ON UNICITY OF MEROMORPHIC FUNCTIONS
DUE TO A RESULT OF YANG - HUA

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Abstract. This paper studies the unicity of meromorphic (resp. entire) functions of the form $f^n f'$ and obtains the following main result: Let $f$ and $g$ be two non-constant meromorphic (resp. entire) functions, and let $a \in \mathbb{C} \setminus \{0\}$ be a non-zero finite value. Then, the condition that $E_3(a, f^n f') = E_3(a, g^n g')$ implies that either $f = d g$ for some $(n+1)$-th root of unity $d$, or $f = c_1 e^{cz}$ and $g = c_2 e^{-cz}$ for three non-zero constants $c, c_1$ and $c_2$ with $(c_1 c_2)^{n+1} c_2^2 = -a^2$ provided that $n \geq 11$ (resp. $n \geq 6$). It improves a result of C. C. Yang and X. H. Hua. Also, some other related problems are discussed.

1. Introduction and main results

In this paper, a meromorphic function will always mean meromorphic in the open complex plane $\mathbb{C}$. We adopt the standard notations in the Nevanlinna’s value distribution theory of meromorphic functions such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$ and the counting function $N(r, f)$ (reduced form $\tilde{N}(r, f)$) of poles. For any non-constant meromorphic function $f$, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$, possibly outside a set of finite linear measure that is not necessarily the same at each occurrence. We refer the reader to Hayman [3], Yang and Yi [8] for more details.

Let $f$ be a non-constant meromorphic function, let $a \in \mathbb{C}$ be a finite value, and let $k \in \mathbb{N} \cup \{+\infty\}$ be a positive integer or infinity. We denote by $E(a, f)$ the set of zeros of $f - a$ and count multiplicities, while by $\tilde{E}(a, f)$ the set of zeros of $f - a$ but ignore multiplicities. Further, we denote by $E_k(a, f)$ the set of zeros of $f - a$ with multiplicities less than or equal to $k$ (counting multiplicities). Obviously, $E(a, f) = E_{+\infty}(a, f)$. Define $E(\infty, f) := E(0, 1/f)$ for the value $\infty$, and define $\tilde{E}(\infty, f)$ and $E_{+\infty}(\infty, f)$ correspondingly. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N_k(r, 1/(f - a))$ the counting function corresponding to the set $E_k(a, f)$, while by $N_{k+1}(r, 1/(f - a))$ the counting function corresponding to the set $E_{k+1}(a, f) := E(a, f) - E_k(a, f)$.

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Also, we denote by $\bar{N}_{k}\left(r, 1/(f-a)\right)$ and $\bar{N}_{k+1}\left(r, 1/(f-a)\right)$ the reduced forms of $N_{k}\left(r, 1/(f-a)\right)$ and $N_{k+1}\left(r, 1/(f-a)\right)$, respectively.

All those foregoing definitions and notations hold well for any small meromorphic function, say, $\alpha$ (i.e., whose characteristic function satisfies $T(r, \alpha) = S(r, f)$), of $f$.

Let $f$ and $g$ be two non-constant meromorphic functions, and let $\alpha$ be a common small meromorphic function of $f$ and $g$. We say that $f$ and $g$ share $\alpha$ CM (resp. IM) provided that $E(\alpha, f) = E(\alpha, g)$ (resp. $E(\alpha, f) = E(\alpha, g)$).

W. K. Hayman proposed the following well-known conjecture in [4].

**Hayman Conjecture.** If an entire function $f$ satisfies $f^n f' \neq 1$ for all positive integers $n \in \mathbb{N}$, then $f$ is a constant.

It has been verified by Hayman himself in [5] for the cases $n > 1$ and Clunie in [1] for the cases $n \geq 1$, respectively.

It is well-known that if two non-constant meromorphic functions $f$ and $g$ share two values CM and other two values IM, then $f$ is a Möbius transformation of $g$. In 1997, C. C. Yang and X. H. Hua studied the unicity of differential monomials of the form $f^n f'$ and obtained the following theorem in [7].

**Theorem A.** Let $f$ and $g$ be two non-constant meromorphic (resp. entire) functions, let $n \geq 11$ (resp. $n \geq 6$) be an integer, and let $a \in \mathbb{C}\{0\}$ be a non-zero finite value. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some $(n+1)$-th root of unity $d$, or $f = c_1 e^{cz}$ and $g = c_2 e^{-cz}$ for three non-zero constants $c, c_1$ and $c_2$ such that $(c_1 c_2)^{n+1} e^{2} = -a^2$.

**Remark 1.** In fact, combining their original argumentations with a more precise calculation on equations (20) and (23) in [7, p.p. 403-404] could reduce the lower bound of the integer $n$ from 7 to 6 [7, Remark 2] if $f$ and $g$ are entire.

In 2000, by using argumentations similar to those in [7], M. L. Fang and H. L. Qiu proved the following uniqueness theorem in [2].

**Theorem B.** Let $f$ and $g$ be two non-constant meromorphic (resp. entire) functions, let $n \geq 11$ (resp. $n \geq 6$) be an integer. If $f^n f'$ and $g^n g'$ share a CM, then either $f = dg$ for some $(n+1)$-th root of unity $d$, or $f = c_1 e^{cz}$ and $g = c_2 e^{-cz}$ for three non-zero constants $c, c_1$ and $c_2$ such that $4(c_1 c_2)^{n+1} e^{2} = -1$.

In this paper, we shall weaken the assumption of sharing the non-zero finite value a CM (i.e., $E(a, f^n f') = E(a, g^n g')$) in Theorem A to $E_3(a, f^n f') = E_3(a, g^n g')$. In fact, we shall prove the following three uniqueness theorems.

**Theorem 1.** Let $f$ and $g$ be two non-constant meromorphic (resp. entire) functions, let $n \geq 11$ (resp. $n \geq 6$) be an integer, and let $a \in \mathbb{C}\{0\}$ be a non-zero finite value. If $E_3(a, f^n f') = E_3(a, g^n g')$, then $f^n f'$ and $g^n g'$ share the value a CM.

**Theorem 2.** Let $f$ and $g$ be two non-constant meromorphic (resp. entire) functions, let $n \geq 15$ (resp. $n \geq 8$) be an integer, and let $a \in \mathbb{C}\{0\}$ be a non-zero
finite value. If $E_2(a, f'f') = E_2(a, g'g')$, then $f'f'$ and $g'g'$ share the value a CM.

**Theorem 3.** Let $f$ and $g$ be two non-constant meromorphic (resp. entire) functions, let $n \geq 19$ (resp. $n \geq 10$) be an integer, and let $a \in \mathbb{C}\{0\}$ be a non-zero finite value. If $E_1(a, f'f') = E_1(a, g'g')$, then $f'f'$ and $g'g'$ share the value a CM.

**Remark 2.** Obviously, Theorem 1 is an improvement of Theorem A.

2. Some lemmas

**Lemma 1.** Let $f$ and $g$ be two non-constant meromorphic functions satisfying $E_k(1, f) = E_k(1, g)$ for some positive integer $k \in \mathbb{N}$. Define $H$ as the following

$$H := \left(\frac{f''}{f'} - 2 \frac{f'}{f-1} - \frac{g''}{g'} - 2 \frac{g'}{g-1}\right).$$

If $H \neq 0$, then

$$N(r, H) \leq N_1\left(r, \frac{1}{f-1}\right) + N_1\left(r, \frac{1}{g-1}\right) + N_0\left(r, \frac{1}{f'}\right) + N_0\left(r, \frac{1}{g'}\right) + \tilde{N}_1\left(r, \frac{1}{f'}\right) + \tilde{N}_1\left(r, \frac{1}{g'}\right) + S(r, f) + S(r, g) + S\left(r, \frac{1}{f} - 1\right) + S\left(r, \frac{1}{g} - 1\right),$$

where $N_0(r, 1/f')$ denotes the counting function of zeros of $f'$ but not the zeros of $f(f - 1)$, and $N_0(r, 1/g')$ is similarly defined.

**Proof.** It is not difficult to see that simple poles of $f$ is not poles of $\frac{f''}{f'} - 2 \frac{f'}{f-1}$ and simple poles of $g$ is not poles of $\frac{g''}{g'} - 2 \frac{g'}{g-1}$. Then, the conclusion follows immediately since we assume $E_k(1, f) = E_k(1, g)$.

**Lemma 2** (see [7, p.p. 397]). Under the condition of Lemma 1, we have

$$N_1\left(r, \frac{1}{f-1}\right) = N_1\left(r, \frac{1}{g-1}\right) \leq N(r, H) + S(r, f) + S(r, g).$$

**Lemma 3** (see [7, p.p. 398] or [9]). Let $f$ be some non-constant meromorphic function on $\mathbb{C}$. Then,

$$N\left(r, \frac{1}{f'}\right) \leq N(r, H) + S(r, f).$$

**Lemma 4** (see [8]). Let $f$ be a non-constant meromorphic function on $\mathbb{C}$, and let $k \in \mathbb{N}$ be a positive integer. Then,

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r, f) + N\left(r, \frac{1}{f'}\right) + S(r, f).$$
3. Proof of Theorem 1

Define $F := \frac{F'}{a}$ and $F_1 := \frac{F'}{a(n+1)}$. Then, $F_1' = F$. Similarly, define $G := \frac{G'}{a}$ and $G_1 := \frac{G'}{a(n+1)}$. Now, by equations (19)–(20) in [7, p.p. 403-404], we have

(3.1) $\tilde{N}(r, F) = \tilde{N}(r, F_1)$,
(3.2) $\tilde{N}(r, G) = \tilde{N}(r, G_1)$,
and

(3.3) $\tilde{N}\left(r, \frac{1}{F}\right) + \tilde{N}\left(r, \frac{1}{F_1}\right) \leq 2 \tilde{N}\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{F}\right)$,
(3.4) $\tilde{N}\left(r, \frac{1}{G}\right) + \tilde{N}\left(r, \frac{1}{G_1}\right) \leq 2 \tilde{N}\left(r, \frac{1}{G}\right) + N\left(r, \frac{1}{G}\right)$.

Then, by the conclusions of Lemma 4, we derive

\begin{align*}
(n + 1)T(r, f) &= T(r, F_1) + O(1) \\
&\leq T(r, F) + \tilde{N}\left(r, \frac{1}{F}\right) - N\left(r, \frac{1}{F}\right) + S(r, f) \\
&\leq T(r, F) + \tilde{N}\left(r, \frac{1}{F_1}\right) - N\left(r, \frac{1}{F_1}\right) + S(r, f).
\end{align*}

(3.5)

Similarly, we obtain

\begin{align*}
(n + 1)T(r, g) &= T(r, G_1) + O(1) \\
&\leq T(r, G) + \tilde{N}\left(r, \frac{1}{G}\right) - N\left(r, \frac{1}{G}\right) + S(r, g).
\end{align*}

(3.6)

Firstly, we suppose that equation (2.1) is not identically zero, that is, $H \not\equiv 0$. Here, we replace the functions $f$ and $g$ in the statement of Lemma 1 by $F$ and $G$, respectively. Combining the conclusions of Lemmas 1 and 2 with the assumption that $E_{33}(1, F) = E_{33}(1, G)$ yields

\begin{align*}
N_1\left(r, \frac{1}{F-1}\right) &\leq \tilde{N}_1(2, F) + \tilde{N}_1(2, G) + \tilde{N}_2\left(r, \frac{1}{F}\right) + \tilde{N}_2\left(r, \frac{1}{G}\right) \\
&+ N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + \tilde{N}_4\left(r, \frac{1}{F-1}\right) \\
&+ \tilde{N}_4\left(r, \frac{1}{G-1}\right) + S(r, f) + S(r, g).
\end{align*}

(3.7)

Applying the second fundamental theorem to the functions $F$ and $G$ with the values 0, 1 and $\infty$, respectively, to conclude that
where \( N \) (from equations (3.1)–(3.6) and (3.10), and noting Lemma 3, we derive

\[
T(r, F) + T(r, G) \leq \hat{N}(r, F) + \hat{N}(r, \frac{1}{F}) + \hat{N}(r, \frac{1}{F-1}) - N_0(r, \frac{1}{F}) + S(r, f) \\
+ \hat{N}(r, G) + \hat{N}(r, \frac{1}{G}) + \hat{N}(r, \frac{1}{G-1}) - N_0(r, \frac{1}{G}) + S(r, g) \\
\leq \hat{N}(r, F) + \hat{N}(r, \frac{1}{F}) + \hat{N}(r, G) + \hat{N}(r, \frac{1}{G}) + N_1(r, \frac{1}{F-1}) \\
\quad + \hat{N}(r, \frac{1}{F-1}) - N_1\left(\frac{1}{F-1}\right) \\
- N_0\left(\frac{1}{F}\right) - N_0\left(\frac{1}{G}\right) + S(r, f) + S(r, g).
\]

(3.8)

Noting that

\[
\hat{N}\left(\frac{1}{F-1}\right) - \frac{1}{2} N_1\left(\frac{1}{F-1}\right) + N_4\left(\frac{1}{F-1}\right) \leq \frac{1}{2} N\left(\frac{1}{F-1}\right), \\
\hat{N}\left(\frac{1}{G-1}\right) - \frac{1}{2} N_1\left(\frac{1}{G-1}\right) + N_4\left(\frac{1}{G-1}\right) \leq \frac{1}{2} N\left(\frac{1}{G-1}\right).
\]

Then, combining the above two equations with \( E_{31}(1, F) = E_{31}(1, G) \) yields

\[
\hat{N}\left(\frac{1}{F-1}\right) + \hat{N}\left(\frac{1}{G-1}\right) + N_4\left(\frac{1}{F-1}\right) + N_4\left(\frac{1}{G-1}\right) \\
- N_1\left(\frac{1}{F-1}\right) \leq \frac{1}{2} (T(r, F) + T(r, G)) + S(r, f) + S(r, g).
\]

(3.9)

Hence, equations (3.7) - (3.9) imply

\[
T(r, F) + T(r, G) \leq 2 \left( N_2(r, F) + N_2(r, G) + N_2\left(\frac{1}{F}\right) + N_2\left(\frac{1}{G}\right) \right) \\
+ S(r, f) + S(r, g).
\]

(3.10)

where \( N_2(r, F) := \hat{N}(r, F) + N_2(r, F) \) and \( N_2(r, 1/F) := \hat{N}(r, 1/F) + N_2(r, 1/F) \), and \( N_2(r, G) \) and \( N_2(r, 1/G) \) are similarly defined.

From equations (3.1)–(3.6) and (3.10), and noting Lemma 3, we derive

\[
(n + 1)(T(r, f) + T(r, g)) \leq 2 \left( N_2(r, F) + N_2(r, G) + N_2\left(\frac{1}{F}\right) + N_2\left(\frac{1}{G}\right) \right) \\
+ N\left(\frac{1}{f}\right) - N\left(\frac{1}{F}\right) + N\left(\frac{1}{g}\right) - N\left(\frac{1}{G}\right) \\
+ S(r, f) + S(r, g) \\
\leq 4(\hat{N}(r, f) + \hat{N}(r, g)) + 5 \left( N\left(\frac{1}{f}\right) + N\left(\frac{1}{g}\right) \right) \\
+ N\left(\frac{1}{f}\right) + N\left(\frac{1}{g}\right) + S(r, f) + S(r, g)
\]

(3.11)
\[
\leq 5(\bar{N}(r, f) + \bar{N}(r, g)) + 6 \left( N\left( r, \frac{1}{f} \right) + N\left( r, \frac{1}{g} \right) \right)
\]

(3.12)

\[+ S(r, f) + S(r, g),\]

which implies \((n-10)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g)\), a contradiction against the assumption that \(n \geq 11\).

In particular, if \(f\) and \(g\) are entire, equation (3.11) turns out to be

(3.13)

\[\leq S(r, f) + S(r, g),\]

since both the terms \(\bar{N}(r, f)\) and \(\bar{N}(r, g)\) equal to \(O(1)\) now. Obviously, it contradicts the assumption that \(n \geq 6\).

Hence, \(H \equiv 0\). Integrating the equation \(H \equiv 0\) twice results in

\[\frac{F'}{F - 1} = k_1 \frac{G'}{G - 1} + k_2 \quad (k_1 \in \mathbb{C}\backslash\{0\}, \quad k_2 \in \mathbb{C}),\]

which implies that \(F\) and \(G\) share the value 1 CM.

This finishes the proof of Theorem 1.

\[\square\]

4. Proof of Theorem 2

From the condition that \(E_2(1, F) = E_2(1, G)\), if we furthermore suppose that \(H \neq 0\), then similar to equation (3.7), we have

\[
N_{11}\left( r, \frac{1}{F - 1} \right) \leq N_{12}(r, F) + N_{12}(r, G) + N_{12}\left( r, \frac{1}{F} \right) + N_{12}\left( r, \frac{1}{G} \right)
\]

\[+ N_0\left( r, \frac{1}{F'} \right) + N_0\left( r, \frac{1}{G'} \right) + \bar{N}_{13}\left( r, \frac{1}{F - 1} \right)
\]

\[+ \bar{N}_{13}\left( r, \frac{1}{G - 1} \right) + S(r, f) + S(r, g).\]

(4.1)

A routine calculation leads to

(4.2)

\[\bar{N}\left( r, \frac{1}{F - 1} \right) \leq \frac{1}{2} N_{11}\left( r, \frac{1}{F - 1} \right) + \frac{1}{2} \bar{N}_{13}\left( r, \frac{1}{F - 1} \right) \leq \frac{1}{2} \bar{N}\left( r, \frac{1}{F - 1} \right),\]

(4.3)

\[\bar{N}\left( r, \frac{1}{G - 1} \right) \leq \frac{1}{2} N_{11}\left( r, \frac{1}{G - 1} \right) + \frac{1}{2} \bar{N}_{13}\left( r, \frac{1}{G - 1} \right) \leq \frac{1}{2} \bar{N}\left( r, \frac{1}{G - 1} \right).\]

Applying the conclusions of Lemma 3 to \(F\) and taking reduced forms of the counting functions on both sides of equation (2.4) to conclude

\[
\bar{N}_{13}\left( r, \frac{1}{F - 1} \right) \leq \bar{N}(r, F) + S(r, f) \leq \bar{N}\left( r, \frac{1}{F} \right) + S(r, f)
\]

\[\leq \bar{N}(r, F) + \bar{N}\left( r, \frac{1}{F} \right) + S(r, f)
\]

\[\leq \bar{N}(r, F) + \bar{N}\left( r, \frac{1}{F} \right) + S(r, f)
\]

(4.4)

\[\leq 2\bar{N}(r, f) + 2\bar{N}\left( r, \frac{1}{f} \right) + S(r, f),\]
and similarly,

\[ \tilde{N}_3 \left( r, \frac{1}{G-1} \right) \leq 2 \tilde{N}(r, g) + 2 \tilde{N} \left( r, \frac{1}{g} \right) + S(r, g). \]

Hence, equations (4.2)–(4.5) yield

\[
\begin{align*}
\tilde{N} \left( r, \frac{1}{F-1} \right) + \tilde{N} \left( r, \frac{1}{G-1} \right) + \tilde{N}_3 \left( r, \frac{1}{F-1} \right) + \tilde{N}_3 \left( r, \frac{1}{G-1} \right) & - \tilde{N}_1 \left( r, \frac{1}{F-1} \right) \leq \frac{1}{2} \left( T(r, F) + T(r, G) \right) \\
& + \left( \tilde{N}(r, f) \tilde{N}(r, g) + \tilde{N} \left( r, \frac{1}{f} \right) + \tilde{N} \left( r, \frac{1}{g} \right) \right) + S(r, f) + S(r, g).
\end{align*}
\]

Analogous to equation (3.10), we have

\[
T(r, F) + T(r, G) \leq 2 \left( N_2 \left( r, F \right) + N_2 \left( r, G \right) + N_2 \left( r, \frac{1}{F} \right) \right) + N_2 \left( r, \frac{1}{G} \right) + 2 \left( \tilde{N}(r, f) \tilde{N}(r, g) + \tilde{N} \left( r, \frac{1}{f} \right) + \tilde{N} \left( r, \frac{1}{g} \right) \right) + S(r, f) + S(r, g).
\]

Combining the above equation with equations (3.1)–(3.6) yields

\[ (n + 1)(T(r, F) + T(r, G)) \leq 7(\tilde{N}(r, f) + \tilde{N}(r, g)) + S(r, f) + S(r, g), \]

which implies that \((n - 14)(T(r, F) + T(r, G)) \leq S(r, f) + S(r, g),\) a contradiction since we assume \(n \geq 15.\) In particular, if \(f\) and \(g\) are entire, then equation (4.6) turns into \((n - 7)(T(r, F) + T(r, G)) \leq S(r, f) + S(r, g).\) Obviously, it contradicts the assumption that \(n \geq 8.\)

Hence \(H \equiv 0,\) and \(F\) and \(G\) share the value 1 CM.

This finishes the proof of Theorem 2. \(\square\)

5. Proof of Theorem 3

From the condition that \(E_{11}(1, F) = E_{11}(1, G),\) if we furthermore assume that \(H \neq 0,\) then similar to equation (3.7), we have

\[ N_1 \left( r, \frac{1}{F-1} \right) \leq \tilde{N}_1 \left( r, F \right) + \tilde{N}_1 \left( r, G \right) + \tilde{N}_1 \left( r, \frac{1}{F} \right) + \tilde{N}_1 \left( r, \frac{1}{G} \right) + N_0 \left( r, \frac{1}{F'} \right) + N_0 \left( r, \frac{1}{G'} \right) + \tilde{N}_1 \left( r, \frac{1}{F-1} \right) + \tilde{N}_1 \left( r, \frac{1}{G-1} \right) + S(r, f) + S(r, g). \]

\[ (n) \]
It is not difficult to see that
\[
\bar{N}\left( r, \frac{1}{F - 1}\right) - \frac{1}{2}N_1\left( r, \frac{1}{F - 1}\right) \leq \frac{1}{2}N\left( r, \frac{1}{F - 1}\right), 
\]
(5.2)
\[
\bar{N}\left( r, \frac{1}{G - 1}\right) - \frac{1}{2}N_1\left( r, \frac{1}{G - 1}\right) \leq \frac{1}{2}N\left( r, \frac{1}{G - 1}\right). 
\]
(5.3)

Also, as shown in inequality (4.4), we have
\[
\bar{N}_{(2)}\left( r, \frac{1}{F - 1}\right) \leq \bar{N}\left( r, \frac{1}{F}\right) + S(r, f) \leq 2 \bar{N}\left( r, \frac{1}{F}\right) + 2 \bar{N}\left( r, \frac{1}{g}\right) + S(r, f). 
\]
(5.4)
\[
\bar{N}_{(2)}\left( r, \frac{1}{G - 1}\right) \leq 2 \bar{N}(r, g) + 2 \bar{N}\left( r, \frac{1}{g}\right) + S(r, g). 
\]
(5.5)

Hence, equations (5.2)–(5.5) yield
\[
\bar{N}\left( r, \frac{1}{F - 1}\right) + \bar{N}\left( r, \frac{1}{G - 1}\right) + \bar{N}_{(2)}\left( r, \frac{1}{F - 1}\right) + \bar{N}_{(2)}\left( r, \frac{1}{G - 1}\right) 
- N_1\left( r, \frac{1}{F - 1}\right) \leq \frac{1}{2}(T(r, F) + T(r, G)) 
+ 2\left( \bar{N}(r, f) + \bar{N}(r, g) + \bar{N}\left( r, \frac{1}{f}\right) + \bar{N}\left( r, \frac{1}{g}\right) \right) 
+ S(r, f) + S(r, g). 
\]

Analogically, we have
\[
T(r, F) + T(r, G) \leq 2\left( N_2(r, F) + N_2(r, G) + N_2\left( r, \frac{1}{F}\right) + N_2\left( r, \frac{1}{G}\right) \right) 
+ 4\left( \bar{N}(r, f) + \bar{N}(r, g) + \bar{N}\left( r, \frac{1}{f}\right) + \bar{N}\left( r, \frac{1}{g}\right) \right) 
+ S(r, f) + S(r, g). 
\]

Hence,
\[
(n + 1)\left( T(r, f) + T(r, g) \right) \leq 9(\bar{N}(r, f) + \bar{N}(r, g)) 
+ 10\left( \bar{N}\left( r, \frac{1}{f}\right) + \bar{N}\left( r, \frac{1}{g}\right) \right) + S(r, f) + S(r, g), 
\]
(5.6)
which implies that \((n - 18)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g)\), a contradiction since we assume \(n \geq 19\). In particularly, if \(f\) and \(g\) are entire, then equation (5.6) turns into \((n - 9)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g)\). Obviously, it contradicts the assumption that \(n \geq 10\).

Hence \(H \equiv 0\), and \(F\) and \(G\) share the value 1 CM.

This finishes the proof of Theorem 3.
6. Related results

**Final Note 1.** If we assume that \( f \) and \( g \) share the value \( \infty \) CM (resp. IM) in the statement of Lemma 1 besides the assumption that \( E_k(1, f) = E_k(1, g) \) for some positive integer \( k \in \mathbb{N} \), then equation (2.2) becomes

\[
N(r, H) \leq N_0(r, \frac{1}{f}) + N_0(r, \frac{1}{g}) + \tilde{N}_0(\frac{r, 1}{f}) + \tilde{N}_0(\frac{r, 1}{g}) + \tilde{N}(r, 1 \frac{1}{f-1}) + \tilde{N}(r, 1 \frac{1}{g-1}) + S(r, f) + S(r, g),
\]

and respectively,

\[
N(r, H) \leq \frac{1}{2} N(r, f) + \frac{1}{2} N(r, g) + N_0(r, \frac{1}{f}) + N_0(r, \frac{1}{g}) + \tilde{N}_0(\frac{r, 1}{f}) + \tilde{N}_0(\frac{r, 1}{g}) + \tilde{N}(r, 1 \frac{1}{f-1}) + \tilde{N}(r, 1 \frac{1}{g-1}) + S(r, f) + S(r, g).
\]

Applying the arguments used in our proofs with equation (2.2a) (resp. (2.2b)) could reduce the lower bounds of the integers \( n \) from \( n \geq 11, 15 \) and \( 19 \) in Theorems 1, 2 and 3 to \( n \geq 9, 13 \) and \( 17 \) (resp. \( n \geq 10, 14 \) and \( 18 \)), respectively, provided that we assume furthermore that \( f \) and \( g \), and thus \( F \) and \( G \), share the value \( \infty \) CM (resp. IM).

**Final Note 2.** Using similar arguments as those in our proofs and replacing the notations \( F, F_1 \) (resp. \( G, G_1 \)) in Section 3 by new ones \( F = f^n f'/z, F_1 = f^{n+1}/(n+1) \) (resp. \( G = g^n g'/z, G_1 = g^{n+1}/(n+1) \)) (then, \( F'_1 = zF \) and \( G'_1 = zG \)), we could weaken the assumption of sharing \( z \) CM (i.e., \( E(z, f^n f') = E(z, g^n g') \)) in the statement of Theorem C to \( E_k(z, f^n f') = E_k(z, g^n g') \) for \( k = 1, 2 \) and 3.

In fact, if \( f \) and \( g \) are transcendental, our original proofs go well, while if \( f \) and \( g \) are rational functions (resp. polynomials), routine calculations on the term \( \log r \) would lead to analogous conclusions. However, in those cases we may have to increase the lower bounds of the integers \( n \) from \( n \geq 11, 15 \) and \( 19 \) (resp. \( n \geq 6, 8 \) and \( 10 \)) to \( n \geq 14, 19 \) and \( 24 \) (resp. \( n \geq 9, 12 \) and \( 15 \)), since now \( f \) and \( g \) have the same growth estimate as that of the function \( z \), in other words, of \( O(\log r) \). Below, we give an outline of the proof for those special cases.

**Proof.** First of all, according to the conclusion of [2, Theorem C], we know that \( f \) is rational whenever \( g \) is, and vice versa. Similarly, we have \( N(r, F) = N(r, f) + \log r \) and \( N_2(r, F) \leq 2N(r, f) + \log r \), and \( N(r, G) = N(r, g) + \log r \) and \( N_2(r, G) \leq 2N(r, g) + \log r \). Furthermore, we have

\[
(n+1)T(r, f) = T(r, F_1) + O(1) \leq T(r, zF) + N\left(r, \frac{1}{F_1}\right) - N\left(r, \frac{1}{zF}\right) + O(1)
\]

\[
\leq T(r, F) + N\left(r, \frac{1}{f}\right) - N\left(r, \frac{1}{F_1}\right) + \log r + O(1),
\]
\[(n+1)T(r, g) = T(r, G_1) + O(1) \leq T(r, zG) + N\left(r, \frac{1}{G_1}\right) - N\left(r, \frac{1}{zG}\right) + O(1) \]
\[
\leq T(r, G) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{g'}\right) + \log r + O(1).
\]

If \(E_{20}(z, f^n f') = E_{20}(z, g^n g')\), then analogous to equation (3.11), we derive
\[(n+1)(T(r, f) + T(r, g)) \leq 5(\bar{N}(r, f) + \bar{N}(r, g)) + 6\left(N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right)\right) + 6 \log r + O(1),
\]
which implies that \((n-10)(T(r, f) + T(r, g)) \leq 6 \log r + O(1).

Noting the discussions in [2, p. p. 437-438] fail here, we may have to suppose that \((n-10)(T(r, f) + T(r, g)) \geq (2n - 20) \log r\), and hence \((2n - 26) \log r \leq O(1)\), a contradiction since we assume \(n \geq 14\).

If \(E_{21}(z, f^n f') = E_{21}(z, g^n g')\) for \(k = 1, 2\), then parallel to equations (4.4)-(4.5) and (5.4)-(5.5), we have
\[
\bar{N}_{i_k}(r, \frac{1}{f - 1}) \leq 2\bar{N}(r, f) + 2\bar{N}\left(r, \frac{1}{f}\right) + \log r + O(1),
\]
\[
\bar{N}_{i_k}(r, \frac{1}{G - 1}) \leq 2\bar{N}(r, g) + 2\bar{N}\left(r, \frac{1}{g}\right) + \log r + O(1).
\]

If \(k = 2\), we have
\[(n+1)(T(r, f) + T(r, g)) \leq 7(\bar{N}(r, f) + \bar{N}(r, g)) + 8\left(N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right)\right) + 8 \log r + O(1),
\]
which means \((2n - 36) \log r \leq O(1)\), a contradiction since we assume \(n \geq 19\).

If \(k = 1\), we have
\[(n+1)(T(r, f) + T(r, g)) \leq 9(\bar{N}(r, f) + \bar{N}(r, g)) + 10\left(N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right)\right) + 10 \log r + O(1),
\]
which shows \((2n - 46) \log r \leq O(1)\), a contradiction since we assume \(n \geq 24\).

If \(f\) and \(g\) are polynomials, then \(N(r, F) = N(r, G) = \log r\), and hence \(\bar{N}(r, F) = N_2(r, F) = \bar{N}(r, G) = N_2(r, G) = \log r\). Similarly, we derive
\[(n+1)(T(r, f) + T(r, g)) \leq 6\left(N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right)\right) + 6 \log r + O(1) \quad (k = 3),
\]
\[(n+1)(T(r, f) + T(r, g)) \leq 8\left(N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right)\right) + 8 \log r + O(1) \quad (k = 2),
\]
\[(n+1)(T(r, f) + T(r, g)) \leq 10\left(N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right)\right) + 10 \log r + O(1) \quad (k = 1).
\]
All the above three equations contradict the assumptions that \( n \geq 9 \) (\( k = 3 \)), \( n \geq 12 \) (\( k = 2 \)) and \( n \geq 15 \) (\( k = 1 \)), respectively.

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References


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