SPECTRAL PROPERTIES OF A CERTAIN CLASS OF CARLEMAN OPERATORS

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Abstract. The object of the present work is to construct all the generalized spectral functions of a certain class of Carleman operators in the Hilbert space \( L^2(X, \mu) \) and establish the corresponding expansion theorems, when the deficiency indices are \((1,1)\). This is done by constructing the generalized resolvents of \( A \) and then using the Stieltjes inversion formula.

1. Preliminaries

The set of generalized resolvents of a symmetric operator \( A \) with defect indices \((1,1)\) was first derived independently by Naimark [15] and Krein [10]. The case of defect indices \((m,m)\), \( m \in \mathbb{N} \) is due to Krein [11]. Saakjan [19] extended Krein’s formula to the general case of defect indices \((m,m)\), \( m \in \mathbb{N} \cup \{\infty\} \). In another form, the generalized resolvent formula for symmetric operators (including the case of non-densely defined operators) has been obtained by Straus [20, 21].

Let \( H \) be a Hilbert space endowed with the inner product \((\cdot, \cdot)\), and let \( A: D(A) \subseteq H \longrightarrow H \) be a densely defined closed linear operator whose range is denoted \( R(A) \).

1.1. Basic Spectral Properties. We say that \( \lambda \in \mathbb{C} \) is a regular point of the operator \( A \) if the resolvent \( R_\lambda = (A - \lambda I)^{-1} \) exists and is a bounded operator defined everywhere in \( H \) (\( I \) denotes the identity operator in \( H \)). In this case we say that \( \lambda \) belongs to \( \rho(A) \), the resolvent set of \( A \). \( R_\lambda \) is an analytic operator function of \( \lambda \) on \( \rho(A) \). The number \( \lambda \in \mathbb{C} \) is said to be an eigenvalue of \( A \) if there exists an \( f \in D(A) \) for which \( f \neq 0 \) and \( Af = \lambda f \). In this case, the operator \( A - \lambda I \) is not injective, i.e., \( \ker(A - \lambda I) \neq \{0\} \). The complement of \( \rho(A) \), in the complex plane, is denoted by \( \sigma(A) \) and is called the spectrum of \( A \).

A resolution of the identity [1] is a one-parameter family \( \{E_t\}, -\infty < t < \infty \), of orthogonal projection operators acting on a Hilbert space \( H \), such that
i) \( E_s \leq E_t \) if \( s \leq t \) ( monotonicity);

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ii) $E_t$ is strongly left continuous, i.e. $E_{t-0} = E_t$ for every $t \in \mathbb{R}$;

iii) $E_t \xrightarrow{\text{as}} 0$ as $t \to -\infty$ and $E_t \xrightarrow{\text{as}} I$ as $t \to \infty$; here 0 and I are the zero and the identity operator on the space $H$.

Condition ii) can be replaced by the condition of strong right continuity at every point $t \in \mathbb{R}$.

From this it follows that, for each fixed $f \in H$, the function $\rho_f : \mathbb{R} \to [0, 1)$ given by

$$\rho_f(t) = (E(t)f, f) = \|E(t)f\|^2$$

is bounded, non-decreasing, left continuous and

$$\lim_{t \to -\infty} \rho_f(t) = 0, \quad \lim_{t \to \infty} \rho_f(t) = \|f\|^2.$$  

In [1] is proven that for each resolution of the identity $E_t$ ($-\infty \leq t \leq +\infty$) corresponds a uniquely defined self adjoint operator $\hat{A}$, admitting the following integral representation

$$\hat{A} = \int_{-\infty}^{+\infty} t \, dE_t,$$

where the integral is understood as the strong limit of the integral sums for each $f \in D(\hat{A})$, and

$$D(\hat{A}) = \left\{ f : \int_{-\infty}^{+\infty} t^2 d (E_t f, f) < \infty \right\}$$

is satisfied. The resolvent $\hat{R}_\lambda$ and the spectral function $E_t$ of a self adjoint operator $\hat{A}$ are bound by the relation

$$\hat{R}_\lambda = \int_{-\infty}^{+\infty} \frac{dE_t}{t - \lambda}, \quad \lambda \in \rho(\hat{A}),$$

in the sense of strong limit.

The resolution of the identity given by the operator $A$ completely determines the spectral properties of that operator, namely:

$\alpha$) a real number $t_0$ is a regular point of $A$ if and only if it is a point of constancy, that is, if there is an $\varepsilon > 0$ such that $E_{t_0+\varepsilon} = E_{t_0-\varepsilon} = 0$;

$\beta$) a real number $t_0$ is an eigenvalue of $A$ if and only if $\lambda$ is a jump point of $E_t$, that is, $E_{t_0+0} - E_{t_0} \neq 0$.

Hence the resolution of the identity determined by the operator is also called the spectral function of this operator.

1.2. Deficiency indices. The defect number is the dimension of the orthogonal complement to $R(A)$

$$d_\lambda = \dim (H \ominus R(A)) = \dim \text{Ker}(A^*),$$

where $A^*$ is the adjoint operator of $A$ and $\text{Ker}(A^*) = \{ f \in D(A^*) : A^*f = 0 \}$, $D(A^*)$ being the domain of $A^*$. 

Let $A$ be a symmetric operator, $\tilde{A}$ its symmetric extension, then the following relation holds

$$A \subset \tilde{A} \subset \tilde{A}^* \subset A^*.$$  

The interest of (1.6) resides in the following conclusion: all symmetrical extension of $A$ comes of a restriction of the domain of $A^*$. So $D(\tilde{A})$ is a subspace between $D(A)$ and $D(A^*)$. To construct the extensions $\tilde{A}$ it is therefore well to examine the structure of the space $D(A^*)$. Let’s put

$$N_\lambda = \ker (A^* - \lambda I) \quad \text{and} \quad \tilde{N}_\lambda = \ker (A^* - \tilde{\lambda} I), \quad (3m\lambda > 0),$$

with respective dimensions $n_+, n_-$. They are called the deficiency indices of the operator $A$ and will be denoted by the ordered pair $(n_+, n_-)$. It being, in the Hilbert space $D(A^*)$ we have the following hilbertienne decomposition [4]

$$D(A^*) = D(A) \oplus N_\lambda \oplus \tilde{N}_\lambda.$$

$A$ possesses self adjoint extensions [6] if and only if $n_+ = n_-$. We get in this case all self adjoints extensions of $A$ from all isometric Cayley transforms $V = (A - \lambda I)(A - \tilde{\lambda} I)^{-1}$ defined from $N_{\tilde{\lambda}}$ to $N_\lambda$.

1.3. Generalized resolvents formulas. In the general case, every symmetric operator $A$ can be prolonged in a selfadjoint operator $A^+$ defined in a wide space $H^+$ containing $H$. If $E_{\sigma+}^+$ (respectively $R_{\sigma+}^+$) is the spectral function (respectively the resolvent) of $A^+$ and $P$ the operator of projection of $H^+$ on $H$ then the functions operators $E_t = P^+ E_{\sigma+}^+$ and $R_\lambda = P^+ R_{\sigma+}^+$ are said, respectively, generalized spectral function and generalized resolvent of the operator $A$. They are joined by the relation

$$R_\lambda = \int_\alpha^\beta \frac{dE_t}{t-\lambda}, \quad \lambda \in \rho (A),$$

in addition, for all real numbers $\alpha, \beta$ ($\alpha < \beta$), we have the Stieltjes inversion formula

$$(E_\alpha - E_\beta) f, g) = \frac{1}{2\pi i} \lim_{\tau \to \infty} \int_\alpha^\beta (R_{\sigma+i\tau} - R_{\sigma-i\tau}) f, g) d\sigma, \quad f, g \in H.$$ 

Moreover, for all $f$ of $D(A)$:

$$A^+ f = \int_{-\infty}^{+\infty} t dE_t f.$$ 

The generalized spectral function $E_t$ satisfy the same conditions (ii) and (iii) of $E_t$ but the first is replaced by

(i') $E_{t_2} - E_{t_1}$, where $t_2 > t_1$, is a bounded positive operator.

The restriction $P^+ A^+$ is said quasi selfadjoint extension of the operator $A$. It is from this notion that Strauss [21] developed his theory of generalized resolvent of a symmetric operator. Let’s designate by $F_\lambda$ the class of all quasi selfadjoint
linear operators defined on \( N_\lambda \) and that apply \( N_\lambda \) to \( \bar{N}_\lambda \). The set of generalized resolvents is defined by

\[
\begin{align*}
\mathcal{R}_\lambda &= (A_{F(\lambda)} - \lambda I)^{-1} \quad \Im m \lambda \Im m \lambda_o > 0, \\
\mathcal{R}_\lambda &= \mathcal{R}_\lambda^* 
\end{align*}
\]

where \( \lambda_o \) is a non real point, \( F(\lambda) \) an analytic function operator in the half plane \((\Im m \lambda \Im m \lambda_o > 0)\) to value in \( \mathcal{H}_\lambda \) and \( A_{F(\lambda)} \) \((\Im m \lambda \Im m \lambda_o > 0)\) a quasi selfadjoint extension of the operator \( A \) defined by

\[
\begin{align*}
D(A_{F(\lambda)}) &= D(A) \oplus [F(\lambda) - I] \mathcal{H}_\lambda, \\
A_{F(\lambda)}(f + F(\lambda) \varphi - \varphi) &= Af + \lambda_o F(\lambda) \varphi - \bar{\lambda}_o \varphi \\
\end{align*}
\]

with \( f \in D(A) \) and \( \varphi \in \mathcal{H}_\lambda \). The adjoint operator \( A_{F(\lambda)}^* \) is defined by

\[
\begin{align*}
D(A_{F(\lambda)}^*) &= D(A) \oplus [F^*(\lambda) - I] \mathcal{H}_\lambda, \\
A_{F(\lambda)}^*(f + F^*(\lambda) \psi - \psi) &= Af + \bar{\lambda}_o F^*(\lambda) \psi - \bar{\lambda}_o \psi \\
\end{align*}
\]

with \( f \in D(A) \) and \( \psi \in \mathcal{H}_\lambda \).

1.4. Some convergences. We call \( t \) a continuity point of \( E_t \) if \( E_{t+0} - E_t = 0 \).

We call [1] convergence in the mean the convergence in the space \( L^2(X, \mu) \) and we denote by

\[
f(x) = l.i.m. f_n(x),
\]

if

\[
\lim_{n \to \infty} \int_X |f(x) - f_n(x)|^2 \, dx = 0, \quad \text{almost everywhere in } X.
\]

(\( l.i.m. \) is an abbreviation for \( \text{limes in medio}, \) i.e. limit in the mean).

2. Carleman operators

One can find necessary information about Carleman operators, for example, in [5, 9, 22, 23, 24]. In this section we shall present only part of it. Let \( X \) be an arbitrary set, \( \mu \) a \( \sigma \)-fini measure on \( X \) ( \( \mu \) is defined on a \( \sigma \)-algebra of subsets of \( X \), we don’t indicate this \( \sigma \)-algebra), \( L_2(X, \mu) \) the Hilbert space of square integrable functions with respect to \( \mu \). Instead of writing \( \mu \)-measurable’, \( \mu \)-almost everywhere’ and \( \{(d\mu (x))\} \) we write ‘measurable’, ‘a e’ and ‘\( dx \)’.

**Definition 1** ([24]). A linear operator \( A: D(A) \to L_2(X, \mu) \), where the domain \( D(A) \) is a dense linear manifold in \( L_2(X, \mu) \), is said to be **integral** if there exists a measurable function \( K \) on \( X \times X \), a kernel, such that, for every \( f \in D(A) \),

\[
Af(x) = \int_X K(x, y) f(y) \, dy \quad \text{a e}.
\]
A kernel $K$ on $X \times X$ is said to be Carleman if $K(x,y) \in L^2(X,\mu)$ for almost every fixed $x$, that is to say
\[ \int_X |K(x,y)|^2 \, dy < \infty \quad a.e. \]

An integral operator $A$ with a kernel $K$ is called **Carleman operator** if $K$ is a Carleman kernel. Every Carleman kernel $K$ defines a Carleman function $k$ from $X$ to $L^2(X,\mu)$ by $k(x) = K(x,\cdot)$ for all $x$ in $X$ for which $K(x,\cdot) \in L^2(X,\mu)$. Self-adjoint Carleman operators have generalized eigenfunction expansions, which can be used in the study of linear elliptic operators, see [14]. A general reference for Carleman operators on $L^2$-spaces is [8]. The notion of a Carleman operator has been extended in many directions. By replacing $L^2$ by an arbitrary Banach function space one obtains the so-called generalized Carleman operators (see [18]) and by considering Bochner integrals and abstract Banach spaces one is lead to the so-called Carleman and Korotkov operators on a Banach space ([7]).

Now we consider the class of integral operators (2.1) that we go studied here generated by the following symmetric Carleman kernel
\[
K(x,y) = \sum_{p=0}^{\infty} a_p \psi_p(x) \overline{\psi_p(y)},
\]
where the overbar denotes complex conjugation. $\{\psi_p(x)\}_{p=0}^{\infty}$ is an orthonormal sequence in $L^2(X,\mu)$ such that
\[
\sum_{p=0}^{\infty} |\psi_p(x)|^2 < \infty \quad a.e.,
\]
and $\{a_p\}_{p=0}^{\infty}$ a real number sequence verifying
\[
\sum_{p=0}^{\infty} a_p^2 |\psi_p(x)|^2 < \infty \quad a.e.
\]
We called $\{\psi_p(x)\}_{p=0}^{\infty}$ a Carleman sequence. Let $L(\psi)$ be the closed set of linear combinations of elements of the orthogonal sequence $\{\psi_p(x)\}_{p=0}^{\infty}$. It is lucid that the orthogonal complement $L^\perp(\psi) = L^2(X,\mu) \ominus L(\psi)$ is contained in $D(A)$ and annul the operator $A$. The following lemma [3] tells us when the Carleman operator $A$ possesses equal deficiency indices.

**Lemma 1** ([3]). The operator $A$ possesses equal deficiency indices $n_+(A) = n_-(A) = m$, $(m < \infty)$, if and only if there exist sequences $\{\gamma_p^{(k)}\}_{p=0}^{\infty}$, $(k = 1,2,\ldots,m)$, verifying
1) For all $k$

\[(2.6) \quad \theta_k (x) = \sum_{p=0}^{\infty} \gamma_p (k) \psi_p (x) \in L^1 (\psi) \quad (k = 1, 2, \ldots, m)\]

2) For all $\lambda \ (\Im \lambda \neq 0)$

\[(2.7) \quad \sum_{p=0}^{\infty} \left| \frac{\gamma_p (k)}{a_p - \lambda} \right|^2 < \infty , \quad (k = 1, 2, \ldots, m)\]

3) The linear space of the sequences \(\left\{ \gamma_p (k) \right\}_{p=0}^{\infty}, \ (k = 1, 2, \ldots, m)\), verifying 1) and 2) is $m$ dimension.

3. Generalized resolvents

We first prove the following important lemma.

**Lemma 2.** Let $B$ be a closed symmetric operator, $\psi$ the eigenvector of $B$ belonging to the eigenvalue $b$. Then $\psi \in D (B)$ if and only if for a certain $\lambda \ (\Im \lambda \neq 0)$ and for all $\varphi_\lambda$ and $\varphi_\lambda^*$

\[(\varphi_\lambda, \psi) = (\varphi_\lambda, \psi) = 0 ,\]

where $\varphi_\lambda$ and $\varphi_\lambda^*$ belong respectively to the defect spaces $N_\lambda$ and $N_\lambda$.

**Proof.** Let $\psi \in D (B)$ and $\varphi_\lambda \in N_\lambda^c \ (\Im \lambda \neq 0)$, then

\[(b \psi, \varphi_\lambda) = (B \psi, \varphi_\lambda) = (\psi, B^* \varphi_\lambda) = \bar{\lambda} (\psi, \varphi_\lambda) .\]

Therefore,

\[(b - \bar{\lambda}) (\psi, \varphi_\lambda) = 0\]

and as $b - \bar{\lambda} \neq 0$, it follows that $(\psi, \varphi_\lambda) = 0$. Now let $h$ be an arbitrary element of $D (B^*)$. By the hilbertienne decomposition we have

\[h = f + \alpha \varphi_\lambda + \beta \varphi_\lambda^* ,\]

with $f \in D (B), \ \varphi_\lambda \in N_\lambda^c, \ \varphi_\lambda^* \in N_\lambda$, and $\alpha, \ \beta$ two complex numbers. Then,

\[(B^* h, \psi) = (B f, \psi) = (f, b \psi) = (h, b \psi) ,\]

that is to say $\psi \in D (B)$. $\square$

Now we suppose that the symmetric Carleman operator $A$ (2.1) -- (2.3) posses equal deficiency indices $n_+ (A) = n_- (A) = 1$. By Lemma 1 there exist a sequence \(\left\{ \gamma_p \right\}_{p=0}^{\infty}\) such that:

\[\sum_{p=0}^{\infty} |\gamma_p|^2 = \infty\]

and verifying the three conditions of the quoted lemma. By (2.6) and (2.7) we conclude that the function

\[(3.1) \quad \varphi_\lambda (x) = \sum_{p=0}^{\infty} \frac{\gamma_p}{a_p - \lambda} \psi_p (x)\]
belongs to the defect space $\mathcal{N}_\lambda$ of the operator $A$. In what follows, to facilitate the writing, we will designate by $\lambda$ the restriction of $A$ on the subspace $L(\psi)$.

Now we consider the following integral equation
\[(3.2) \quad \int_X \sum_{p=0}^{\infty} a_p \psi_p(x) \overline{\psi_p(y)} Y(y) \, dy - \lambda Y(x) = f(x).\]

Let $f(x) = \sum_{p=0}^{\infty} c_p \psi_p(x) \left( \sum_{p=0}^{\infty} |c_p|^2 < \infty \right)$, then the solution of the equation (3.2) will be the function
\[(3.3) \quad Y(x, \lambda) = \sum_{p=0}^{\infty} \frac{c_p}{a_p - \lambda} \psi_p(x).\]

Let's notice that the formula (3.3) gives the resolvent of the self-adjoint extension $A$ of the operator $A$ which possesses a complete system of eigenfunctions $\{\psi_k(x)\}$ of the space $L(\psi)$. The resolvent $R_\lambda$ of the operator $A$ is an integral operator defined on the space $L(\psi)$:
\[(3.4) \quad R_\lambda f = \int_X \tilde{K}(x, y; \lambda) f(y) \, dy ,\]

where
\[ \tilde{K}(x, y; \lambda) = \sum_{p=0}^{\infty} \frac{1}{a_p - \lambda} \psi_p(x) \overline{\psi_p(y)}.\]

Any solution of the equation (3.2) in $D(A^*)$ admits the following representation
\[(3.5) \quad Y(x, \lambda) = R_\lambda f(x) + c \varphi_i(x) ,\]

where $c$ is an any complex number.

Let's put $\lambda_0 = i$, then $F(\lambda)$ (subsection 1.3) can be given by the formula
\[ F(\lambda) \varphi_{-i} = \omega(\lambda) \varphi_i \]

with $\omega(\lambda)$ an analytic function in the upper half plane and $|\omega(\lambda)| \leq 1$.

The operator $A_{F(\lambda)}$ is defined on the set $D(A_{F(\lambda)})$ as
\[(3.6) \quad \begin{cases} f = x + \omega(\lambda) \varphi_{-i} - \varphi_i, \quad x \in D(A), \\ A_{F(\lambda)} f = A x + i \omega(\lambda) \varphi_i + \varphi_{-i}, \end{cases}\]

then
\[(3.7) \quad \begin{cases} D(A_{F(\lambda)}) = \{ g \in L(\psi) : g = f + [\omega(\lambda) \varphi_i - \varphi_{-i}] c, \; f \in D(A) \}, \\ D(A_{F^*(\lambda)}) = \{ h \in L(\psi) : h = f + [\overline{\omega(\lambda)} \varphi_{-i} - \varphi_i] c, \; f \in D(A) \}. \end{cases}\]

We introduce the following function
\[ \nu_\lambda = \frac{\omega(\lambda) \varphi_{-i} - \varphi_i} ,\]

then $D(A_{F(\lambda)})$ is defined as the set of $y \in D(A^*)$ for which
\[ (A^* y, \nu_\lambda) = (y, A^* \nu_\lambda) .\]
While choosing in (3.5) for all \(\lambda (\Im \lambda > 0)\) \(c = C(\lambda)\), as we have the equality
\[
(A^*Y, \nu_\lambda) = (Y, A^*\nu_\lambda),
\]
we get a formula giving the set of generalized resolvents in terms of analytic functions \(\omega(\lambda)\). By (3.8) we have
\[
C(\lambda) = \frac{1 - \omega(\lambda)(f, \varphi_\lambda)}{\omega(\lambda)(\lambda - 1)(\lambda + i)(\varphi_\lambda, \varphi_i)} \quad (\Im \lambda > 0),
\]
where
\[
\chi(\lambda) = \frac{\lambda - i (\varphi_\lambda, \varphi_{-i})}{\lambda + i (\varphi_\lambda, \varphi_i)}
\]
denote the characteristic function \([1]\) of operator \(A\). If we substitute (3.9) in (3.5), we get the formula of generalized resolvents
\[
R_\lambda = R_\lambda^* + \frac{1 - \omega(\lambda)}{\omega(\lambda)(\lambda - 1)(\lambda + i)(\varphi_\lambda, \varphi_i)\varphi_\lambda} \quad (\Im \lambda > 0).
\]
While taking account that \(R_\lambda = R_\lambda^*\), it is easy to have
\[
R_\lambda f = \tilde{R}_\lambda f + \frac{1 - \omega(\lambda)}{\omega(\lambda)(\lambda - 1)(\lambda + i)(\varphi_\lambda, \varphi_i)\varphi_\lambda} \quad (\Im \lambda > 0).
\]
So we have demonstrated

**Theorem 1.** Formulas (3.11) and (3.12) establish a bijective correspondence between the set of generalized resolvents of the operator \(A\) and the set of the analytic functions \(\omega(\lambda)\) as \(|\omega(\lambda)| \leq 1 (\Im \lambda > 0)\). These formulas define the resolvent of a selfadjoint extension of \(A\) in the space \(L(\psi)\) if and only if, \(\omega(\lambda) = \omega(\text{constant}), |\omega| = 1\).

4. **Generalized spectral functions**

Let’s consider the function \(\chi(\lambda)\) given by the formula (3.10):
\[
\chi(\lambda) = \frac{\lambda - i}{\lambda + i} \sum_{p=0}^\infty \frac{1}{(a_p - \lambda)(a_p - 1)},
\]
it’s an analytic function in the half plane \(\Pi^+ = \{\lambda \in \mathbb{C} : \Im \lambda \geq 0\}\) and take its values on the unit disk \(D = \{\zeta \in \mathbb{C} : |\zeta| \leq 1\}\), so that the real axis \(\mathbb{R}\) turns into the unit circle \(C = \{\zeta \in \mathbb{C} : |\zeta| = 1\}\). Thus, for all \(p = 0, 1, 2, \ldots\), \(\chi(a_p) = 1\).

Let’s put
\[
\zeta = \frac{\lambda - i}{\lambda + i}.
\]
We can write \([1]\) \(\chi(\lambda)\) under the form
\[
\chi(\lambda) = \chi\left(\frac{1 + \zeta}{1 - \zeta}\right) = \omega(\zeta) = \frac{\zeta(\bar{U} - \zeta I)^{-1}\varphi_i, \varphi_i)}{(\bar{U} - \zeta I)^{-1}\bar{U}\varphi_i, \varphi_i)} = \Phi(\zeta) - ||\varphi_i||^2, \Phi(\zeta) + ||\varphi_i||^2,
\]
where

$$\tilde{U} = (\hat{A} - iI)(\hat{A} + iI^{-1})$$

is the unitary Cayley transform of the self-adjoint operator \(\hat{A}\) and

$$\Phi(\zeta) = \int_{0}^{2\pi} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} d(\tilde{E}_s \varphi, \varphi_i) ,$$

\(\tilde{E}_s\) being the resolution of the identity of the unitary operator \(\tilde{U}\). For \(|\zeta| = 1\), we have

\[\Ree \Phi(\zeta) = 0 .\]

From the equality

\[\Ree \Phi(\zeta) = \frac{i}{\lambda + 1} \left[ \Phi(\zeta) + \| \varphi_i \|^2 \right] \]

we conclude that

\[(\varphi_\lambda, \varphi_i) \neq 0 \forall \lambda, \quad \Im \lambda \geq 0 .\]

Formulas (4.1) and (4.2) imply that

\[\Im \left[ (\sigma + i)(\varphi_\sigma, \varphi_i) \right] = \| \varphi_i \|^2 \quad (\Im \sigma = 0) .\]

Now, we introduce the following useful lemmas:

**Lemma 3.** For all \(f, g \in H\), the functions \((\tilde{R}_\lambda f, g), (\varphi_\lambda, \varphi_i), (f, \varphi_\lambda^\top)\) and \((\varphi_\lambda, g)\) are regular on all the complex plane except to points \(a_p\) \((p = 0, 1, 2, \ldots)\), where they admit simple poles. Besides, the following equalities are true:

\[
\res_{\lambda = a_p} (\tilde{R}_\lambda f, g) = \res_{\lambda = a_p} \frac{(f, \varphi_\lambda^\top)(\varphi_\lambda, g)}{(\lambda - i)(\varphi_\lambda, \varphi_i)} = (f, \psi_p)(\psi_p, g) ,
\]

\[
\res_{\lambda = a_p} \frac{(f, \varphi_\lambda^\top)(\varphi_\lambda, g)}{(\lambda - i)(\varphi_\lambda, \varphi_i)} = \res_{\lambda = a_p} \frac{(f, \varphi_\lambda^\top)(\varphi_\lambda, g)}{(\lambda - i)(\varphi_\lambda, \varphi_i)} = (f, \psi_p)(\psi_p, g) .
\]

**Proof.** The fact that the mentioned functions are regular on the complex plane except to points \(a_p\) \((p = 0, 1, 2, \ldots)\) result from formulas (3.1) and

\[\tilde{R}_\lambda f, g = \sum_{p=0}^{\infty} \frac{(f, \psi_p)(\psi_p, g)}{a_p - \lambda} .\]

Furthermore we have:

\[\res_{\lambda = a_p} (\tilde{R}_\lambda f, g) = (f, \psi_p)(\psi_p, g) ,\]

it is easy to see that the function

\[
\frac{(f, \varphi_\lambda^\top)(\varphi_\lambda, g)}{(\lambda - i)(\varphi_\lambda, \varphi_i)} = \left[ \sum_{p=0}^{\infty} \frac{\gamma_p(f, \psi_p)}{a_p - \lambda} \right] \left[ \sum_{p=0}^{\infty} \frac{\gamma_p(\psi_p, g)}{a_p - \lambda} \right] ,
\]

admits the same residue to the point \(\lambda = a_p\).

The second equality can be verified in the same way.
Lemma 4 ([21]). Let $\varphi (\lambda)$ an analytic function in the half-plane $\Pi^+$ with a positive imaginary part and $\psi (\lambda)$ an analytic function in a certain domain containing the interval $[\alpha, \beta]$. Then we have the formula
\[
\frac{1}{2 \pi i} \lim_{\tau \to +0} \int_{\alpha}^{\beta} \left[ \varphi (\lambda) \psi (\lambda) - \varphi (\lambda) \psi (\lambda) \right] d\sigma = \int_{\alpha}^{\beta} \psi (\sigma) d\rho (\sigma) \quad (\lambda = \sigma + i\tau) ,
\]
with
\[
\rho (\sigma) = \frac{1}{\pi} \lim_{\tau \to +0} \int_{0}^{\sigma} 3m \varphi (t + i\tau) dt .
\]

Let $\omega (\lambda)$ be an arbitrary analytic function who applies the half-plane $\Pi^+$ on the unit disk $D$. It is known that the spectral function $E_t$ is uniform and that we can get it by the formula of Stieltjes (1.9):

for all $f (s)$ and $g (s)$ of $L$ and for all reals $\alpha$ and $\beta$ $(\alpha < \beta)$ we have the equality
\[
(E_{\alpha, \beta} f, g) = \frac{1}{2 \pi i} \lim_{\tau \to +0} \int_{\alpha}^{\beta} \left[ R_{\sigma + i\tau} f, R_{\sigma - i\tau} g \right] d\sigma
\]
with
\[
E_{\alpha, \beta} = (E_\beta + E_{\beta + 0}) / 2 - (E_\alpha + E_{\alpha + 0}) / 2 .
\]

Let’s consider the difference
\[
R_\lambda f - R_\bar{\lambda} f = \left[ R_\lambda f - R_\bar{\lambda} f \right] + \left[ C (\lambda) \varphi_\lambda - C (\bar{\lambda}) \varphi_{\bar{\lambda}} \right] .
\]

While holding in account (3.3) and (3.4), we get
\[
\lim_{\tau \to +0} \int_{\alpha}^{\beta} \left[ R_\lambda f - R_\bar{\lambda} f \right] d\sigma = \sum_{\alpha k \in (\alpha, \beta)} c_k \psi_k (s) \quad (\lambda = \sigma + i\tau) ,
\]
where
\[
f (s) = \sum_{k=0}^{\infty} c_k \psi_k (s) .
\]

Let’s consider the second member of (4.4):
\[
C (\lambda) \varphi_\lambda - C (\bar{\lambda}) \varphi_{\bar{\lambda}} = \frac{-i}{i} \left[ \frac{1}{(\lambda - i) (\varphi \lambda, \varphi_{-i})} - \frac{1}{(\lambda + i) (\varphi \lambda, \varphi_{i})} \right]
\]
\[
\times \frac{1}{(f, \varphi_{\lambda}) \varphi_\lambda} - \frac{i}{(\lambda - i) (\varphi \lambda, \varphi_{-i})} \left[ \frac{1}{(\lambda + i) (\varphi_{\bar{\lambda}}, \varphi_{i})} \right]
\]
\[
\times \frac{1}{(f, \varphi_{\lambda}) \varphi_{\bar{\lambda}}} - \left[ \frac{(f, \varphi_{\lambda}) \varphi_\lambda}{(\lambda - i) (\varphi \lambda, \varphi_{-i})} - \frac{(f, \varphi_{\lambda}) \varphi_{\bar{\lambda}}}{(\lambda + i) (\varphi_{\bar{\lambda}}, \varphi_{i})} \right] .
\]

Let’s put
\[
\frac{(f, \varphi_\lambda)}{(\lambda - i) (\varphi \lambda, \varphi_{-i})} = f_1 (\lambda) ; \quad \frac{(f, \varphi_{\bar{\lambda}})}{(\lambda + i) (\varphi_{\bar{\lambda}}, \varphi_{i})} = f_2 (\lambda) \quad (\Im \lambda > 0) .
\]
Then \((\lambda = \sigma + i\tau)\)

\[
\frac{1}{2\pi i} \lim_{\tau \to +0} \int_{\alpha}^{\beta} \left[ f_1(\lambda) - f_2(\lambda) \right] d\sigma = \frac{1}{2\pi i} \int_{\alpha}^{\beta} \frac{2i\Im \left[ \sigma + i(\varphi_{\sigma}, \varphi_i) \right] (f, \varphi_{\sigma})}{(\sigma^2 + 1) ||(\varphi_{\sigma}, \varphi_i)||^2} d\sigma \\
+ \sum_{\alpha_k \in (\alpha, \beta)} c_k \psi_k(s),
\]

c_k being coefficients in the development (4.5).

Now, we notice that for all analytic function \(\omega(\lambda)\) in the half-plane \(\Pi^+\) as \(|\omega(\lambda)| \leq 1\), we obtain

\[
\Im i \omega(\lambda) \chi(\lambda) > 0 \quad (\Im \lambda > 0).
\]

After this, while using the Lemma 2 and the equality (4.3), we get

\[(4.6) \quad E_{\alpha, \beta} f = \frac{1}{2\pi i} \lim_{\tau \to +0} \int_{\alpha}^{\beta} \left[ R_{\lambda} - R_{\bar{\lambda}} \right] f d\sigma = \int_{\alpha}^{\beta} \frac{(f, \varphi_{\sigma}) \varphi_{\sigma}}{(\sigma^2 + 1) ||(\varphi_{\sigma}, \varphi_i)||^2} d\rho(\sigma),
\]

with

\[(4.7) \quad \rho(\sigma) = \frac{1}{\pi} \lim_{\tau \to +0} \int_{0}^{\sigma} \left[ \Im \omega(\lambda) \chi(\lambda) - 1 \right] d\lambda, \quad (\lambda = t + i\tau)
\]

and

\[
\varphi_i(s) = \frac{\varphi_i(s)}{||\varphi_i||}.
\]

The function \(\rho(\sigma)\) is decreasing because

\[
\Re \omega(\lambda) \chi(\lambda) - 1 \geq \frac{1}{1 + ||\omega(\lambda)\chi(\lambda)||} \geq \frac{1}{2}.
\]

Thus, we have demonstrated the theorem

**Theorem 2.** Let \(\omega(\lambda)\) be an analytic function in the half-plane \(\Pi^+\) and \(E_t(-\infty < t < +\infty)\) the spectral function of the operator \(A\). Then for all \(f(s)\) of \(L(\psi)\) and for all reals \(\alpha\) and \(\beta\) \((\alpha < \beta)\) we have the relation (4.6) and the following equalities

\[
(E_{\alpha, \beta} f, f) = \int_{\alpha}^{\beta} \frac{||f, \varphi_{\sigma}||^2}{(\sigma^2 + 1) ||(\varphi_{\sigma}, \varphi_i)||^2} d\rho(\sigma),
\]

\[
f(s) = \lim_{\alpha \to -\infty, \beta \to +\infty} \int_{\alpha}^{\beta} \frac{(f, \varphi_{\sigma}) \varphi_{\sigma}(s)}{(\sigma^2 + 1) \left| \left( \varphi_{\sigma}, \varphi_i \right) \right|^2} d\rho(\sigma),
\]

\[
(f, f) = \int_{-\infty}^{+\infty} \frac{||f, \varphi_{\sigma}||^2}{(\sigma^2 + 1) \left| \left( \varphi_{\sigma}, \varphi_i \right) \right|^2} d\rho(\sigma),
\]

where \(\rho(\sigma)\) is defined by the formula (4.7) for \(\lambda = \sigma + i\tau, \Im \lambda > 0\).
Corollary 1. In order that $t \ (−\infty < t < +\infty)$ be a continuity point of the spectral function $E_t$ of the operator $A$ it is necessary and sufficient that it is a continuity point of the function $\rho(\sigma)$.

Let’s consider the formula (4.7). The function $\chi(\lambda)$ applies all interval $(ap_k, ap_{k+1})$ (we suppose that $ap_k$ and $ap_{k+1}$ are neighboring points) in the unit disk. The homographic transform $\frac{1+z}{1-z}$ applies the circle $|z| = r \leq 1$ in the not euclidean circle of center $i$ such that the image of $r = 0$ will be the point $i$ and the image of $r = 1$ will be the real axis $\mathbb{R}$. Therefore, for $\omega(\lambda) = 1$, $\rho(\sigma)$ is a jumps function with points jumps $ap_k$ and for $\omega(\lambda) = \kappa$ ($\kappa$ = constant with $|\kappa| < 1$), $\rho(\sigma)$ is absolutely continuous.

With the help of the self-adjoints extensions ($\omega(\lambda) = \kappa = \exp(i\varphi)$) $\rho(\sigma)$ will be a jumps function with points jumps $\sigma_p$ for whom $\chi(\sigma_p) = \exp(-i\varphi)$.

Of the pace of the function $\rho(\sigma)$ we are convinced of the following findings.

Corollary 2. The quasi-self-adjoint extension associated to the analytical function $\omega(\lambda)$ ($|\omega(\lambda)| \leq 1$ in $\Pi^+$ and $|\omega(\sigma)| = 1$ for $\Im \sigma = 0$) admits a merely point spectrum.

Corollary 3. The interval $(c, d)$ ($−\infty \leq c < d \leq +\infty$) doesn’t contain the spectrum points of the self-adjoint extension of the operator $A$ generated by the functions $\omega(\lambda)$ if and only if $\omega(\lambda)$ verify the following conditions:

a) $\omega(\lambda)$ is analytic in $\Pi^+$ and $|\omega(\lambda)| \leq 1 (\Im \lambda > 0)$;

b) $\omega(\lambda)$ admits an extension by continuity from $\Pi^+$ on $(c, d)$;

c) $|\omega(\sigma)| = 1$, if $\sigma \in (c, d)$;

d) $\omega(\sigma) \neq \overline{\chi(\sigma)}$ for $\sigma \in (c, d)$.

If we suppose in (2.3) that $ap > 0$, then $A$ will be a positive operator. Thus the Corollary 3 give the criteria to get the positive spectral functions. In particular self-adjoint extension possessed a positive spectral function if it is generated by functions $\omega(\lambda) = \kappa = \exp(i\varphi)$, $0 \leq \varphi \leq \varphi_0$, $\chi(0) = \exp(-i\varphi_0)$.

References


