FROM EULER-LAGRANGE EQUATIONS TO CANONICAL NONLINEAR CONNECTIONS

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Abstract. The aim of this paper is to construct a canonical nonlinear connection $\Gamma = (\mathcal{M}^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$ on the 1-jet space $J^1(T, M)$ from the Euler-Lagrange equations of the quadratic multi-time Lagrangian function

$$L = h^{\alpha\beta}(t)g_{ij}(t,x)x^i_\alpha x^j_\beta + U^{(i)}_{(j)}(t,x)x^i_\alpha + F(t,x).$$

1. Kronecker $h$-regularity

We start our study considering a smooth multi-time Lagrangian function $L : E \to \mathbb{R}$, expressed locally by

$$E \ni (t^a, x^i, x^i_\alpha) \to L(t^a, x^i, x^i_\alpha) \in \mathbb{R},$$

whose fundamental vertical metrical $d$-tensor is defined by

$$G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2} \frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta}.$$

In the sequel, let us fix $h = (h_{\alpha\beta})$ a semi-Riemannian metric on the temporal manifold $T$ and let $g_{ij}(t^\gamma, x^k, x^k_\gamma)$ be a symmetric $d$-tensor on $E = J^1(T, M)$, of rank $n$ and having a constant signature.

Definition 1.1. A multi-time Lagrangian function $L : E \to \mathbb{R}$, having the fundamental vertical metrical $d$-tensor of the form

$$G^{(\alpha)(\beta)}_{(i)(j)}(t^\gamma, x^k, x^k_\gamma) = h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k, x^k_\gamma),$$

is called a Kronecker $h$-regular multi-time Lagrangian function.

In this context, we can introduce the following important concept:

Definition 1.2. A pair $ML^n_p = (J^1(T, M), L)$, $p = \dim T$, $n = \dim M$, consisting of the 1-jet fibre bundle and a Kronecker $h$-regular multi-time Lagrangian function $L : J^1(T, M) \to \mathbb{R}$, is called a multi-time Lagrange space.

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Remark 1.3. i) In the particular case \((T, h) = (\mathbb{R}, \delta)\), a multi-time Lagrange space is called a relativistic rheonomic Lagrange space and is denoted by

\[ RL^n = (J^1(\mathbb{R}, M), L). \]

For more details about the relativistic rheonomic Lagrangian geometry, the reader may consult [14].

ii) If the temporal manifold \(T\) is 1-dimensional one, then, via a temporal reparametrization, we have

\[ J^1(T, M) \equiv J^1(\mathbb{R}, M). \]

In other words, a multi-time Lagrange space, having \(\dim T = 1\), is a reparametrized relativistic rheonomic Lagrange space.

Example 1.4. Let us suppose that the spatial manifold \(M\) is also endowed with a semi-Riemannian metric \(g = (g_{ij}(x))\). Then, the multi-time Lagrangian function

\[
L_1 : E \to \mathbb{R}, \quad L_1 = h^{\alpha\beta}(t)g_{ij}(x)x^i_\alpha x^j_\beta
\]

is a Kronecker \(h\)-regular one. It follows that the pair

\[ E\mathbb{SML}^n_p = (J^1(T, M), L_1) \]

is a multi-time Lagrange space. It is important to note that the multi-time Lagrangian \(L_1 = L_1\sqrt{|h|}\) is exactly the “energy” Lagrangian, whose extremals are the harmonic maps between the semi-Riemannian manifolds \((T, h)\) and \((M, g)\) [4]. At the same time, the multi-time Lagrangian that governs the physical theory of bosonic strings is of kind of the Lagrangian \(L_1\) [6].

Example 1.5. In the above notations, taking \(U^{(\alpha)}(i)(t, x)\) a \(d\)-tensor field on \(E\) and \(F : T \times M \to \mathbb{R}\) a smooth function, the more general multi-time Lagrangian function

\[
L_2 : E \to \mathbb{R}, \quad L_2 = h^{\alpha\beta}(t)g_{ij}(x)x^i_\alpha x^j_\beta + U^{(\alpha)}(t, x)x^i_\alpha + F(t, x),
\]

is also a Kronecker \(h\)-regular one. The multi-time Lagrange space

\[ E\mathbb{DML}^n_p = (J^1(T, M), L_2) \]

is called the autonomous multi-time Lagrange space of electrodynamics. This is because, in the particular case \((T, h) = (\mathbb{R}, \delta)\), the space \(E\mathbb{DML}^n_1\) naturally generalizes the classical Lagrange space of electrodynamics [10], that governs the movement law of a particle placed concomitently into a gravitational field and an electromagnetic one. In a such context, from a physical point of view, the semi-Riemannian metric \(h_{\alpha\beta}(t)\) (resp. \(g_{ij}(x)\)) represents the gravitational potentials of the manifold \(T\) (resp. \(M\)), the \(d\)-tensor \(U^{(\alpha)}(t, x)\) play the role of the electromagnetic potentials, and \(F\) is a potential function. The non-dynamical character of the spatial gravitational potentials \(g_{ij}(x)\) motivates us to use the term “autonomous”.
Example 1.6. More general, if we consider the symmetrical d-tensor \( g_{ij}(t, x) \) on \( E \), of rank \( n \) and having a constant signature on \( E \), we can define the Kronecker \( h \)-regular multi-time Lagrangian function

\[
L_3 : E \to \mathbb{R}, \quad L_3 = h^{\alpha\beta}(t)g_{ij}(t, x)x^i_\alpha x^j_\beta + U^{(\alpha)}_i(t) x^i_\alpha + F(t, x).
\]

The multi-time Lagrange space

\[ \mathcal{NEDML}^n_p = (J^1(T, M), L_3) \]

is called the non-autonomous multi-time Lagrange space of electrodynamics. From a physical point of view, we remark that the spatial gravitational potentials \( g_{ij}(t, x) \) are dependent of the temporal coordinates \( t^\gamma \). For that reason, we use the term “non-autonomous”, in order to emphasize the dynamical character of \( g_{ij}(t, x) \).

2. The characterization theorem of multi-time Lagrange spaces

An important role and, at the same time, an obstruction in the subsequent development of the theory of the multi-time Lagrange spaces, is played by

Theorem 2.1 (of characterization of multi-time Lagrange spaces). If \( p = \dim T \geq 2 \), then the following statements are equivalent:

i) \( L \) is a Kronecker \( h \)-regular Lagrangian function on \( J^1(T, M) \).

ii) The multi-time Lagrangian function \( L \) reduces to a multi-time Lagrangian function of non-autonomous electrodynamic kind, that is

\[
L = h^{\alpha\beta}(t)g_{ij}(t, x)x^i_\alpha x^j_\beta + U^{(\alpha)}_i(t) x^i_\alpha + F(t, x).
\]

Proof 1. ii) ⇒ i) It is obvious.

i)⇒ ii) Let us suppose that \( L \) is a Kronecker \( h \)-regular multi-time Lagrangian function, that is

\[
\frac{1}{2} \frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta} = h^{\alpha\beta}(t)g_{ij}(t, x^k_\gamma, x^k_\gamma).
\]

For the beginning, let us suppose that there are two distinct indices \( \alpha \) and \( \beta \) from the set \( \{1, \ldots, p\} \), such that \( h^{\alpha\beta} \neq 0 \). Let \( k \) (resp. \( \gamma \)) be an arbitrary element of the set \( \{1, \ldots, n\} \) (resp. \( \{1, \ldots, p\} \)). Deriving the above relation with respect to the variable \( x^k_\gamma \) and using the Schwartz theorem, we obtain the equalities

\[
\frac{\partial g_{ij}^{(\alpha\beta)}}{\partial x^k_\gamma} = \frac{\partial g_{jk}^{(\beta\gamma)}}{\partial x^i_\alpha} = \frac{\partial g_{ik}^{(\gamma\alpha)}}{\partial x^j_\beta}, \quad \forall \alpha, \beta, \gamma \in \{1, \ldots, p\}, \forall i, j, k \in \{1, \ldots, n\}.
\]

Contracting now with \( h_{\gamma\mu} \), we deduce

\[
\frac{\partial g_{ij}^{(\alpha\beta)}}{\partial x^k_\gamma} h_{\gamma\mu} = 0, \quad \forall \mu \in \{1, \ldots, p\}.
\]

In these conditions, the supposing \( h^{\alpha\beta} \neq 0 \) implies that \( \frac{\partial g_{ij}}{\partial x^k_\gamma} = 0 \) for all two arbitrary indices \( k \) and \( \gamma \). Consequently, we have \( g_{ij} = g_{ij}(t^\alpha, x^m) \).
Supposing now that \( h^{\alpha\beta} = 0, \forall \alpha \neq \beta \in \{1, \ldots, p\} \), it follows that we have \( h^{\alpha\beta} = h^{\alpha} \delta_{\beta}^{\alpha} \), \( \forall \alpha, \beta \in \{1, \ldots, p\} \). In other words, we use an orthogonal system of coordinates on the manifold \( T \). In these conditions, the relations

\[
\frac{\partial^2 L}{\partial x^\alpha \partial x^\beta} = 0, \quad \forall \alpha \neq \beta \in \{1, \ldots, p\}, \quad \forall i, j \in \{1, \ldots, n\},
\]

hold good. If we fix now an indice \( \alpha \) in the set \( \{1, \ldots, p\} \), from the first relation we deduce that the local functions \( \frac{\partial L}{\partial x^\alpha} \) depend only by the coordinates \( (t^\mu, x^m, x^m_\mu) \).

Considering \( \beta \neq \alpha \) in the set \( \{1, \ldots, p\} \), the second relation implies

\[
\frac{1}{2h^{\alpha}(t)} \frac{\partial^2 L}{\partial x^\alpha \partial x^\beta} = \frac{1}{2h^{\beta}(t)} \frac{\partial^2 L}{\partial x^\beta \partial x^\alpha} = g_{ij}(t^\mu, x^m, x^m_\mu), \quad \forall \alpha \in \{1, \ldots, p\}, \quad \forall i, j \in \{1, \ldots, n\}.
\]

Because the first term of the above equality depends by \( (t^\mu, x^m, x^m_\mu) \), while the second term is dependent only by the coordinates \( (t^\mu, x^m, x^m_\beta) \), and because we have \( \alpha \neq \beta \), we conclude that \( g_{ij} = g_{ij}(t^\mu, x^m) \).

Finally, the equality

\[
\frac{1}{2} \frac{\partial^2 L}{\partial x^\alpha \partial x^\beta} = h^{\alpha\beta}(t^\gamma) g_{ij}(t^\gamma, x^k), \quad \forall \alpha, \beta \in \{1, \ldots, p\}, \quad \forall i, j \in \{1, \ldots, n\}
\]

implies without difficulties that the multi-time Lagrangian function \( L \) is one of non-autonomous electrodynamic kind.

\( \square \)

**Corollary 2.2.** The fundamental vertical metrical d-tensor of an arbitrary Kronecker \( h \)-regular multi-time Lagrangian function \( L \) is of the form

\[
G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2} \frac{\partial^2 L}{\partial x^\alpha \partial x^\beta} = \begin{cases} h^{11}(t)g_{ij}(t, x^k, y^k), & p = \dim T = 1 \\ h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k), & p = \dim T \geq 2. \end{cases}
\]

**Remark 2.3.** i) It is obvious that the preceding theorem is an obstruction in the development of a fertile geometrical theory for the multi-time Lagrange spaces. This obstruction will be surpassed in the paper [12], when we will introduce the more general notion of a **generalized multi-time Lagrange space**. The generalized multi-time Riemann-Lagrange geometry on \( J^1(T, M) \) will be constructed using only a Kronecker \( h \)-regular vertical metrical d-tensor \( G^{(\alpha)(\beta)}_{(i)(j)} \) and a nonlinear connection \( \Gamma \), “a priori” given on the 1-jet space \( J^1(T, M) \).

ii) In the case \( p = \dim T \geq 2 \), the preceding theorem obliges us to continue our geometrical study of the multi-time Lagrange spaces, seawing our attention upon the non-autonomous multi-time Lagrange spaces of electrodynamics.
3. Canonical nonlinear connection $\Gamma$

Let $ML^p_n = (J^1(T, M), L)$, where $\dim T = p$, $\dim M = n$, be a multi-time Lagrange space whose fundamental vertical metrical d-tensor metric is

$$G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2} \frac{\partial^2 L}{\partial x^i_{\alpha} \partial x^j_{\beta}} = \begin{cases} h^{11}(t)g_{ij}(t, x^k, y^k), & p = 1 \\ h^{\alpha\beta}(r)g_{ij}(t', x^k), & p \geq 2. \end{cases}$$

Supposing that the semi-Riemannian temporal manifold $(T, h)$ is compact and orientable, by integration on the manifold $T$, we can define the energy functional associated to the multi-time Lagrange function $L$, taking

$$\mathcal{E}_L : C^\infty(T, M) \to \mathbb{R}, \quad \mathcal{E}_L(f) = \int_T L(t^\alpha, x^i, x'^i_\alpha) \sqrt{|h|} \, dt^1 \wedge dt^2 \wedge \ldots \wedge dt^p,$$

where the smooth map $f$ is locally expressed by $(t^\alpha) \to (x^i(t^\alpha))$ and $x^i_{\alpha} = \frac{\partial x^i}{\partial t^\alpha}$.

It is obvious that, for each index $i \in \{1, 2, \ldots, n\}$, the extremals of the energy functional $\mathcal{E}_L$ verify the Euler-Lagrange equations

$$(3.1) \quad 2G^{(\alpha)(\beta)}_{(i)(j)} x^j_{\alpha\beta} + \frac{\partial^2 L}{\partial x^i_{\alpha} \partial x^j_{\beta}} x^j_{\alpha\beta} = \frac{\partial L}{\partial x^i_{\alpha}} - \frac{\partial^2 L}{\partial t^\alpha \partial x^i_{\alpha}} + \frac{\partial L}{\partial x^i_{\beta}} H^i_{\alpha\gamma} = 0,$$

where $x^j_{\alpha\beta} = \frac{\partial^2 x^j}{\partial t^\alpha \partial t^\beta}$ and $H^i_{\alpha\beta}$ are the Christoffel symbols of the semi-Riemannian temporal metric $h_{\alpha\beta}$.

Taking into account the Kronecker $h$-regularity of the Lagrangian function $L$, it is possible to rearrange the Euler-Lagrange equations of the Lagrangian $L = L \sqrt{|h|}$ in the following generalized Poisson form:

$$(3.2) \quad \Delta_h x^k + 2g^{k(\alpha)}(t^\nu, x^m, x'^m_\nu) = 0,$$

where

$$\Delta_h x^k = h^{\alpha\beta} \{ x^j_{\alpha\beta} - H^j_{\alpha\beta} x^k_{\gamma} \},$$

$$2g^{k(\alpha)} = \frac{g^{ki}}{2} \left\{ \frac{\partial^2 L}{\partial x^i_{\alpha} \partial x^j_{\beta}} x^j_{\alpha\beta} - \frac{\partial L}{\partial x^i_{\alpha}} + \frac{\partial^2 L}{\partial t^\alpha \partial x^i_{\alpha}} + \frac{\partial L}{\partial x^i_{\beta}} H^i_{\alpha\gamma} + 2g_{j\gamma} h^{\alpha\beta} H^j_{\alpha\beta} x^k_{\gamma} \right\}.$$
verify the following transformation laws:

\[ 2S^p = 2S^r \frac{\partial x^p}{\partial x^r} + h^{\alpha \mu} \frac{\partial x^p}{\partial t^\mu} \frac{\partial x^q}{\partial \tau^\mu} r^q_{\alpha}, \]

\[ 2H^p = 2H^r \frac{\partial x^p}{\partial x^r} + h^{\alpha \mu} \frac{\partial x^p}{\partial t^\mu} \frac{\partial x^q}{\partial \tau^\mu} s^q_{\alpha}, \]

\[ 2J^p = 2J^r \frac{\partial x^p}{\partial x^r} - h^{\alpha \mu} \frac{\partial x^p}{\partial t^\mu} \frac{\partial x^q}{\partial \tau^\mu} t^q_{\alpha}. \]

It follows that the local entities \( 2G^p = 2S^p + 2H^p + 2J^p \) modify by the transformation laws

\[ 2\tilde{G}^r = 2G^r \frac{\partial x^r}{\partial x^p} - h^{\alpha \mu} \frac{\partial x^r}{\partial t^\mu} \frac{\partial x^q}{\partial \tau^\mu} r^q_{\alpha}, \]

that is what we were looking for.

ii) In the particular case \( \dim T = 1 \), any spatial \( h \)-spray \( G = (G^i) \) is the \( h \)-trace of a spatial spray \( G = (G^i_{(1)1}) \), where \( G^i_{(1)1} = h_{11}G^i \). In other words, the equality \( G^i = h_{11}G^i_{(1)1} \) is true.

On the other hand, in the case \( \dim T \geq 2 \), the Theorem of characterization of the Kronecker \( h \)-regular Lagrangian functions ensures us that

\[ L = h^{\alpha \beta}(t)g_{ij}(t, x)x^i_x^j \alpha x^j + U^{(\alpha)}(t, x)x^j_x^j + F(t, x). \]

In this particular situation, by computations, the expressions of the entities \( S^i \), \( H^i \) and \( J^i \) reduce to

\[ 2S^i = h^{\alpha \beta} \Gamma^i_{jk} x^i_x^j + \frac{g^{ij}}{2} [U^{(i)j}] x^j + \frac{\partial F}{\partial x^i}, \]

\[ 2H^i = -h^{\alpha \beta} H^i_{\alpha \beta} x^i + \frac{g^{ij}}{2} [2h^{\alpha \beta} \frac{\partial g_{ij}}{\partial t^\alpha} + \frac{\partial U^{(i)}}{\partial t^\alpha} + U^{(i)}(\alpha)] x^j, \]

\[ 2J^i = h^{\alpha \beta} H^i_{\alpha \beta} x^i, \]

where

\[ \Gamma^i_{jk} = \frac{g^{ij}}{2} (\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}) \]

are the generalized Christoffel symbols of the multi-time dependent metric \( g_{ij} \) and

\[ U^{(i)j} = \frac{\partial U^{(i)}}{\partial x^j} - \frac{\partial U^{(i)j}}{\partial x^i}. \]

Consequently, the expression of the spatial \( h \)-spray \( G = (G^i) \) becomes

\[ 2G^p = 2S^p + 2H^p + 2J^p = h^{\alpha \beta} \Gamma^i_{jk} x^i_x^j + 2T^i, \]

where the local components

\[ 2T^i = \frac{g^{ij}}{2} [2h^{\alpha \beta} \frac{\partial g_{ij}}{\partial t^\alpha} + U^{(i)j} x^j + \frac{\partial U^{(i)}}{\partial t^\alpha} + U^{(i)} H^i_{\alpha \beta} - \frac{\partial F}{\partial x^i}]. \]
represent the components of a tensor d-field $T = (T^i)$ on $J^1(T, M)$. It follows that the d-tensor $T$ can be written as the $h$-trace of the d-tensor

$$T_{(\alpha)\beta}^{(i)} = \frac{h_{\alpha\beta}}{p} T^i,$$

where $p = \dim T$. In other words, the relation $T^i = h^{\alpha\beta} T_{(\alpha)\beta}^{(i)}$ is true. Obviously, this writing is not unique but represents a natural extension of the case $\dim T = 1$.

Finally, we can conclude that the spatial $h$-spray $G = (G^i)$ is the $h$-trace of the spatial spray

$$G_{(\alpha)\beta}^{(i)} = \frac{1}{2} \Gamma_{jk}^i x^j_\alpha x^k_\beta + T_{(\alpha)\beta}^{(i)},$$

that is the relation $G^i = h^{\alpha\beta} G_{(\alpha)\beta}^{(i)}$ holds good. □

Following previous reasonings and the preceding result, we can regard the equations (3.2) as being the equations of the harmonic maps of a multi-time dependent spray.

**Theorem 3.2.** The extremals of the energy functional $E_L$ attached to the Kronecker $h$-regular Lagrangian function $L$ are harmonic maps on $J^1(T, M)$ of the multi-time dependent spray $(H, G)$ defined by the temporal components

$$H_{(\alpha)\beta}^{(i)} = \begin{cases} -\frac{1}{2} H_{11}^i (t) y^i, & p = 1 \\ -\frac{1}{2} H_{\gamma}^\alpha x^\gamma_i, & p \geq 2 \end{cases}$$

and the local spatial components $G_{(\alpha)\beta}^{(i)} =$

$$\begin{cases} \frac{1}{4} h_{11}^{ik} \left[ \frac{\partial^2 L}{\partial x^j \partial y^k} y^j - \frac{\partial L}{\partial x^k} + \frac{\partial^2 L}{\partial t \partial y^k} + \frac{\partial L}{\partial x^k} H_{11}^i + 2 h_{11} H_{11} g_{ki} y^j \right], & p = 1 \\ \frac{1}{2} \Gamma_{jk}^i x^j_\alpha x^k_\beta + T_{(\alpha)\beta}^{(i)}, & p \geq 2, \end{cases}$$

where $p = \dim T$.

**Definition 3.3.** The multi-time dependent spray $(H, G)$ constructed in the preceding Theorem is called the canonical multi-time spray attached to the multi-time Lagrange space $ML^n_p$.

In the sequel, by local computations, the canonical multi-time spray $(H, G)$ of the multi-time Lagrange space $ML^n_p$ induces naturally a nonlinear connection $\Gamma$ on $J^1(T, M)$.

**Theorem 3.4.** The canonical nonlinear connection

$$\Gamma = (M_{(\alpha)\beta}^{(i)}, N_{(\alpha)\beta}^{(i)})$$
of the multi-time Lagrange space $ML^p$ is defined by the temporal components

\[(3.10)\]

\[
M^{(i)}_{(\alpha)\beta} = 2H^{(i)}_{(\alpha)\beta} = \begin{cases} 
-H_{11}^{(i)}y^{\prime}, & p = 1 \\
-H_{\alpha\beta}^{(i)}x^{\prime}, & p \geq 2, 
\end{cases}
\]

and the spatial components

\[(3.11)\]

\[
N^{(i)}_{(\alpha)j} = \frac{\partial \mathcal{G}^i}{\partial x^j} h_{\alpha\gamma} = \begin{cases} 
h_{11} \frac{\partial \mathcal{G}^i}{\partial y^{\prime}}, & p = 1 \\
\Gamma^{i}_{jk} x^{\prime}_{\alpha} + g^{ik} \frac{\partial g_{jk}}{\partial t^{\alpha}} + g^{ik} h_{\alpha\gamma} U^{(\gamma)}_{(k)j}, & p \geq 2,
\end{cases}
\]

where $\mathcal{G}^i = h^{\alpha\beta} G^{(i)}_{(\alpha)\beta}$.

**Remark 3.5.** In the particular case $(T, h) = (\mathbb{R}, \delta)$, the canonical nonlinear connection $\Gamma = (0, N^{(i)}_{(\alpha)j})$ of the relativistic rheonomic Lagrange space

\[
RL^n = (J^1(\mathbb{R}, M), L)
\]

generalizes naturally the canonical nonlinear connection of the classical rheonomic Lagrange space $L^n = (\mathbb{R} \times TM, L)$ [10].

**References**


