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ON UNIQUE RANGE SETS OF MEROMORPHIC FUNCTIONS IN $\mathbb{C}^m$

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Abstract. By considering a question proposed by F. Gross concerning unique range sets of entire functions in $\mathbb{C}$, we study the unicity of meromorphic functions in $\mathbb{C}^m$ that share three distinct finite sets CM and obtain some results which reduce $5 \leq c_3(\mathcal{M}(\mathbb{C}^m)) \leq 9$ to $5 \leq c_3(\mathcal{M}(\mathbb{C}^m)) \leq 6$.

1. Introduction and main results

Let f be a non-constant meromorphic function in $\mathbb{C}$, and let $a \in \mathbb{C}$ be a finite value. Define $E_f(a)$ to be the set of zeros of $f - a = 0$, each one counted according to its multiplicity. For $a = \infty$, we define $E_f(\infty) := E_{1/f}(0)$. Let $\mathcal{S} \subset \mathbb{P}^1 := \mathbb{C} \bigcup \{\infty\}$ be a non-empty set with distinct elements. Set $E_f(\mathcal{S}) = \bigcup_{a \in \mathcal{S}} E_f(a)$. If, for another non-constant meromorphic function g in $\mathbb{C}$, we have $E_f(\mathcal{S}) = E_g(\mathcal{S})$, then we say that f and g share the set $\mathcal{S}$ CM. In particular, when $\mathcal{S}$ contains only one element, it coincides with the usual definition of CM shared values. We refer the reader to books [7] or [11] for more details on *Nevanlinna's value distribution theory* of meromorphic functions with single variable and its applications.

In 1968, it was F. Gross who first studied the uniqueness problem of meromorphic functions in $\mathbb{C}$ that share distinct sets instead of values in [5]. From then on, he, as well as many other mathematicians, has studied and obtained a lot of results on this topic and its related problems (see, e.g., [8] or [11]).

In 1976, F. Gross asked the following two questions.

Question 1 (see [6] or [12]). *Can one find two distinct finite sets $\mathcal{S}_1$ and $\mathcal{S}_2$ such that any two non-constant entire functions f and g in $\mathbb{C}$ sharing them CM will be identically equal to each other?*

Question 2 (see [6] or [12]). *If the answer to Question 1 is affirmative, then it would be interesting to know how large both sets would have to be?*

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Questions 1 and 2 have been answered by H.-X. Yi completely in 1998. In fact, he proved the following two theorems.

Theorem A (see [12]). *Let f and g be two non-constant entire functions in $\mathbb{C}$, and let $\mathcal{S}_1 = \{0\}$ and $\mathcal{S}_2 = \big\{\omega|\omega^2(\omega+a) - b = 0\big\}$ be two sets, where a and b are two non-zero constants such that $\frac{4a^3}{27} \neq b$. If f and g share the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ CM, then $f \equiv g$.*

Theorem B (see [12]). *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are unique range sets of non-constant entire functions in $\mathbb{C}$, then $\max\big\{\imath(\mathcal{S}_1),\ \imath(\mathcal{S}_2)\big\} \geq 3$ and $\min\big\{\imath(\mathcal{S}_1),\ \imath(\mathcal{S}_2)\big\} \geq 1$, where $\imath(\mathcal{S}_j)$ denotes the cardinality of the set $\mathcal{S}_j$ for $j = 1, 2$.*

Here, we say that $\mathcal{S}_1$, $\mathcal{S}_2$, …, $\mathcal{S}_n$ are unique range sets of entire or meromorphic functions, if the condition that any two non-constant entire or meromorphic functions f and g sharing $\mathcal{S}_1$, $\mathcal{S}_2$, …, $\mathcal{S}_n$ CM implies that $f \equiv g$. Also, examples are given in [12] to show that the conclusions of Theorem B is sharp.

P.-C. Hu and C.-C. Yang generalized the above two theorems to holomorphic functions in $\mathbb{C}^m$, and obtained the following result.

Theorem C (see [8, Theorem 3.42]). *Let f and g be two non-constant holomorphic functions in $\mathbb{C}^m$, and let $\mathcal{S}_1 = \{0\}$ and $\mathcal{S}_2 = \big\{\omega|\omega^n + a\omega^p - b = 0\big\}$ be two sets, where n and p are two relatively prime integers such that $n > p \geq 2$ and $2p > n$, and a and b are two non-zero constants such that $\frac{a^n}{b^{n-p}} \neq \frac{(-1)^p n^n}{p^p(n-p)^{n-p}}$. If f and g share the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ CM, then $f \equiv g$. Obviously, $\min\big\{\imath(\mathcal{S}_2)\big\} = 3$.*

Also, they studied unique range sets of meromorphic functions in $\mathbb{C}^m$, and obtained the following extension of Theorems A-C.

Theorem D (see [8, Theorem 3.43]). *Let f and g be two non-constant meromorphic functions in $\mathbb{C}^m$, and let $\mathcal{S}_1 = \{0\}$, $\mathcal{S}_2 = \big\{\omega|\omega^n + a\omega^p - b = 0\big\}$ and $\mathcal{S}_3 = \{\infty\}$ be three sets, where n and p are two relatively prime integers such that $n > p+1 \geq 3$ and $2p > n+2$, and a and b are two non-zero constants such that $\frac{a^n}{b^{n-p}} \neq \frac{(-1)^p n^n}{p^p(n-p)^{n-p}}$. If f and g share the sets $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ CM, then $f \equiv g$. Obviously, $\min\big\{\imath(\mathcal{S}_2)\big\} = 7$.*

Remark. Please see Section 2 for the definition of meromorphic functions of several variables and that of the corresponding CM shared sets.

Example. Let f and g be two non-constant distinct meromorphic functions in $\mathbb{C}^m$ with the following expressions

$$f = -\frac{ae^{\alpha}(e^{n\alpha}-1)}{e^{(n+1)\alpha}-1} \quad \text{and} \quad g = -\frac{a(e^{n\alpha}-1)}{e^{(n+1)\alpha}-1}.$$

Then $f/g = e^{\alpha}$, where α is a non-constant entire function in $\mathbb{C}^m$. So, f and g share the values 0, ∞ CM. Also, $f^n(f+a) \equiv g^n(g+a)$, which means f and g sharing the set $\mathcal{S} = \big\{\omega|\omega^n(\omega+a) - b = 0\big\}$ CM for any $n \in \mathbb{N}$ and any $a(\neq 0)$, $b \in \mathbb{C}$.

Hence, the above example shows that the assumption "$n > p + 1$" in Theorem D is sharp. Further, it also shows that, in order to reduce the cardinality of the set $\mathcal{S}_2$, we may have to increase the cardinalities of the sets $\mathcal{S}_1$ or $\mathcal{S}_3$.

Define $(\mathcal{S})_n := \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\}$, where the non-empty sets $\mathcal{S}_j \subset \mathbb{P}^1$ are of distinct elements for $j = 1, 2, \ldots, n$, and $\mathcal{S}_j \bigcap \mathcal{S}_k = \emptyset$ whenever $j \neq k$. Define $\imath\big((\mathcal{S})_n\big) := \sum_{j=1}^{n} \imath(\mathcal{S}_j)$ to be the *total cardinality* of the sets $\mathcal{S}_j$ for $1, 2, \ldots, n$. If the sets $\mathcal{S}_j$ $(j = 1, 2, \ldots, n)$ are unique range sets of meromorphic functions in $\mathbb{C}^m$, then we define $c_n\big(\mathcal{M}(\mathbb{C}^m)\big) := \min\big\{\imath\big((\mathcal{S})_n\big)\big\}$, where $\mathcal{M}(\mathbb{C}^m)$ denotes the set of meromorphic functions in $\mathbb{C}^m$ and obviously, is a field.

Apparently, $c_3\big(\mathcal{M}(\mathbb{C}^m)\big) \geq 5$, since, under some trivial transformation, any three non-intersecting sets containing four pairwise distinct elements totally will assume the form $\mathcal{S}_1 = \{0\}$, $\mathcal{S}_2 = \big\{\omega | \omega(\omega + a) - b = 0\big\}$ and $\mathcal{S}_3 = \{\infty\}$ for two constants a and $b(\neq 0)$ such that $\frac{a^2}{4} + b \neq 0$. If $a = 0$, then $f = -g$ will be in our benefit, while if $a \neq 0$, then our aforesaid *Example* will help to this purpose.

In this paper, we shall reduce the upper bound $5 \leq c_3\big(\mathcal{M}(\mathbb{C}^m)\big) \leq 9$ to $5 \leq c_3\big(\mathcal{M}(\mathbb{C}^m)\big) \leq 6$. For admissible meromorphic functions in the unit disc $\triangle \subset \mathbb{C}$, the corresponding result has been obtained by M.-L. Fang in [2], where he stated that the same conclusion holds well for meromorphic functions in $\mathbb{C}$. In this paper, by employing his main ideas, we shall prove the following two theorems.

Theorem 1. *Let f and g be two non-constant meromorphic functions in $\mathbb{C}^m$, and let $\mathcal{S}_1 = \{0, c\}$, $\mathcal{S}_2 = \big\{\omega | \omega^n(\omega + a) - b = 0\big\}$ and $\mathcal{S}_3 = \{\infty\}$ be three sets, where n is a positive integer such that $n \geq 2$, and a, b and c are three non-zero constants such that $c = -\frac{na}{n+1}$, $\frac{(-1)^n n^n a^{n+1}}{(n+1)^{n+1}} \neq b$, $2b$, and $\frac{(-1)^n n^n (n+2) a^{n+1}}{2^{n+1}(n+1)^{n+1}} \neq b$. If f and g share the sets $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ CM, then $f \equiv g$. Obviously,* $\min\big\{\imath(\mathcal{S}_2)\big\} = 3$.

Theorem 2. *Let f and g be two non-constant meromorphic functions in $\mathbb{C}^m$, and let $\mathcal{S}_1 = \{0\}$, $\mathcal{S}_2 = \big\{\omega | \omega(\omega^n + a) - b = 0\big\}$ and $\mathcal{S}_3 = \{\infty, c\}$ be three sets, where n is a positive integer such that $n \geq 2$, and a, b and c are three non-zero constants such that $c = \frac{(n+1)b}{na}$, $\frac{n^n a^{n+1}}{(n+1)^{n+1} b^n} \neq -1$, -2, and $\frac{n^n (n+2) a^{n+1}}{2^{n+1}(n+1)^{n+1} b^n} \neq -1$. If f and g share the sets $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ CM, then $f \equiv g$. Obviously,* $\min\big\{\imath(\mathcal{S}_2)\big\} = 3$.

2. Preliminaries and some lemmas

If f is a holomorphic function on an open connected neighborhood of $\mathfrak{z}_0 \in \mathbb{C}^m$ and $f \not\equiv 0$, then a series

$$f(\mathfrak{z}) = \sum_{j=\nu}^{\infty} \mathcal{P}_j(\mathfrak{z} - \mathfrak{z}_0)$$

converges uniformly on some neighborhood of $\mathfrak{z}_0$ and represents f on this neighborhood. Here, $\mathcal{P}_j$ denotes a homogeneous polynomial of degree j and $\mathcal{P}_\nu \not\equiv 0$. The non-negative integer ν, uniquely determined by f and $\mathfrak{z}_0$, is called the *zero multiplicity* (or *order*) of f at $\mathfrak{z}_0$ and denoted by $\mathfrak{D}_f^0(\mathfrak{z}_0)$.

Let f be a non-constant meromorphic function in $\mathbb{C}^m$. Then, for each $\mathfrak{z} \in \mathbb{C}^m$, there exists an open connected neighborhood $U_\mathfrak{z}$ of $\mathfrak{z}$ and two holomorphic functions

$g \not\equiv 0$ and $h \not\equiv 0$ on $U_{\mathfrak{z}}$, coprime at $\mathfrak{z}$ (i.e., the germs of g and h have no common factors in the local ring of germs of holomorphic functions at $\mathfrak{z}$), such that $hf \equiv g$ on $U_{\mathfrak{z}}$. Then, in $U_{\mathfrak{z}}$ and for $a \in \mathbb{P}^1$,

$$\mathfrak{D}_f^a(\mathfrak{z}) := \mathfrak{D}_{g-ah}^0(\mathfrak{z}) \qquad (a \in \mathbb{C}),$$
$$\mathfrak{D}_f^\infty(\mathfrak{z}) := \mathfrak{D}_h^0(\mathfrak{z}) \qquad (a = \infty)$$

is well defined and called the *a-multiplicity* of f. The function

$$\mathfrak{D}_f^a : \mathbb{C}^m \to \mathbb{Z}^+$$

is called the *a-divisor* of f, where $\mathbb{Z}^+$ denotes the set of non-negative integers. If f is a meromorphic function in $\mathbb{C}^m$, then it is considered as a holomorphic map into the Riemann sphere $\mathbb{P}^1$ outside its set of *indeterminacy* that is usually denoted by $\mathcal{I}_f$. For $a \in \mathbb{P}^1$, we define

$$f^{-1}(a) := \operatorname{supp}(\mathfrak{D}_f^a),$$

where $\operatorname{supp}(\mathfrak{D}_f^a)$ is the support of $\mathfrak{D}_f^a$, defined as the closed set $\overline{(\mathfrak{D}_f^a)^{-1}\big(\mathbb{Z}^+ \backslash \{0\}\big)}$.

Define the *differential form*

$$\eta := dd^c|\mathfrak{z}|^2,$$

where $d := \partial + \bar{\partial}$ and $d^c := \dfrac{1}{4\pi i}(\partial - \bar{\partial})$. For a meromorphic function f in $\mathbb{C}^m$ and $a \in \mathbb{P}^1$, we define the *counting function* of the *a-divisor* of f as

$$n_f^a(r) := \sum_{|\mathfrak{z}| \le r} \mathfrak{D}_f^a(\mathfrak{z}) \quad \text{for} \quad m = 1,$$

and

$$n_f^a(r) := r^{2-2m} \int_{|\mathfrak{z}| \le r} \mathfrak{D}_f^a(\mathfrak{z}) \eta^{m-1} \quad \text{for} \quad m > 1.$$

Write $n\big(r, \frac{1}{f-a}\big) = n_f^a(r)$ for $a \in \mathbb{C}$, and $n(r, f) = n_f^\infty(r)$ for $a = \infty$. Define the *valence function* of the *a-divisor* of f to be

$$N\Big(r, \frac{1}{f-a}\Big) := \int_{r_o}^r \frac{n\big(r, \frac{1}{f-a}\big)}{t}\, dt, \qquad a \in \mathbb{C}, r \ge r_0 > 0;$$
$$N(r, f) := \int_{r_o}^r \frac{n(r, f)}{t}\, dt, \qquad a = \infty, r \ge r_0 > 0.$$

The *compensation function* of $f - a$ for $a \in \mathbb{P}^1$ is defined as

$$m\Big(r, \frac{1}{f-a}\Big) := \frac{1}{V_m(r)} \int_{S_m(r)} \log \frac{1}{\|f(\mathfrak{z}), a\|}\, d\sigma_r, \qquad a \in \mathbb{C},$$
$$m(r, f) := \frac{1}{V_m(r)} \int_{S_m(r)} \log \sqrt{1 + |f(\mathfrak{z})|^2}\, d\sigma_r, \qquad a = \infty,$$

where $\|f(\mathfrak{z}), a\|$ is the chordal distance between $f(\mathfrak{z})$ and a in the Riemann sphere $\mathbb{P}^1$ $\big($for $a = \infty$, it is $\frac{1}{\sqrt{1+|f(\mathfrak{z})|^2}}\big)$, $V_m(r) = \frac{2\pi^m r^{2m-1}}{(m-1)!}$, $S_m(r) = \big\{\mathfrak{z} \in \mathbb{C}^m \,\big|\, |\mathfrak{z}| = r\big\}$, and $d\sigma_r$ is the positive element of volume on $S_m(r)$ such that $\int_{S_m(r)} d\sigma_r = V_m(r)$.

As for the sphere $S_m(r)$, it is considered as a $(2m-1)$-dimensional real manifold that orients to the exterior of the ball $B_m(r) = \{\mathfrak{z} \in \mathbb{C}^m \big| |\mathfrak{z}| < r\}$.

The *Nevanlinna characteristic function* of f is defined as

$$T(r,f) := m(r,f) + N(r,f)\,.$$

In particular, when $m = 1$, the difference between this definition and that in [7] or [11] is an $O(1)$, i.e., a bounded quantity.

Nevanlinna's first main theorem states that for any $a \in \mathbb{C}$,

$$T(r,f) - m(r_0,f) = T\Big(r, \frac{1}{f-a}\Big) - m\Big(r_0, \frac{1}{f-a}\Big)\,,$$

i.e., $T(r,f) = T\big(r, \frac{1}{f-a}\big) + O(1)$.

For $k \in \mathbb{Z}^+ \backslash \{0\}$, define the *truncated a-divisor* of f as

$$\mathfrak{D}^a_{f,k}(\mathfrak{z}) := \min\big\{\mathfrak{D}^a_f(\mathfrak{z}), k\big\}\,,$$

and define the *truncated counting function* $n_k\big(r, \frac{1}{f-a}\big)$ $(a \in \mathbb{C})$, $n_k(r,f)$ $(a = \infty)$, and the *truncated valence function* $N_k\big(r, \frac{1}{f-a}\big)$ $(a \in \mathbb{C})$, $N_k(r,f)$ $(a = \infty)$ generated by $\mathfrak{D}^a_{f,k}(\mathfrak{z})$ similarly.

Nevanlinna's second main theorem states that for pairwise distinct values $a_j \in \mathbb{C}$ $(j = 1, 2, \ldots, q)$,

$$\begin{aligned}(q-1)T(r,f) &\le \sum_{j=1}^{q} N\Big(r, \frac{1}{f-a_j}\Big) + N(r,f) - N_{\mathrm{Ram}}(r,f)\\ &\quad + O\Big(\log \frac{\rho^{2m-1}T(R,f)}{r^{2m-1}(\rho-r)}\Big)\\ &\le \sum_{j=1}^{q} N_1\Big(r, \frac{1}{f-a_j}\Big) + N_1(r,f) + O\Big(\log \frac{\rho^{2m-1}T(R,f)}{r^{2m-1}(\rho-r)}\Big)\,,\end{aligned}$$

where $N_{\mathrm{Ram}}(r,f)$ is called the *ramification term* and $r_0 < r < \rho < R < +\infty$.

Let f and g be two non-constant meromorphic functions in $\mathbb{C}^m$, and let a be a value in $\mathbb{P}^1$. If $\mathfrak{D}^a_f(\mathfrak{z}) \equiv \mathfrak{D}^a_g(\mathfrak{z})$ for all $\mathfrak{z} \in \mathbb{C}^m \setminus \mathcal{I}_f \cup \mathcal{I}_g$, then we say that f and g share the value a CM. For a non-empty set $\mathcal{S} \subset \mathbb{P}^1$, define

$$\mathfrak{D}^{\mathcal{S}}_f(\mathfrak{z}) := \sum_{a\in\mathcal{S}} \mathfrak{D}^a_f(\mathfrak{z})\,.$$

If $\mathfrak{D}^{\mathcal{S}}_f(\mathfrak{z}) \equiv \mathfrak{D}^{\mathcal{S}}_g(\mathfrak{z})$ for all $\mathfrak{z} \in \mathbb{C}^m \setminus \mathcal{I}_f \cup \mathcal{I}_g$, then we say that f and g share the set $\mathcal{S}$ CM.

We refer the reader to [1] or [8] for details on *Nevanlinna's value distribution theory* of meromorphic functions with several variables.

Now, let's introduce several lemmas.

Lemma 1 (see [4] or [8, Theorem 1.26]). *Let f be a non-constant meromorphic function in $\mathbb{C}^m$, and let $P(z)$ and $Q(z)$ be two coprime polynomials with constant* *coefficients and of degrees* p *and* q*, respectively. Write*

$$R(f) := \frac{P(f)}{Q(f)} = \frac{a_p f^p + a_{p-1} f^{p-1} + \cdots + a_0}{b_q f^q + b_{q-1} f^{q-1} + \cdots + b_0} \qquad a_p b_q \neq 0\,.$$

Then, we have

$$T\big(r, R(f)\big) = \max\{p, q\} T(r, f) + O(1)\,.$$

Lemma 2 (see [8, Lemma 1.39] or [10, Equation (5.1)])**.** *Let* $f_0, f_1, \dots, f_n$ *be* $n+1$ *meromorphic functions in* $\mathbb{C}^m$ *such that they are linearly independent. Write* $f := (f_0, f_1, \dots, f_n)$*. Then, there are multi-indices* $\nu_j \in \mathbb{Z}_+^m$ $(j = 1, 2, \dots, n)$ *such that* $0 < |\nu_j| \leq j$ *and* $f, \partial^{\nu_1} f, \partial^{\nu_2} f, \dots, \partial^{\nu_n} f$ *are linearly independent over* $\mathbb{C}^m$*, where* $\mathbb{Z}_+^m$ *denotes the* m*-th Descartes' product of* $\mathbb{Z}^+$*.*

Lemma 3 (see [3] or [9])**.** *Let* f *and* g *be two non-constant meromorphic functions in* $\mathbb{C}^m$*, and let* $a_j \in \mathbb{P}^1$ $(j = 1, 2, 3, 4)$ *be four distinct values. If* f *and* g *share* a_j $(j = 1, 2, 3, 4)$ *CM, then* f *is some Möbius transformation of* g*.*

Remark. Since only Borel's Lemma was involved in [3] and [9] for the proof of their main results, it's straightforward to get the conclusions of our Lemma 3.

Lemma 4 (see [8, Lemma 3.36])**.** *Let* f *and* g *be two non-constant meromorphic functions in* $\mathbb{C}^m$ *such that they share the value* 1 *CM. If there exists a real number* $\lambda \in [0, \frac{1}{2})$ *such that*

$$\| \; N_2\Big(r, \frac{1}{f}\Big) + N_2\Big(r, \frac{1}{g}\Big) + N_2(r, f) + N_2(r, g) \leq \big(\lambda + o(1)\big)\big(T(r, f) + T(r, g)\big)\,,$$

then we have either $f \equiv g$ *or* $fg \equiv 1$*.*

Here and in the following, the notation " $\|$ " denotes that an (in)equality holds as $r \to +\infty$ outside a possible set of finite linear measure.

3. Proof of Theorem 1

Define

$$F := \frac{f^n(f + a)}{b} \quad \text{and} \quad G := \frac{g^n(g + a)}{b}\,.$$

Then, from the assumptions of Theorem 1, we know that F and G share the values 1 and ∞ CM. By Lemma 1, we have

$$T(r, F) = (n + 1) T(r, f) + O(1)$$

and

$$T(r, G) = (n + 1) T(r, g) + O(1)\,. \tag{3.1}$$

We now distinguish the following two cases for discussions.

Case 1. $F - 1$ and $G - 1$ are linearly dependent. Then, there exists a non-zero constant $k \in \mathbb{C}$ such that

$$F - 1 \equiv k(G - 1)\,,$$

which implies that

$$f^n(f+a)-b\equiv k\big(g^n(g+a)-b\big)\,. \tag{3.2}$$

Set $E(f,g)=\big\{\mathfrak{z}|\ \text{either}\ \ f(\mathfrak{z})=g(\mathfrak{z})=0\ \ \text{or}\ \ f(\mathfrak{z})=g(\mathfrak{z})=c\big\}\subset\mathbb{C}^m\setminus\mathcal{I}_f\cup\mathcal{I}_g$.

Subcase 1.1. $E(f,g)\neq\emptyset$.

By (3.2), noting $b\neq 0$ and $b\neq\frac{(-1)^n n^n a^{n+1}}{(n+1)^{n+1}}$, we have $k=1$. Thus,

$$f^n(f+a)\equiv g^n(g+a)\,, \tag{3.3}$$

which means that f and g share the values 0, $-a$ and c CM, since we assume that f and g share the set $\mathcal{S}_1$ CM. Also, by assumption, f and g share the value ∞ CM. By the conclusions of Lemma 3, f is some Möbius transformation of g, say,

$$f=\frac{Ag+B}{Cg+D}\qquad AD-BC\neq 0\,. \tag{3.4}$$

Substituting (3.4) into (3.3) yields

$$g^{n+1}+ag^n\equiv\frac{(Ag+B)^n\big((A+aC)g+(B+aD)\big)}{(Cg+D)^{n+1}}\,.$$

Obviously, $C=0$ and thus $AD\neq 0$. Noting $a\neq 0$ and $n\geq 2$, a routine calculation on the like terms of g leads to $B=0$ and $A=D$. Hence, $f\equiv g$.

Subcase 1.2. $E(f,g)=\emptyset$.

Since f and g share the set $\mathcal{S}_1$ CM, then $\mathfrak{D}^0_f(\mathfrak{z})\equiv\mathfrak{D}^c_g(\mathfrak{z})$ and $\mathfrak{D}^c_f(\mathfrak{z})\equiv\mathfrak{D}^0_g(\mathfrak{z})$.

Subcase 1.2.1. Both $f^{-1}(0)=g^{-1}(c)=\emptyset$ and $f^{-1}(c)=g^{-1}(0)=\emptyset$.

Since f and g share the value ∞ CM, we know that

$$\frac{f}{f-c}\quad\text{and}\quad\frac{g-c}{g}$$

are non-vanishing holomorphic functions and share the value 1 CM. According to the conclusions of Lemma 4, we have either $\frac{f}{f-c}\equiv\frac{g-c}{g}$ or $\frac{f}{f-c}\frac{g-c}{g}\equiv 1$, which implies that either $f+g\equiv c$ or $f\equiv g$. However, if $f+g\equiv c$, then by (3.2) and the fact that $a\neq -c$, $\frac{(-1)^n n^n a^{n+1}}{(n+1)^{n+1}}\neq 2b$, it is self-contradicted. So, $f\equiv g$.

Subcase 1.2.2. Either $f^{-1}(0)=g^{-1}(c)\neq\emptyset$ or $f^{-1}(c)=g^{-1}(0)\neq\emptyset$.

Without loss of generality, we may assume that $f^{-1}(0)=g^{-1}(c)\neq\emptyset$ while $f^{-1}(c)=g^{-1}(0)=\emptyset$. Hence, from (3.2), we get $k=\frac{b}{b-ac^n-c^{n+1}}$.

Obviously, c is the only double root of the equation $z^{n+1}+az^n-c^{n+1}-ac^n=0$ while the remaining $n-1$ roots, say, a_j $(j=1,2,\dots,n-1)$, are all simple.

Let b_k $(k=1,2,\dots,n+1)$ be the $n+1$ roots of the following equation

$$z^{n+1}+az^n-b=-\frac{b-ac^n-c^{n+1}}{k}=-\frac{(b-ac^n-c^{n+1})^2}{b}\,. \tag{3.5}$$

By our hypothesis that $a\neq -c$, $\frac{(-1)^n n^n a^{n+1}}{(n+1)^{n+1}}\neq b$ and $\frac{(-1)^n n^n a^{n+1}}{(n+1)^{n+1}}\neq 2b$, equation (3.5) has no multiple roots at all, since neither 0 nor c is a root of it. Hence, b_k

($k = 1, 2, \dots, n+1$) are pairwise distinct such that $\prod_{k=1}^{n+1} b_k \neq 0$. By (3.2), noting that $f^{-1}(c) = \emptyset$, we have $\sum\limits_{j=1}^{n-1} \mathfrak{D}^{a_j}_{f,1}(\mathfrak{z}) \equiv \sum\limits_{k=1}^{n+1} \mathfrak{D}^{b_k}_{g,1}(\mathfrak{z})$.

Applying *Nevanlinna's second main theorem* to g, and noting that $f^{-1}(c) = g^{-1}(0) = \emptyset$, and f and g share the value ∞ CM, we conclude that

$$\begin{aligned}\| \quad (n+1)T(r,g) &\le N_1(r,g) + N_1\Big(r, \frac{1}{g}\Big) + \sum_{k=1}^{n+1} N_1\Big(r, \frac{1}{g-b_k}\Big) + o\big(T(r,g)\big) \\ &\le N_1(r,f) + N_1\Big(r, \frac{1}{f-c}\Big) + \sum_{j=1}^{n-1} N_1\Big(r, \frac{1}{f-a_j}\Big) + o\big(T(r,g)\big) \\ &\le nT(r,f) + o\big(T(r,g)\big)\,. \qquad (3.6)\end{aligned}$$

However, by (3.2), we have $T(r,f) = T(r,g) + O(1)$. Combining it with (3.6) yields $\| \ T(r,g) = o\big(T(r,g)\big)$, a contradiction.

If $f^{-1}(0) = g^{-1}(c) = \emptyset$ and $f^{-1}(c) = g^{-1}(0) \neq \emptyset$, interchanging the positions of f and g yields a contradiction, too. Hence, Subcase 1.2.2 can be ruled out.

Subcase 1.2.3. Neither $f^{-1}(0) = g^{-1}(c) = \emptyset$ nor $f^{-1}(c) = g^{-1}(0) = \emptyset$.

By a similar way as above, we have $k = \frac{b}{b-ac^n-c^{n+1}}$ and $k = \frac{b-ac^n-c^{n+1}}{b}$, which yields $\frac{(-1)^n n^n a^{n+1}}{(n+1)^{n+1}} = 2b$ since we assume that $a \neq -c$, and a contradiction against our hypothesis follows immediately.

Case 2. $F-1$ and $G-1$ are linearly independent. In this case, we have $F \not\equiv G$.

From the conclusions of Lemma 2, there exists an integer $j_0 \in \{1, 2, \dots, m\}$ such that $(F-1, G-1)$ and $(\partial_{z_{j_0}} F, \partial_{z_{j_0}} G)$ are linearly independent, i.e.,

$$W = \begin{vmatrix} F-1 & G-1 \\ \partial_{z_{j_0}} F & \partial_{z_{j_0}} G \end{vmatrix} \not\equiv 0\,.$$

Define

$$H := \frac{W}{(F-1)(G-1)} = \frac{\partial_{z_{j_0}} G}{G-1} - \frac{\partial_{z_{j_0}} F}{F-1}\,. \qquad (3.7)$$

By the *lemma of the logarithmic derivative* (see [8, Lemma 1.34] or [10]),

$$\| \quad m(r,H) = o\big(T(r,f) + T(r,g)\big)\,.$$

Define $\mathcal{I}_{F-1}$ to be the set of *indeterminacy* of $F-1$. For each $\mathfrak{z} \in \mathbb{C}^m$, there exists an open connected neighborhood $U_{\mathfrak{z}}$ of $\mathfrak{z}$ and two holomorphic functions $F_1 \not\equiv 0$ and $F_2 \not\equiv 0$ on $U_{\mathfrak{z}}$, coprime at $\mathfrak{z}$, such that $F_1(F-1) \equiv F_2$,

$$\dim_{\mathfrak{z}} F_1^{-1}(0) \bigcap F_2^{-1}(0) \le m-2$$

and

$$\mathcal{I}_{F-1} \bigcap U_{\mathfrak{z}} \equiv F_1^{-1}(0) \bigcap F_2^{-1}(0)\,.$$

Define $\mathcal{E}_1 := \{\mathrm{supp}(\mathfrak{D}^0_{F-1})\}_s$ to be the set of *singular points* of the analytic set $\mathrm{supp}(\mathfrak{D}^0_{F-1})$. Then, $\mathcal{E}_1$ is of co-dimension at least 2. Define $\mathcal{E}_2 := \{\mathrm{supp}(\mathfrak{D}^\infty_{F-1})\}_s$, $\mathcal{I}_{G-1}$, $\mathcal{E}_3 := \{\mathrm{supp}(\mathfrak{D}^0_{G-1})\}_s$ and $\mathcal{E}_4 := \{\mathrm{supp}(\mathfrak{D}^\infty_{G-1})\}_s$ similarly. Write $\mathcal{E} := \mathcal{I}_{F-1}\bigcup\mathcal{I}_{G-1}\bigcup\mathcal{E}_1\bigcup\mathcal{E}_2\bigcup\mathcal{E}_3\bigcup\mathcal{E}_4$. Then, $\dim_{\mathfrak{z}}\mathcal{E}\le m-2$.

Take $\mathfrak{z}_0\in\mathbb{C}^m\backslash\mathcal{E}$ to be a point such that $\mathfrak{D}^0_{F-1}(\mathfrak{z}_0)=p\in\mathbb{Z}^+\backslash\{0\}$. Hence, $\mathfrak{D}^0_{G-1}(\mathfrak{z}_0)=p$ by our hypothesis that F and G share the value 1 CM. By a result of Bernstein-Chang-Li [1, Lemma 2.3], there exists a holomorphic coordinate system $\mathfrak{u}=(u_1,u_2,\dots,u_m)$ of $\mathfrak{z}_0$ on $U_{\mathfrak{z}_0}\subset\mathbb{C}^m\backslash\mathcal{E}$ such that

$$U_{\mathfrak{z}_0}\bigcap\mathrm{supp}(\mathfrak{D}^0_{F-1})=\{\mathfrak{z}\in U_{\mathfrak{z}_0}\mid u_1(\mathfrak{z})=0\}$$

and

$$\mathfrak{u}(\mathfrak{z}_{\mathfrak{o}})=\big(u_1(\mathfrak{z}_0),u_2(\mathfrak{z}_0),\dots,u_m(\mathfrak{z}_{\mathfrak{o}})\big)=\mathfrak{o}\in\mathbb{C}^m\,.$$

Hence, there exists a biholomorphic coordinate transformation

$$z_j=z_j(u_1,u_2,\dots,u_m)\qquad j=1,2,\dots,m$$

around $\mathfrak{o}\in\mathbb{C}^m$ such that $\mathfrak{z}_0=\mathfrak{z}(\mathfrak{o})=\big(z_1(\mathfrak{o}),z_2(\mathfrak{o}),\dots,z_m(\mathfrak{o})\big)$. So, we can write

$$F(\mathfrak{z})-1=u_1^pF^*(u_1,u_2,\dots,u_m)$$

and

$$G(\mathfrak{z})-1=u_1^pG^*(u_1,u_2,\dots,u_m)\,,$$

where F^* and G^* are holomorphic functions around $\mathfrak{o}\in\mathbb{C}^m$ and do not vanish along the analytic set $U_{\mathfrak{z}_0}\bigcap\mathrm{supp}(\mathfrak{D}^0_{F-1})$. A routine calculation leads to

$$\frac{\partial_{z_{j_0}}F}{F-1}\bigg|_{\mathfrak{z}_0}=\frac{p}{u_1}\frac{\partial u_1}{\partial z_{j_0}}\bigg|_{\mathfrak{o}}+\frac{1}{F^*}\sum_{t=1}^m\frac{\partial F^*}{\partial u_t}\frac{\partial u_t}{\partial z_{j_0}}\bigg|_{\mathfrak{o}}\tag{3.8}$$

and

$$\frac{\partial_{z_{j_0}}G}{G-1}\bigg|_{\mathfrak{z}_0}=\frac{p}{u_1}\frac{\partial u_1}{\partial z_{j_0}}\bigg|_{\mathfrak{o}}+\frac{1}{G^*}\sum_{t=1}^m\frac{\partial G^*}{\partial u_t}\frac{\partial u_t}{\partial z_{j_0}}\bigg|_{\mathfrak{o}}\,.\tag{3.9}$$

Hence, $H|_{\mathfrak{z}_0}=O(1)$, i.e., $\mathfrak{D}^\infty_H(\mathfrak{z}_0)=0$.

Take $\mathfrak{z}_\infty\in\mathbb{C}^m\backslash\mathcal{E}$ to be a point such that $\mathfrak{D}^\infty_{F-1}(\mathfrak{z}_0)=q\in\mathbb{Z}^+\backslash\{0\}$. Similarly, $\mathfrak{D}^\infty_{G-1}(\mathfrak{z}_0)=q$ and $\mathfrak{D}^\infty_H(\mathfrak{z}_\infty)=0$. Hence, H is holomorphic on $\mathbb{C}^m$ and

$$\|\ \ N(r,H)=o\big(T(r,f)+T(r,g)\big)\,.$$

Therefore, we obtain

$$\|\ \ T(r,H)=o\big(T(r,f)+T(r,g)\big)\,.\tag{3.10}$$

It is not difficult to show that

$$\begin{aligned}\|\ \ N\Big(r,\frac{1}{f}\Big)&\le N\Big(r,\frac{1}{H}\Big)+o\big(T(r,f)+T(r,g)\big)\\&\le T(r,H)+o\big(T(r,f)+T(r,g)\big)\le o\big(T(r,f)+T(r,g)\big)\,.\end{aligned}$$

Analogically, $\|\ N\big(r,\frac{1}{g}\big)\le o\big(T(r,f)+T(r,g)\big)$, $\|\ N\big(r,\frac{1}{f-c}\big)\le o\big(T(r,f)+T(r,g)\big)$ and $\|\ N\big(r,\frac{1}{g-c}\big)\le o\big(T(r,f)+T(r,g)\big)$.

Combining the method used in Subcase 1.2.1, the estimate that $N_2(r,\cdot) \le N(r,\cdot) + O(1)$ on valence functions, and the conclusions of Lemma 4 yields either $f+g \equiv c$ or $f \equiv g$. The former case implies that $f^{-1}(0) = g^{-1}(c)$ and $f^{-1}(c) = g^{-1}(0)$. Since f and g share the set $\mathcal{S}_2$ CM, and 0 and c are the only two Picard values of both f and g, and $\frac{(-1)^n n^n a^{n+1}}{(n+1)^{n+1}} \ne b$ (then, all the zeros, say, $\omega_1, \omega_2, \ldots, \omega_{n+1}$, of the equation $\omega^n(\omega + a) - b = 0$ are simple and distinct from 0, c) and $\frac{(-1)^n n^n (n+2) a^{n+1}}{2^{n+1}(n+1)^{n+1}} \ne b$ (then, $\omega_j \ne \frac{c}{2}$ for $j = 1, 2, \ldots, n+1$), so, without loss of generality, we might assume that $\omega_1 + \omega_2 = c$, $\omega_2 + \omega_3 = c$, $\ldots$, $\omega_n + \omega_{n+1} = c$, $\omega_{n+1} + \omega_1 = c$. Noting $n \ge 2$, we derive that $\omega_2 = \omega_{n+1}$, a contradiction. On the other hand, the latter case yields $F \equiv G$, a contradiction, too. The proof of Theorem 1 finishes here completely. □

4. Proof of Theorem 2

Define $f^* := 1/f$ and $g^* := 1/g$. By the assumptions of Theorem 2 and the conclusions of Theorem 1, we have $f^* \equiv g^*$. Hence, $f \equiv g$. □

Final Note. From a recent discussion with M. Shirosaki in a conference at Hiroshima University, the second author was informed that any three non-intersecting sets of the form $\mathcal{S}_1 = \{a_1, a_2\}$, $\mathcal{S}_2 = \{b_1, b_2\}$ and $\mathcal{S}_3 = \{c\}$ are necessarily not unique range sets, see also H.-X. Yi's paper [12, Examples 3 and 4] with $\mathcal{S}_3 = \{\infty\}$. On the other hand, our aforementioned several examples show that, under some trivial transformation, the only possible triple unique range sets of five elements might be $\mathcal{S}_1 = \{0\}$, $\mathcal{S}_2 = \{\omega | \omega^3 + a\omega^2 + b\omega + c = 0\}$ and $\mathcal{S}_3 = \{\infty\}$ such that the polynomial in $\mathcal{S}_2$ has no multiple zeros and $bc \ne 0$. Further, we think new method, say, that of algebraic curve, might be involved to completely solve this problem. Also see related works of A. Boutabaa and A. Escassut on p-adic fields.

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