τ-SUPPLEMENTED MODULES AND τ-WEAKLY SUPPLEMENTED MODULES

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Abstract. Given a hereditary torsion theory \( \tau = (T, F) \) in \( \text{Mod}-R \), a module \( M \) is called \( \tau \)-supplemented if every submodule \( A \) of \( M \) contains a direct summand \( C \) of \( M \) with \( A/C \tau \)-torsion. A submodule \( V \) of \( M \) is called \( \tau \)-supplement of \( U \) in \( M \) if \( U + V = M \) and \( U \cap V \leq \tau(V) \) and \( M \) is \( \tau \)-weakly supplemented if every submodule of \( M \) has a \( \tau \)-supplement in \( M \). Let \( M \) be a \( \tau \)-weakly supplemented module. Then \( M \) has a decomposition \( M = M_1 \oplus M_2 \) where \( M_1 \) is a semisimple module and \( M_2 \) is a module with \( \tau(M_2) \leq E M_2 \). Also, it is shown that; any finite sum of \( \tau \)-weakly supplemented modules is a \( \tau \)-weakly supplemented module.

Introduction

Throughout this paper, we assume that \( R \) is an associative ring with unity, \( M \) is a unital right \( R \)-module. The symbols, “\( \leq \)” will denote a submodule, “\( \leq_d \)” a module direct summand, “\( \leq_e \)” an essential submodule, “\( \ll \)” small submodule and “\( \text{Rad} (M) \)” the Jacobson radical of \( M \).

Let \( \tau = (T, F) \) be a torsion theory. Then \( \tau \) is uniquely determined by its associated class \( T \) of \( \tau \)-torsion modules \( T = \{ M \in \text{Mod}-R \mid \tau(M) = M \} \) where for a module \( M \), \( \tau(M) = \sum \{ N \mid N \leq M, N \in T \} \) and \( F \) is referred as \( \tau \)-torsion free class and \( F = \{ M \in \text{Mod}-R \mid \tau(M) = 0 \} \). A module in \( T \) (or \( F \)) is called a \( \tau \)-torsion module (or \( \tau \)-torsionfree module). Every torsion class \( T \) determines in every module \( M \) a unique maximal \( T \)-submodule \( \tau(M) \), the \( \tau \)-torsion submodule of \( M \), and \( \tau(M/\tau(M)) = 0 \). In what follows \( \tau \) will represent a hereditary torsion theory, that is, if \( \tau = (T, F) \) then the class \( T \) is closed under taking submodules, direct sums, homomorphic images and extensions by short exact sequences, equivalently the class \( F \) is closed under submodules, direct products, injective hulls and isomorphic copies.

Let \( N \) and \( K \) be submodules of \( M \). \( N \) is said to be a supplement submodule of \( K \) in \( M \) if \( M = N + K \) and \( N \cap K \ll N \). \( M \) is called a weakly supplemented module.

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if every submodule of $M$ has a supplement in $M$. The module $M$ is called a $\oplus$-
supplemented module if every submodule of $M$ has a supplement that is a direct
summand of $M$. Supplemented modules and its variations have been discussed
by several authors in the literature and these modules are useful in characterizing
semiperfect modules and rings.

Given a hereditary torsion theory $\tau = (T, F)$ in Mod-$R$, $\tau$-complemented mod-
ules are studied in [8]. Dually, a module $M$ is said to be a $\tau$-supplemented module
if every submodule $A$ of $M$ contains a direct summand $C$ of $M$ with $A/C \tau$-torsion
[4]. Some further properties of $\tau$-supplemented were studied in [4] and [5].

In this note, we define $\tau$-supplement and $\tau$-weakly supplemented modules. In
Section 2, we will show that

**Theorem.** Let $M$ be a $\tau$-weakly supplemented module. Then

(1) If $M$ is $\tau$-torsionfree, then $M$ is $\tau$-weakly supplemented if and only if $M$
is semisimple.

(2) Every homomorphic image of $M$ is again a $\tau$-weakly supplemented module.

(3) $M/\tau(M)$ is semisimple

and

**Theorem.** Any finite sum of $\tau$-weakly supplemented modules is a $\tau$-weakly sup-
plemented module.

In [6], the authors defined and characterized perfect module and ring relative to
a torsion theory. In this note, we define semiperfect module relative to a torsion
theory and we will prove that

**Theorem.** $M$ is a $\tau$-semiperfect module if and only if $M$ is a $\tau$-weakly supple-
mented module and each $\tau$-supplement submodule of $M$ is a $\tau$-projective cover.

We refer the reader to [3] and [9] as torsion theoretic sources sufficient for our
purposes and [1] and [10] for the other notations in this paper.

1. $\tau$-SUPPLEMENTED MODULES AND $\tau$-WEAKLY SUPPLEMENTED MODULES

Let $\tau = (T, F)$ be a hereditary torsion theory in Mod-$R$ and $M$ be a right
$R$-module. Following [4], $M$ is said to be a $\tau$-supplemented module if every
submodule $A$ of $M$ contains a direct summand $C$ of $M$ with $A/C \tau$-torsion.

Firstly, we give some properties of $\tau$-supplemented modules:

**Theorem 1.1.**

(1) Let $M$ be a module. Then the following are equivalent

(a) $M$ is a $\tau$-supplemented module.

(b) Every submodule $A$ of $M$ can be written as $A = B \oplus C$ with $B$ a direct
summand of $M$ and $\tau(C) = C$.

(c) For every submodule $A$ of $M$, there exist a decomposition $M = X \oplus X'$
with $X \leq A$ and $X' \cap A \leq \tau(X')$.

(d) For every submodule $A$ of $M$, there is an idempotent $e \in \text{End } (M_R)$ such
that $e(M) \subseteq A$ and $(1 - e)(A) \leq \tau((1 - e)A)$.
Let $M$ be a $\tau$-supplemented module. Then
(a) Every submodule of $M$ is a $\tau$-supplemented module.
(b) Every $\tau$-torsionfree submodule of $M$ is a direct summand of $M$.
(c) Every submodule $N$ of $M$ with $N \cap \tau(M) = 0$ is a direct summand of $M$. In particular, if $M$ is $\tau$-torsionfree, then $M$ is $\tau$-supplemented if and only if $M$ is semisimple.
(d) $M/\tau(M)$ is semisimple.
(e) For any submodules $K, N$ of $M$ such that $M = N + K$, there exist a submodule $X$ of $N$ with $M = X + K$ and $K \cap X \subseteq \tau(X)$.
(f) $\text{Rad}(M) \leq \tau(M)$.
(g) If $\tau(M) \neq \text{Rad}(M)$, then $M$ has a nonzero direct summand with $\tau$-torsion.
(h) $\tau(M) = \text{Rad}(M)$ or $M$ has a nonzero $\tau$-torsion submodule that is a direct summand of $M$.

Proof. (1)(a)$\Leftrightarrow$(b) and (2)(a) are [4, Lemma 2.1].
(1)(a)$\Leftrightarrow$ (c) and (a)$\Leftrightarrow$(d) are obvious.
(2)(b) Is [4, Lemma 2.5].
(2)(c) Is [4, Corollary 2.6].
(2)(d) By [5, Theorem 4.8].
(2)(e) Let $M$ be a $\tau$-supplemented and $K, N$ be submodules of $M$ with $M = N + K$. By (2)(a), $N$ is a $\tau$-supplemented module. Then there exist a submodule $X$ of $N$ such that $N = N \cap K + X$ and $N \cap K \cap X$ is $\tau$-torsion and so $N \cap K \cap X \subseteq \tau(X)$. Note that $M = X + K$. It is clear that $K \cap X = N \cap K \cap X \subseteq \tau(X)$.
(2)(f) By (2)(d), $M/\tau(M)$ is semisimple and so $\text{Rad}(M) \leq \tau(M)$.
(2)(g) Assume that $\tau(M) \neq \text{Rad}(M)$. Then there exist a maximal submodule $P$ of $M$ such that $\tau(M)$ is not contained in $P$. Since $M$ is $\tau$-supplemented, there exists a submodule $X$ of $K$ such that $M = X \oplus X'$ and $P \cap X' \leq \tau(X')$ by (1)(c). Note that $P \cap X'$ is also maximal submodule of $X'$. We may assume that $\tau(X') = X'$. Thus $M = X \oplus X'$, where $X' = \tau(X')$.
(2)(h) Clear from (2)(d) and (g). Also, it follows from [5, Theorem 4.9].

As we mentioned in introduction, a submodule $V$ of $M$ is called supplement of $U$ in $M$ if $V$ is a minimal element in the set of submodules $L$ of $M$ with $U + L = M$. So $V$ is a supplement of $U$ if and only if $U + V = M$ and $U \cap V$ is small in $V$. An $R$-module $M$ is weakly supplemented if every submodule of $M$ has a supplement in $M$.

After considering several possible definitions for a supplement module in a torsion theory, by Theorem 2.1, we propose as; a submodule $V$ of $M$ is called $\tau$-supplement of $U$ in $M$ if $U + V = M$ and $U \cap V \leq \tau(V)$ and $M$ is said to be a $\tau$-weakly supplemented module if every submodule of $M$ has a $\tau$-supplement in $M$. Clearly, every $\tau$-supplemented is a $\tau$-weakly supplemented.
Lemma 1.2. Let $M$ be a module and $V \leq M$.

1. If $V$ is a $\tau$-torsionfree $\tau$-supplement submodule, then $V$ is a direct summand of $M$.
2. If $\tau(M) = 0$, then every $\tau$-supplement submodule of $M$ is a direct summand.
3. If $V$ is a $\tau$-supplement submodule of $M$ and $V' \subseteq V$, then $V/V'$ is also a $\tau$-supplement submodule of $M/M'$.

Proof. Trivial.

Theorem 1.3. Let $M$ be a $\tau$-weakly supplemented module. Then

(a) If $M$ is $\tau$-torsionfree, then $M$ is $\tau$-weakly supplemented if and only if $M$ is semisimple.
(b) Every homomorphic image of $M$ is again a $\tau$-weakly supplemented module.
(c) $M/\tau(M)$ is semisimple.

Proof. They are consequences of Lemma 2.2.

The class of $\tau$-supplemented module is not closed under direct sums. Therefore, there are some decompositions theorems for $\tau$-supplemented modules, for example:

A $\tau$-supplemented module $M$ has a decomposition $M = M_1 \oplus M_2$ where $M_1$ is a semisimple module and $M_2$ is a $\tau$-supplemented module with $\tau(M_2) \leq_\tau M_2$ (see [4, Lemma 2.7]).

Lemma 1.4.

1. Let $M$ be a $\tau$-weakly supplemented module. Then $M$ has a decomposition $M = M_1 \oplus M_2$ where $M_1$ is a semisimple module and $M_2$ is a module with $\tau(M_2) \leq_\tau M_2$.
2. For submodules $N, K$ of $M$, if $N$ is a $\tau$-weakly supplemented module and $N + K$ has a $\tau$-supplement in $M$ then $K$ has a $\tau$-supplement in $M$.

Proof. (1) For the proof, we completely follow the proof of [4, Lemma 2.7]. If $\tau(M) \leq_\tau M$, then proof is clear. Assume not. Let $N \leq_\tau M$ be a complement of $\tau(M)$. Therefore $N \oplus \tau(M) \leq_\tau M$. By Theorem 2.3, $N$ is a semisimple module. Since $M$ is $\tau$-supplemented module, there exists a submodule $X$ of $M$ such that $M = N + X$ and $N \leq_\tau X$. Note that $N \leq_\tau X = N \cap (N \cap X) \leq_\tau N \cap \tau(X)$ because $\tau(N) = 0$. Therefore, we have $\tau(X) \leq_\tau X$.

(2) Because $N + K$ has a $\tau$-supplement in $M$, let $A$ be a submodule of $M$ with $M = (N + K) + A$ and $(N + K) \cap A \leq_\tau A$. Since $N$ is $\tau$-weakly supplemented module, there exists a submodule $B$ of $N$ such that $[(K + A) \cap N] + B = N$ and $[(K + A) \cap N] \cap B \leq_\tau B$. Hence $M = K + A + B$ and $B$ is a $\tau$-supplement of $K + A$. We claim that $A + B$ is a $\tau$-supplement of $K$ in $M$. Since $B + K \leq_\tau N + K$, we have $A \cap (B + K) \leq_\tau A$. Now, $(A + B) \cap K \leq_\tau (A + B) \leq_\tau (A + B)$.

The following theorem generalizes a part of [2, 17.13].

Theorem 1.5. Any finite sum of $\tau$-weakly supplemented modules is $\tau$-weakly supplemented module.
Proof. Let $M_1$ and $M_2$ be $\tau$-weakly supplemented modules and $M = M_1 + M_2$. Let $N$ be a submodule of $M$. Clearly, $M_1 + M_2 + N$ has a $\tau$-supplement $0$ in $M$.

By Lemma 2.4, $M_2 + N$ has a $\tau$-supplement in $M$. Again by Lemma 2.4, $N$ has a $\tau$-supplement in $M$. This implies that $M = M_1 + M_2$ is $\tau$-weakly supplemented module.

We recall that a module $M$ is $\tau$-projective if and only if it is projective with respect to every $R$-epimorphism having a $\tau$-torsion kernel [3].

Lemma 1.6. Let $M$ be a module and $L$ a direct summand of $M$ and $K$ a submodule of $M$ such that $M/K$ is $\tau$-projective and $M = L + K$ and $L \cap K$ is $\tau$-torsion. Then $L \cap K$ is direct summand of $M$.

Proof. Let $M = L \oplus L'$ and $\alpha : M/L' \to L$ be the isomorphism and $\beta : L \to M/K \cong L/(L \cap K)$ the epimorphism that having $L \cap K$ as kernel. Then we have epimorphism $\beta \alpha : M/L' \to M/K$ having kernel $((L \cap K) \oplus L')/L' \cong L \cap K$ which is $\tau$-torsion. Since $M/K$ is $\tau$-projective, there exists $g : M/K \to M/L'$ such that $1 = \beta \alpha g$. Hence $L \cap K$ is direct summand.

An epimorphism $f : P \to M$ is called a $\tau$-projective cover of $M$ if $P$ is $\tau$-projective and $\text{Ker}(f)$ is small $\tau$-torsion submodule of $P$ (see [3, Page 117]).

Lemma 1.7.

(1) If $f : P \to N$ is a $\tau$-projective cover and $g : N \to M$ is a $\tau$-projective cover, then $gf : P \to M$ is a $\tau$-projective cover.

(2) The following are equivalent for a module $M$ and $N \leq M$.

(a) If $M/N$ has a $\tau$-projective cover.

(b) $N$ has a $\tau$-supplement $K$ in $M$ which has a $\tau$-projective cover.

(c) If $N'$ is a submodule of $M$ with $M = N + N'$, then $N$ has a $\tau$-supplement $X$ such that $X \leq N'$ and $X$ has a $\tau$-projective cover.

Proof. (1) For the proof, we claim that $\text{Ker}(gf)$ is small $\tau$-torsion. By [7, Lemma 4.2], $\text{Ker}(gf)$ is small. Let $x \in \text{Ker}(gf)$. Then $f(x) \in \text{Ker}(g) \leq \tau(N) = f(\tau(P))$. For any $p \in \tau(P)$, we have $f(x) = f(p)$, and so $x - p \in (f)\tau(P)$, that is $x \in \tau(P)$.

(2)(a)$\Rightarrow$(c) is [6, Lemma 3.1].

(2)(a)$\Rightarrow$(b) is [6, Lemma 3.3].

(2)(c)$\Rightarrow$(b) is clear.

(2)(b)$\Rightarrow$(a) assume $N$ has a $\tau$-supplement $K$ in $M$ which has a $\tau$-projective cover, that is $f : P \to K$ with $\text{Ker}(f)$ is small $\tau$-torsion. Let $g : K \to K/(N \cap K)$. It is easy to see that, $\text{Ker}(g)$ small $\tau$-torsion. Since $N/N \cap K = M/N$, we have $gf : P \to M/N$ is $\tau$-projective cover of $M/N$ by (1).

Following [6], a module $M$ is said to be a $\tau - \oplus$-supplemented when for every submodule $N$ of $M$ there exists a direct summand $K$ of $M$ such that $M = N + K$ and $N \cap K$ is $\tau$-torsion, and $M$ is called a completely $\tau - \oplus$-supplemented if every direct summand of $M$ is $\tau - \oplus$-supplemented and the module $M$ is called strongly $\tau - \oplus$-supplemented if for any submodule $N$ of $M$ there exists a direct summand $K$ of $M$ with $M = N + K$ and $N \cap K$ is small $\tau$-torsion in $K$ by [6].
Theorem 1.8. Let $P$ be a projective $R$-module. Then the following are equivalent:

1. $P$ is $\tau-$supplemented.
2. $P$ is $\tau-\oplus-$supplemented.

Proof. (1) $\Rightarrow$ (2) Clear from definitions.
(2) $\Rightarrow$ (1) Let $N$ be submodule of $P$. By (2), there exists a direct summand $K$ of $P$ such that $P = N + K = K' \oplus K$ and $N \cap K$ is $\tau-$torsion. By [7, Lemma 4.47], there exists a direct summand $L$ of $P$ such that $P = L \oplus K$ and $L \leq N$. Since $N/L$ is isomorphic to $N \cap K$, $N/L$ is $\tau-$torsion. (2) follows.

In [6], a ring $R$ is called a right $\tau$-perfect ring if every right $R$-module has a $\tau$-projective cover (compare with [11, Remark 4.5]). Every right $\tau$-perfect ring is right perfect, and any strongly $\tau-\oplus-$supplemented module is $\tau-\oplus$-supplemented.

Theorem 1.9. Let $R$ be a ring. Then the following are equivalent.

1. $R$ is a right $\tau$-perfect ring.
2. Every projective $R$-module is a strongly $\tau-\oplus-$supplemented module.

Proof. (1) $\Rightarrow$ (2) Let $M$ be any $R$-module, $P$ a projective module and $f$ an epimorphism $f : P \longrightarrow M$. By (2), $P$ has direct summands $K$ and $K'$ so that $P = \text{Ker}(f) + K = K' \oplus K$ with $\text{Ker}(f) \cap K$ small and $\tau$-torsion in $K$. Hence $K$ is the required $\tau$-projective cover of $M$.

Similar to $\tau$-perfect module, we call a module $M$ $\tau$-semiperfect if every homomorphic image of $M$ has a $\tau$-projective cover.

Theorem 1.10. The following are equivalent for a module $M$

1. $M$ is a $\tau$-semiperfect module;
2. $M$ is a $\tau$-weakly supplemented module and each $\tau$-supplement submodule of $M$ has $\tau$-projective cover.
3. For any submodules $K, N$ of $M$ such that $M = N + K$, there exist a $\tau$-supplement submodule $X$ of $N$ that $X$ has a $\tau$-projective cover.

Proof. Clear from Lemma 2.7 and Theorem 2.1.

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