IDEAL AMENABILITY OF MODULE EXTENSIONS OF BANACH ALGEBRAS

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Abstract. Let $A$ be a Banach algebra. $A$ is called ideally amenable if for every closed ideal $I$ of $A$, the first cohomology group of $A$ with coefficients in $I^*$ is zero, i.e., $H^1(A, I^*) = \{0\}$. Some examples show that ideal amenability is different from weak amenability and amenability. Also for $n \in \mathbb{N}$, $A$ is called $n$-ideally amenable if for every closed ideal $I$ of $A$, $H^1(A, I^{(n)}) = \{0\}$.

In this paper we find the necessary and sufficient conditions for a module extension Banach algebra to be 2-ideally amenable.

1. Introduction

Let $A$ be a Banach algebra and let $X$ be a Banach $A$-bimodule. Then $X^*$, the dual space of $X$, with the following module actions is a Banach $A$-bimodule:

$$\langle x, a \cdot x^* \rangle = \langle x \cdot a, x^* \rangle, \quad \langle x, x^* \cdot a \rangle = \langle a \cdot x, x^* \rangle, \quad (a \in A, \ x \in X, \ x^* \in X^*) .$$

In particular, if $I$ is a closed ideal in $A$, then $I$ and $I^*$ will be a Banach $A$-bimodule and a dual Banach $A$-bimodule respectively. A bounded linear operator $D : A \to X$ is called a derivation if

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in A) .$$

For $x \in X$, we put $\delta_x : A \to X$ by

$$\delta_x(a) = a \cdot x - x \cdot a \quad (a \in A) .$$

It is clear that $\delta_x$ is a derivation. Derivations of this form are called inner derivations. A Banach algebra $A$ is amenable if for every Banach $A$-bimodule $X$, every derivation from $A$ into $X^*$ is inner; i.e., $H^1(A, X^*) = \{0\}$, where $H^1(A, X^*)$ is the first cohomology group of $A$ with coefficients in $X^*$. Johnson has introduced the concept of amenability of Banach algebras [12]. A Banach algebra $A$ is weakly amenable if $H^1(A, A^*) = \{0\}$ (see [3], [9], [10] and [13]). Bade, Curtis and Dales [1] defined the concept of weak amenability for commutative Banach algebras. Let $n \in \mathbb{N}$; a Banach algebra $A$ is called $n$-weakly amenable if $H^1(A, A^{(n)}) = \{0\}$.

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Dales, Ghahramani and Grønbæk brought the concept of \( n \)-weak amenability of Banach algebras in [2].

**Definition 1.1.** A Banach algebra \( A \) is called ideally amenable if for every closed ideal \( I \) of \( A \); \( H^1(A, I^*) = \{0\} \).

**Definition 1.2.** A Banach algebra \( A \) is called \( n \)-ideally amenable if for every closed ideal \( I \) of \( A \); \( H^1(A, I^{(n)}) = \{0\} \).

2. Some Examples

Obviously, amenability implies ideal amenability and ideal amenability implies weak amenability. However, the following examples show that the converse is not valid.

**Example 2.1.** Consider the algebra \( A = B(H) \) of bounded linear operators on some infinite-dimensional separable Hilbert space \( H \). Then \( A \) has exactly two nonzero closed ideals \( I_0 = K(H) \), the compact operators on \( H \), and \( I_1 = B(H) \). Denoting by \( N(H) \) the space of nuclear or trace-class operators on \( H \), we have

\[
H^1(A, I_0^*) = H^1(B(H), N(H)) = \{0\},
\]

\[
H^1(A, I_1^*) = H^1(B(H), (B(H))^*) = \{0\}.
\]

To prove (1), take any bounded derivation \( D : B(H) \to N(H) \). The restriction of \( D \) to \( K(H) \) is a derivation \( D_0 : K(H) \to N(H) = K(H)^* \) and hence of the form \( D_0(T) = AT - TA \) \((T \in K(H))\), for some \( A \in N(H) \) [11, Corollary 4.2]. But \( D_0 \) being weakly compact, the above equation extends to all of \( B(H) \), such that \( D(T) = AT - TA \) \((T \in B(H))\), showing that \( D \) is inner. The proof of (2) follows directly from the result of Haagerup just quoted. Thus \( A = B(H) \) is an example of an ideally amenable Banach algebra that is not amenable [15].

We know that \( B(H) \) is a \( C^* \)-algebra. So, one might wonder about the ideal amenability of \( C^* \)-algebras. Here we have:

**Example 2.2.** All \( C^* \)-algebras \( A \) are ideally amenable. Indeed let \( I \) be a closed two-sided ideal in \( A \) and let \( D : A \to I^* \) be a derivation. Since the restriction of \( D \) to \( I \) is again a derivation and \( I \) is a \( C^* \)-algebra in its own right, there exists by [11] an \( f \in I^* \) such that \( Db = bf - fb \) for all \( b \in I \). We have to show that this holds true for all \( a \in A \). For an approximate identity \( (e_\alpha) \) in \( I \) and \( b \in I \) and \( a \in A \) we have

\[
\langle e_\alpha b, D(a) \rangle = \langle b, D(a)e_\alpha \rangle = \langle b, D(ae_\alpha) - aD(e_\alpha) \rangle = \langle b, (ae_\alpha)f - f(ae_\alpha) \rangle = \langle (ba)e_\alpha - a(e_\alpha b), f \rangle = \langle (ba)e_\alpha - e_\alpha (ba), f \rangle.
\]

Such that in the limit

\[
\langle b, D(a) \rangle = \langle ba - ab, f \rangle = \langle ba, f - fa \rangle,
\]
i.e. $Da = af - fa$ for all $a \in A$. This means $H^1(A, I^\ast) = \{0\}$. Therefore $A$ is ideally amenable.

**Remark 2.3.** A $C^\ast$-algebra is amenable if and only if it is nuclear [11]. So, a non-nuclear $C^\ast$-algebra is not amenable, but ideally amenable.

Let $n \in \mathbb{N}$, then the following assertions hold.

a) Every $n$-ideally amenable Banach algebra is $n$-weakly amenable.

b) An amenable Banach algebra is $n$-ideally amenable.

c) Every $(n + 2)$-ideally amenable Banach algebra is $n$-ideally amenable.

d) Every weakly amenable commutative Banach algebra is $n$-ideally amenable.

e) A commutative Banach algebra $A$ is weakly amenable if and only if $A$ is $(2n - 1)$-ideally amenable.

The assertions (a) and (b) above are obvious. Assertion (c) is Theorem 1.5 of [7]. Also (d) and (e) follow from Theorem 1.5 of [1].

**Example 2.4.** Let $0 < \alpha < \frac{1}{2}$, and let $(K, d)$ be an infinite compact metric space. Then $A = \text{lip}_\alpha(K)$ is weakly amenable Banach algebra that is not amenable [1]. $A$ is commutative, then by assertion (d) above, $A$ is ideally amenable.

There are also some examples of Banach algebras which show that ideal amenability is not equivalent to weak amenability. In the following we give one of them.

**Example 2.5.** Let $A = L^1(G)$, where $G = SL(2, \mathbb{R})$, the set of elements in $M_2(\mathbb{R})$ with determinant one. Also let $I = \{ f \in L^1(G) : \int_G f(g)dm_G(g) = 0 \}$, the augmentation ideal of $A$. By Theorem 5.2 of [14]: $H^1(A, I^\ast) \neq \{0\}$. So, $A$ is not ideally amenable. On the other hand, for every locally compact group $G$, $L^1(G)$ is weakly amenable [13]. Thus $A$ is weakly amenable.

For more examples see [5] and [8].

### 3. Module extension Banach algebras

Let $A$ and $X$ be a Banach algebra and a Banach $A$-bimodule respectively. Consider $A \oplus X$ as a Banach space with the following norm

$$\|(a, x)\| = \|a\| + \|x\| \quad (a \in A, x \in X).$$

Then $A \oplus X$ is a Banach algebra with the product

$$(a_1, x_1)(a_2, x_2) = (a_1a_2, x_1 \cdot a_2 + a_1 \cdot x_2).$$

$A \oplus X$ is called a module extension Banach algebra. Since $(A \oplus X)^\ast = (0 \oplus X)^\perp + (A \oplus 0)^\perp$, where $\perp$ denotes the direct $A$-bimodule $l_\infty$-sum, and $(0 \oplus X)^\perp$ (respectively, $(A \oplus 0)^\perp$) is isometrically isomorphic to $A^\ast$ (respectively, $X^\ast$) as $A$-bimodules, for convenience, we simply identify the corresponding terms and write

$$(A \oplus X)^\ast = A^\ast + X^\ast.$$

Take $A^{(n)} \oplus X^{(n)}$ as the underlying space of $(A \oplus X)^{(n)}$. The sum is an $l_1$-sum when $n$ is even and is an $l_\infty$-sum when $n$ is odd. One can verify that the $(A \oplus X)$-bimodule
actions on \((A \oplus X)^{(n)}\) for \((a, x) \in A \oplus X\) and \((a^{(n)}, x^{(n)}) \in A^{(n)} \oplus X^{(n)} = (A \oplus X)^{(n)}\) are formulated as follows:

\[
(a, x)((a^{(n)}, x^{(n)}) = (aa^{(n)} + xa^{(n)} + ax^{(n)})
\]

and

\[
(a^{(n)}, x^{(n)})(a, x) = (a^{(n)}a + x^{(n)}a, x^{(n)})
\]

where \(n\) is odd, and

\[
(a, x)((a^{(n)}, x^{(n)}) = (aa^{(n)}a + xa^{(n)} + xa^{(n)})
\]

and

\[
(a^{(n)}, x^{(n)})(a, x) = (a^{(n)}a, a^{(n)}x + x^{(n)}a)
\]

where \(n\) is even.

We need the following lemma for the main result of paper.

**Lemma 3.1.** Let \(A\) be a Banach algebra and let \(X\) be a Banach \(A\)-bimodule. Then \(J\) is a closed two sided ideal of \(A \oplus X\), if and only if there exist a closed ideal \(I\) of \(A\) and a closed \(A\)-submodule \(Y\) of \(X\) such that \(J = I \oplus Y\) and \(IX \cup XI \subseteq Y\).

Yong Zhang in [16] found a necessary and sufficient condition for a module extension Banach algebra to be \(n\)-weakly amenable \((n = 1, 2, \ldots)\). Also in [6, Theorem 2.4], it has been proved that:

**Theorem 3.2.** \(A \oplus X\) is ideally amenable if and only if for arbitrary ideal \(I \oplus Y\) of \(A \oplus X\) the following conditions hold:

1. \(H^1(A, I^*) = \{0\}\);
2. \(H^1(A, Y^*) = \{0\}\);
3. For every continuous \(A\)-bimodule morphism \(\Gamma: X \to I^*\), there exists \(F \in Y^*\) such that \(aF - Fa = 0\) for \(a \in A\) and \(\Gamma(x) = xF - Fx\) for \(x \in X\);
4. The only continuous \(A\)-bimodule morphism \(T: X \to Y^*\) for which \(xT(y) + T(x)y = 0\) \((x, y \in X)\) in \(I^*\) is \(T = 0\).

We prove the similar argument for \(n\)-ideal amenability when \(n = 2\).

**Lemma 3.3.** Suppose that \(T: X \to Y^{**}\) is a continuous \(A\)-bimodule morphism. Then \(T: A \oplus X \to (I \oplus Y)^{**}\), defined by \(\hat{T}(a, x) = (0, T(x))\) is a continuous derivation. \(T\) is inner if and only if there exists \(u \in I^{**}\) such that \(ua = au\) for \(a \in A\) and \(T(x) = xu - ux\) for all \(x \in X\).

**Proof.** Let \((a, x), (b, y) \in A \oplus X\). We have

\[
\hat{T}((a, x) \cdot (b, y)) = \hat{T}((ab, ay + xb)) = (0, T(ay + xb)) = (0, aT(y) + T(x)b).
\]

On the other hand

\[
\hat{T}((a, x)) \cdot (b, y) = (0, T(x)) \cdot (b, y) = (0, 0 + T(x)b)
\]
and
\[(a, x) \cdot \bar{T}((b, y)) = (a, x) \cdot (0, T(y)) = (0, aT(y) + 0) .\]

It is clear that \(\bar{T}\) is continuous and thus \(\bar{T}\) is a derivation. Let \(\bar{T}\) be inner, then there exist \(u \in I^{**}\) and \(F \in Y^{**}\) such that
\[
\bar{T}((a, x)) = (a, x) \cdot (u, F) - (u, F) \cdot (a, x) = (au - ua, aF - Fa + xu - ux) ,
\]
but
\[
(0, T(x)) = \bar{T}(0, x)) = (0, xu - ux)
\]
and
\[
(0, 0) = \bar{T}((a, 0)) = (au - ua, aF - Fa).
\]

It shows that \(au = ua\) and therefore there exists \(u \in I^{**}\) such that \(T(x) = xu - ux\) \((x \in X)\). For converse, let \(au = ua\) and there exists \(u \in I^{**}\) such that \(T(x) = xu - ux\) \((x \in X)\). We have
\[
\bar{T}((a, x)) = (0, T(x)) = (au - ua, xu - ux)
\]
and hence \(\bar{T}((a, x)) = (a, x) \cdot (u, 0) - (u, 0) \cdot (a, x)\), where \((u, 0) \in (I \oplus Y)^{**}\). Then \(\bar{T}\) is inner and proof is complete.

If \(D: A \to Y^{**}\) is a continuous derivation, we define \(\tilde{D}: A \oplus X \to (I \oplus Y)^{**}\) by \(\tilde{D}((a, x)) = (0, D(a))\). Also, if \(T: X \to I^{**}\) is a continuous \(A\)-bimodule morphism such that \(xT(y) + T(x)y = 0\), we define \(\bar{T}: A \oplus X \to (I \oplus Y)^{**}\) by \(\bar{T}((a, x)) = (T(x), 0)\).

Lemma 3.4. The operators \(\tilde{D}\) and \(\bar{T}\) defined above are continuous derivations. Furthermore, the derivation \(\tilde{D}\) is inner if and only if \(D\) is inner, and \(\bar{T}\) is inner if and only if \(T = 0\).

Proof. It is clear that \(\tilde{D}\) and \(\bar{T}\) are continuous derivations. Let \(\tilde{D}\) be inner and \((a, x) \in A \oplus X\) be arbitrary. There exist \(u \in I^{**}\), \(F \in Y^{**}\) such that \(\tilde{D}((a, x)) = (a, x) \cdot (u, F) - (u, F) \cdot (a, x) = (au - ua, aF - Fa + xu - ux)\). But
\[
(0, D(a)) = \tilde{D}((0, 0)) = (au - ua, aF - Fa)
\]
and
\[
(0, 0) = \tilde{D}((0, x)) = (0, xu - ux).
\]

Then \(D(a) = aF - Fa\) for some \(F \in Y^{**}\) and so \(D\) is inner. For converse, let \(D\) be inner. There exists \(F \in Y^{**}\) such that \(D(a) = aF - Fa\) \((a \in A)\). Then
\[
\tilde{D}((a, x)) = (0, D(a)) = (0, aF - Fa) = (a, x) \cdot (0, F) - (0, F) \cdot (a, x) .
\]
This means that there exists \(\xi = (0, F) \in (I \oplus Y)^{**}\) such that \(\tilde{D}((a, x)) = (a, x) \cdot \xi - \xi \cdot (a, x) \((a, x) \in A \oplus X\)\). Then \(\tilde{D}\) is inner. Now let \(\bar{T}\) be inner. There exist \(u \in I^{**}\), \(F \in Y^{**}\) such that for each \((a, x) \in A \oplus X\),
\[
\bar{T}((a, x)) = (a, x) \cdot (u, F) - (u, F) \cdot (a, x) = (au - ua, aF - Fa + xu - ux) .
\]
But \[(T(x), 0) = \bar{T}((0, x)) = (0, xu - ux)\]
and so \(T(x) = 0\), for every \(x \in X\). The converse is trivial. \qed

Now we find a necessary and sufficient condition for a module extension Banach algebra to be 2-ideally amenable.

**Theorem 3.5.** \(A \oplus X\) is 2-ideally amenable if and only if for every arbitrary ideal \(I \oplus Y\) of \(A \oplus X\) the following conditions hold:

1. the only continuous derivations \(D: A \to I^{**}\) for which there is a continuous operator \(T: X \to Y^{**}\) such that \(T(ax) = D(a)x + aT(x)\) and \(T(xa) = xD(a) + T(x)a\) (\(a \in A, x \in X\)) are the inner derivations;
2. \(H^1(A, Y^{**}) = \{0\};\)
3. the only continuous \(A\)-bimodule morphism \(\Gamma: X \to I^{**}\) for which \(x \Gamma(y) + \Gamma(x)y = 0\) (\(x, y \in X\)) in \(Y^{**}\) is zero;
4. for every continuous \(A\)-bimodule morphism \(T: X \to Y^{**}\), there exists \(u \in I^{**}\) for which \(au = ua\) for \(a \in A\) and \(T(x) = xu - ux\) for \(x \in X\).

**Proof.** Let \(I \oplus Y\) be an arbitrary ideal of \(A \oplus X\). Denote by \(\tau_1\) and \(\tau_2\) the inclusion mappings from, respectively, \(A\) and \(X\) into \(A \oplus X\), and denote by \(\Delta_1\) and \(\Delta_2\) the natural projections from \((I \oplus Y)^{**}\) onto \(I^{**}\) and \(Y^{**}\), respectively. These are \(A\)-bimodule morphisms. To prove the sufficiency we assume that Conditions 1–4 hold.

Let \(D: A \oplus X \to (I \oplus Y)^{**}\) be a continuous derivation. Then \(\Delta_1 \circ D \circ \tau_1: A \to I^{**}\) and \(\Delta_2 \circ D \circ \tau_2: A \to Y^{**}\) are continuous derivations.

**Claim 1:** \(\Delta_1 \circ D \circ \tau_2: X \to I^{**}\) is trivial.

Let \(\Gamma = \Delta_1 \circ D \circ \tau_2\). To prove Claim 1, by Condition 3 it suffices to show that \(\Gamma\) is an \(A\)-bimodule morphism satisfying \(x \Gamma(y) + \Gamma(x)y = 0\) (\(x, y \in X\)).

\[
0 = D((0, 0)) = D((0, x) \cdot (0, y)) = D((0, x)) \cdot (0, y) + (0, x) \cdot D((0, y)) = (0, \Gamma(x)y) + (0, x \Gamma(y))
\]

Thus \(x \Gamma(y) + \Gamma(x)y = 0\). On the other hand,

\[
\Gamma(ax) = \Delta_1 \circ D((0, ax)) = \Delta_1 \circ D((a, 0) \cdot (0, x)) = \Delta_1(D((a, 0) \cdot (0, x)) + (a, 0) \cdot D((0, x))) = \Delta_1((a, 0) \cdot D((0, x))) = \Delta_1(aD \circ \tau_2(x)) = a \Gamma(x).
\]

Similarly, \(\Gamma xa = \Gamma(x)a\) and so \(\Gamma\) is an \(A\)-bimodule morphism. Therefore claim 1 is true. Now let \(T = \Delta_2 \circ D \circ \tau_2: X \to Y^{**}\) and \(D_1 = \Delta_1 \circ D \circ \tau_1: A \to I^{**}\).

**Claim 2:** \(T(ax) = D_1(a)x + aT(x)\) and \(T(xa) = xD_1(a) + T(x)a\) for \(a \in A\) and \(x \in X\).

\[
(0, T(ax)) = (0, \Delta_2 \circ D((0, ax))) = D((0, ax)) = D((a, 0) \cdot (0, x)) = D((a, 0) \cdot (0, x)) = (0, D_1(a)x + a(0, T(x))) = (0, D_1(a)x + aT(x)).
\]
Similarly, for every \( a \in \mathcal{A} \) and \( x \in X \), we have \( (0, T(ax)) = (0, xD_1(a) + T(x)a) \). Thus Claim 2 holds. Therefore by Condition 1, \( D_1 = \Delta_1 \circ D \tau_1 \) is inner.

Now suppose that \( u \in I^{**} \) satisfies \( D_1(a) = au - ua \) for \( a \in \mathcal{A} \). Let \( T_1 : X \to Y^{**} \) be defined by \( T_1(x) = xu - ux \) for \( x \in X \). Then \( T - T_1 : X \to Y^{**} \) is a continuous \( \mathcal{A}\)-bimodule morphisms. In fact, from Claim 2, for every \( a \in \mathcal{A} \) and \( x \in X \), we have
\[
(T - T_1)(ax) = T(ax) - T_1(ax) = (D_1(a)x + aT(x)) - (axu - uax)
\]
\[
= (au - ua)x + aT(x) - (axu - uax) = a(wx - xu) + aT(x)
\]
\[
= a(T - T_1)(x).
\]
Similarly, \( T - T_1 \) is a right \( \mathcal{A}\)-bimodule morphism. From Condition 4, there is a \( v \in I^{**} \) such that \( av = va \) for \( a \in \mathcal{A} \) and \( (T - T_1)(x) = xv - vx \) for \( x \in X \). By Lemma 3.2, we know that
\[
\overline{T - T_1} : \mathcal{A} \oplus X \to (I \oplus Y)^{**}, \quad (a, x) \mapsto (0, (T - T_1)(x))
\]
is an inner derivation. Since \( \Delta_2 \circ D \circ \tau_1 : A \to Y^{**} \) is a continuous derivation, it is inner by Condition 2. By Lemma 2.4, the mapping
\[
\overline{\Delta_2 \circ D \circ \tau_1} : \mathcal{A} \oplus X \to (I \oplus Y)^{**}, \quad (a, x) \mapsto (0, \Delta_2 \circ D \circ \tau_1(a))
\]
is also inner derivation. Using Claim 1, we now have
\[
D((a, x)) = (D_1(a), \Delta_2 \circ D \circ \tau_1(a) + T(x))
\]
\[
= \Delta_2 \circ D \circ \tau_1((a, x)) + \overline{T - T_1}((a, x)) + (D_1(a), T(x)).
\]
Since
\[
(D_1(a), T_1(x)) = (au - ua, xu - ux) = (a, x) \cdot (u, 0) - (u, 0) \cdot (a, x)
\]
for \( a \in \mathcal{A} \) and \( x \in X \), it gives an inner derivation from \( \mathcal{A} \oplus X \) into \( (I \oplus Y)^{**} \). Hence as a sum of three inner derivations, \( D \) is inner. Thus under Conditions 1-4, \( \mathcal{A} \oplus X \) is 2-ideally amenable.

Now we prove the necessity. Suppose that \( \mathcal{A} \oplus X \) is 2-ideally amenable. Let \( D : \mathcal{A} \to I^{**} \) be a continuous derivation with the property given in Condition 1. We define \( \tilde{D} : \mathcal{A} \oplus X \to (I \oplus Y)^{**} \) by
\[
\tilde{D}((a, x)) = (D(a), T(x)) \quad ((a, x) \in \mathcal{A} \oplus X).
\]
\( \tilde{D} \) is a continuous derivation. \( \tilde{D} \) is inner, so there exists \( (u, F) \in (I \oplus Y)^{**} \) such that
\[
\tilde{D}((a, x)) = (a, x) \cdot (u, F) - (u, F) \cdot (a, x),
\]
and then for some \( u \in I^{**} \), we have \( (D(a), T(x)) = (au - ua, xF - Fx) \), thus \( D(a) = au - ua \), this means that \( D \) is inner, and Condition 1 holds. Conditions 2 and 3 hold by Lemma 3.4. Also Condition 4 holds by Lemma 3.3. \( \square \)

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