mu^n(M)/\mu^{n+1}(M) < \ell(R)/\ell(\text{Soc } (R))$ for any $n \geq 1$.

Proof. Let $I = \text{Soc } (R)$. From the exact sequence
\[ 0 \to I \to R \to R/I \to 0, \]
it follows that for any $n \geq 1$,
\[ \text{Ext}^{n+1}_R(R/I, M) \cong \text{Ext}^n_R(I, M) \cong \mathbb{Q}_n(R/m, M), \]
where the numbers of copies in the direct sum is $\ell(I)$. Hence
\[ \ell(\text{Ext}^{n+1}_R(R/I, M)) = \ell(I)\mu^n(M) \quad \text{for } n \geq 1. \]
Thus, by Lemma 1.1, for $n \geq 2$,
\[ \ell(I)\mu^n(M) - \mu^{n-1}(M) \geq \mu^{n+1}(M) - \ell(R/I)\mu^n(M). \]
Therefore, $\ell(I)\mu^{n-1}(M) + \mu^{n+1}(M) \leq \ell(R)\mu^n(M)$. By [7, Theorem 1.1], $\mu^n(M) > 0$ for depth $R/M \leq i \leq \text{inj.dim } R/M$. Since $R$ is Artinian, depth $R/M = 0$. Thus for any $n, n \geq 2$, $\mu^n(M)$ and $\mu^{n-1}(M)$ are positive integer and hence $\mu^{n+1}(M)/\mu^n(M) < \ell(R)$. Moreover, if $2 \leq n$, then $\mu^n(M)$ and $\mu^{n+1}(M)$ are positive integers and thus $\mu^{n-1}(M)/\mu^n(M) < \ell(R)/\ell(\text{Soc } (R))$. \hfill \Box

Corollary 1.5. Let $R$ be a zero dimensional ring. Let $M$ be a finitely generated $R$-module. For any prime ideal $p$ with $M_p$ non-injective $R_p$-module, the following hold:
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1.6. \( \mu^{n+1}(p, M)/\mu^n(p, M) < \ell(R_p) \) for any \( n \geq 2 \),

2. \( \mu^n(p, M)/\mu^{n+1}(p, M) < \ell(R_p)/\ell(\text{Soc}(R_p)) \) for any \( n \geq 1 \).

Remark 1.6. To the best of the knowledge of the authors, there is no condition (yet!) which implies that \( \pi_n(p, M) > 0 \). This is the reason that we could not give a similar result as Theorem 1.4 for the dual notion of Bass numbers.

References


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