ON THREE EQUIVALENCES CONCERNING
PONOMAREV-SYSTEMS

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Abstract. Let \( \{P_n\} \) be a sequence of covers of a space \( X \) such that \( \{st(x,P_n)\} \) is a network at \( x \) in \( X \) for each \( x \in X \). For each \( n \in \mathbb{N} \), let \( \mathcal{P}_n = \{P_\beta : \beta \in \Lambda_n\} \) and \( \Lambda_n \) be endowed the discrete topology. Put \( M = \{b = (\beta_n) \in \Pi_{n \in \mathbb{N}} \Lambda_n : \{P_\beta\} \text{ forms a network at some point } x_b \text{ in } X\} \) and \( f : M \longrightarrow X \) by choosing \( f(b) = x_b \) for each \( b \in M \). In this paper, we prove that \( f \) is a sequentially-quotient (resp. sequence-covering, compact-covering) mapping if and only if each \( P_n \) is a cs\(^*\)-cover (resp. fcs-cover, cf\( p \)-cover) of \( X \). As a consequence of this result, we prove that \( f \) is a sequentially-quotient, s-mapping if and only if it is a sequence-covering, s-mapping, where "s" can not be omitted.

1. Introduction

A space is called a Baire’s zero-dimensional space if it is a Tychonoff-product space of countable many discrete spaces. In [9], Ponomarev proved that each first countable space can be characterized as an open image of a subspace of a Baire’s zero-dimensional space. More precisely, he obtained the following result.

Theorem 1.1. Let \( X \) be a space with the topology \( \tau = \{P_\beta : \beta \in \Lambda\} \). For each \( n \in \mathbb{N} \), put \( \Lambda_n = \Lambda \) and endow \( \Lambda_n \) the discrete topology. Put \( Z = \Pi_{n \in \mathbb{N}} \Lambda_n \), which is a Baire’s zero-dimensional space, and put \( M = \{b = (\beta_n) \in Z : \{P_\beta\} \text{ forms a neighbourhood base at some point } x_b \text{ in } X\} \). Define \( f : M \longrightarrow X \) by choosing \( f(b) = x_b \) for each \( b \in M \). Then

(1) \( f \) is a mapping.
(2) \( f \) is continuous and onto.
(3) If \( X \) is first countable, then \( f \) is an open mapping.

Recently, while generalizing the Ponomarev’s methods, Lin ([6]) introduced Ponomarev-systems \( (f, M, X, \{P_n\}) \) as in the following definition.

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Definition 1.2.

(1) Let \( \mathcal{P} = \{ \mathcal{P}_x : x \in X \} \) be a cover of a space \( X \), where \( \mathcal{P}_x \subset (\mathcal{P})_x = \{ P \in \mathcal{P} : x \in P \} \). \( \mathcal{P} \) is called a network of \( X \) ([8]), if for each \( x \in U \) with \( U \) open in \( X \), there exists \( P \in \mathcal{P}_x \) such that \( x \in P \subset U \), where \( \mathcal{P}_x \) is called a network at \( x \) in \( X \).

(2) Let \( \{ \mathcal{P}_n \} \) be a sequence of covers of a space \( X \). \( \{ \mathcal{P}_n \} \) is called a point-star network of \( X \) ([7]), if \( \{ \text{st}(x, \mathcal{P}_n) \} \) is a network at \( x \) in \( X \) for each \( x \in X \), where \( \text{st}(x, \mathcal{P}) = \bigcup \{ P \in \mathcal{P} : x \in P \} \).

(3) Let \( \{ \mathcal{P}_n \} \) be a point-star network of a space \( X \). For each \( n \in \mathbb{N} \), put \( \mathcal{P}_n = \{ P_\beta : \beta \in \Lambda_n \} \) and endow \( \Lambda_n \) the discrete topology. Put \( M = \{ b = (\beta_n) \in \Pi_{n \in \mathbb{N}} \Lambda_n : \{ P_\beta \} \text{ forms a network at some point } x_b \text{ in } X \} \), then \( M \), which is a subspace of the product space \( \Pi_{n \in \mathbb{N}} \Lambda_n \), is a metric space and \( x_b \) is unique for each \( b \in M \). Define \( f : M \rightarrow X \) by choosing \( f(b) = x_b \), then \( f \) is a continuous and onto mapping. \( \{ f, M, X, \{ \mathcal{P}_n \} \} \) is called a Ponomarev-system ([7, 10]).

In a Ponomarev-system \( \{ f, M, X, \{ \mathcal{P}_n \} \} \), the following results have been obtained.

Theorem 1.3 ([6, 7, 10]). Let \( \{ f, M, X, \{ \mathcal{P}_n \} \} \) be a Ponomarev-system. Then the following hold.

(1) If each \( \mathcal{P}_n \) is a point-finite (resp. point-countable) cover of \( X \), then \( f \) is a compact mapping (resp. s-mapping).

(2) If each \( \mathcal{P}_n \) is a cs*-cover (resp. cf p-cover) of \( X \), then \( f \) is a sequentially-quotient (resp. compact-covering) mapping.

Take Theorem 1.3 into account, the following question naturally arises.

Question 1.4. Can implications (1) and (2) in Theorem 1.3 be reversed?

In this paper, we investigate the Ponomarev-system \( \{ f, M, X, \{ \mathcal{P}_n \} \} \) to answer Question 1.4 affirmatively. We also prove that, in a Ponomarev-system \( \{ f, M, X, \{ \mathcal{P}_n \} \} \), \( f \) is a sequence-covering mapping if and only if each \( \mathcal{P}_n \) is an fcs-cover. As a consequence of these results, \( f \) is a sequentially-quotient, s-mapping if and only if it is a sequence-covering, s-mapping, where “s” can not be omitted.

Throughout this paper, all spaces are assumed to be regular and \( T_1 \), and all mappings are continuous and onto. \( \mathbb{N} \) denotes the set of all natural numbers, \( \{ x_n \} \) denotes a sequence, where the \( n \)-th term is \( x_n \). Let \( X \) be a space and let \( A \) be a subset of \( X \). We call that a sequence \( \{ x_n \} \) converging to \( x \) in \( X \) is eventually in \( A \) if \( \{ x_n : n > k \} \cup \{ x \} \subset A \) for some \( k \in \mathbb{N} \). Let \( \mathcal{P} \) be a family of subsets of \( X \) and let \( x \in X \). \( \bigcup \mathcal{P} \), \( \text{st}(x, \mathcal{P}) \) and \( (\mathcal{P})_x \) denote the union \( \bigcup \{ P : P \in \mathcal{P} \} \), the union \( \bigcup \{ P \in \mathcal{P} : x \in P \} \) and the subfamily \( \{ P \in \mathcal{P} : x \in P \} \) of \( \mathcal{P} \) respectively. For a sequence \( \{ \mathcal{P}_n : n \in \mathbb{N} \} \) of covers of a space \( X \) and a sequence \( \{ P_n : n \in \mathbb{N} \} \) of subsets of a space \( X \), we abbreviate \( \{ \mathcal{P}_n : n \in \mathbb{N} \} \) and \( \{ P_n : n \in \mathbb{N} \} \) to \( \{ \mathcal{P}_n \} \) and \( \{ P_n \} \) respectively. A point \( b = (\beta_n)_{n \in \mathbb{N}} \) of a Tychonoff-product space is abbreviated to \( (\beta_n) \), and the \( n \)-th coordinate \( \beta_n \) of \( b \) is also denoted by \( (b)_n \).
2. The main results

Definition 2.1. Let \( f : X \rightarrow Y \) be a mapping. 
(1) \( f \) is called a sequentially-quotient mapping ([1]) if for each convergent sequence \( S \) in \( Y \), there exists a convergent sequence \( L \) in \( X \) such that \( f(L) = S \).
(2) \( f \) is called a sequence-covering mapping ([4]) if for each convergent sequence \( S \) converging to \( y \) in \( Y \), there exists a compact subset \( K \) of \( X \) such that \( f(K) = S \cup \{ y \} \).
(3) \( f \) is called a compact-covering mapping ([8]) if for each compact subset \( L \) of \( Y \), there exists a compact subset \( K \) of \( X \) such that \( f(K) = L \).

Remark 2.2. (1) Compact-covering mapping \( \Rightarrow \) sequence-covering mapping \( \Rightarrow \) (if the domain is metric) sequentially-quotient mapping ([6]).
(2) “sequence-covering mapping” in Definition 2.1(2) was also called “pseudo-sequence-covering mapping” by Ikeda, Liu and Tanaka in [5].

Definition 2.3. Let \( (X,d) \) be a metric space, and let \( f : X \rightarrow Y \) be a mapping. \( f \) is called a \( \pi \)-mapping ([9]), if for each \( y \) in \( Y \) and for each neighbourhood \( U \) of \( y \) in \( Y \), \( d(f^{-1}(y), X - f^{-1}(U)) > 0 \).

Remark 2.4. (1) For a Ponomarev-system \( \{f, M, X, \{P_n\}\} \), \( f : M \rightarrow X \) is a \( \pi \)-mapping ([7, 10]).
(2) Recall a mapping \( f : X \rightarrow Y \) is a compact mapping (resp. \( s \)-mapping), if \( f^{-1}(y) \) is a compact (resp. separable) subset of \( X \) for each \( y \in Y \). It is clear that each compact mapping from a metric space is an \( s \)- and \( \pi \)-mapping.

Definition 2.5. Let \( \mathcal{P} \) be a cover of a space \( X \).
(1) \( \mathcal{P} \) is called a \( cs^* \)-cover of \( X \) ([6]) if for each convergent sequence \( S \) in \( X \), there exists \( P \in \mathcal{P} \) and a subsequence \( S' \) of \( S \) such that \( S' \) is eventually in \( P \).
(2) \( \mathcal{P} \) is called an \( fcs \)-cover of \( X \) ([3]) if for each sequence \( S \) converging to \( x \) in \( X \), there exists a finite subfamily \( \mathcal{P}' \) of \( (\mathcal{P})_x \) such that \( S \) is eventually in \( \bigcup \mathcal{P}' \).
(3) \( \mathcal{P} \) is called a \( cfps \)-cover of \( X \) ([7]) if for each compact subset \( K \), there exists a finite family \( \{K_n : n \leq m\} \) of closed subsets of \( K \) and \( \{P_n : n \leq m\} \subset \mathcal{P} \) such that \( K = \bigcup\{K_n : n \leq m\} \) and each \( K_n \subset P_n \).

Lemma 2.6. Let \( \{f, M, X, \{P_n\}\} \) be a Ponomarev-system and let \( U = (\Pi_{n \in \mathbb{N}} \Gamma_n) \cap M \), where \( \Gamma_n \subset \Lambda_n \) for each \( n \in \mathbb{N} \). Then \( f(U) \subset \bigcup\{P_\beta : \beta \in \Gamma_k\} \) for each \( k \in \mathbb{N} \).

Proof. Let \( b = (\beta_k) \in U \) and let \( k \in \mathbb{N} \). Then \( \{P_{\beta_k}\} \) forms a network at \( f(b) \) in \( X \) and \( \beta_k \in \Gamma_k \). So \( f(b) \in P_{\beta_k} \subset \bigcup\{P_\beta : \beta \in \Gamma_k\} \). This proves that \( f(U) \subset \bigcup\{P_\beta : \beta \in \Gamma_k\} \).

Theorem 2.7. Let \( \{f, M, X, \{P_n\}\} \) be a Ponomarev-system. Then the following hold.
(1) \( f \) is a compact mapping (resp. \( s \)-mapping) if and only if \( \mathcal{P}_m \) is point-finite (resp. point-countable) cover of \( X \) for each \( m \in \mathbb{N} \).
(2) \( f \) is a sequentially-quotient mapping if and only if \( \mathcal{P}_m \) is a \( cs^* \)-cover of \( X \) for each \( m \in \mathbb{N} \).
(3) $f$ is a compact-covering mapping if and only if $\mathcal{P}_m$ is a cf$p$-cover of $X$ for each $m \in \mathbb{N}$.

**Proof.** By Theorem 1.3, we only need to prove necessities of (1), (2) and (3). Let $m \in \mathbb{N}$.

(1) We only give a proof for the parenthetic part. If $\mathcal{P}_m$ is not point-countable, then, for some $x \in X$, there exists an uncountable subset $\Gamma_m$ of $\Lambda_m$ such that $\Gamma_m = \{ \beta \in \Lambda_m : x \in P_\beta \}$. For each $\beta \in \Gamma_m$, put $U_\beta = ((\Pi_{n<m} \Lambda_n) \times \{ \beta \} \times (\Pi_{n>m} \Lambda_n)) \cap M$. Then $U_\beta : \beta \in \Gamma_m$ covers $f^{-1}(x)$. If not, there exists $c = (\gamma_n) \in f^{-1}(x)$ and $c \not\in U_\beta$ for each $\beta \in \Gamma_m$, so $\gamma_m \not\in \Gamma_m$. Thus $x \not\in P_\gamma_m$ from construction of $\Gamma_m$. But $x = f(c) \in P_\gamma_m$ from Lemma 2.6. This is a contradiction. Thus $\{ U_\beta : \beta \in \Gamma_m \}$ is an uncountable open cover of $f^{-1}(x)$, but it has not any proper subcover. So $f^{-1}(x)$ is not separable, hence $f$ is not an $s$-mapping.

(2) Let $f$ be a sequentially-quotient mapping, and let $\{ x_n \}$ be a sequence converging to $x$ in $X$. Then there exists a sequence $\{ b_k \}$ converging to $b$ in $M$ such that $f(b_k) = x_n$ for each $k \in \mathbb{N}$. Let $b = (\beta_n) \in (\Pi_{n \in \mathbb{N}} \Lambda_n) \cap M$. We claim that the subsequence $\{ x_{n_k} \}$ of $\{ x_n \}$ is eventually in $P_{\beta_m}$. In fact, put $U = ((\Pi_{n<m} \Lambda_n) \times \{ \beta_m \} \times (\Pi_{n>m} \Lambda_n)) \cap M$, then $U$ is an open neighbourhood of $b$ in $M$. So sequence $\{ b_k \}$ is eventually in $U$, hence sequence $\{ x_{n_k} \}$ is eventually in $f(U)$. $f(U) \subset P_{\beta_m}$ from Lemma 2.6, so $\{ x_{n_k} \}$ is eventually in $P_{\beta_m}$. Note that $\beta_m \in \Lambda_m$, so $P_{\beta_m} \subset \mathcal{P}_m$. This proves that $\mathcal{P}_m$ is a $cs^*$-cover of $X$.

(3) Let $f$ be a compact-covering mapping, and let $C$ be a compact subset of $X$. Then there exists a compact subset $K$ of $M$ such that $f(K) = C$. For each $a \in K$, put $U_a = ((\Pi_{n<m} \Lambda_n) \times \{(a)_{m} \} \times (\Pi_{n>m} \Lambda_n)) \cap M$, where $(a)_{m} \in \Lambda_m$ is the $m$-th coordinate of $a$, then $U_a \cap K$ is an open (in subspace $K$) neighbourhood of $a$. So there exists an open (in subspace $K$) neighbourhood $V_a$ of $a$ such that $a \in V_a \subset Cl_K(V_a) \subset U_a \cap K$, where $Cl_K(V_a)$ is the closure of $V_a$ in subspace $K$. Note that $\{ V_a : a \in K \}$ is an open cover of subspace $K$ and $K$ is compact in $M$, so there exists a finite subset $\{ a_1, a_2, \ldots, a_s \}$ of $K$ such that $\{ V_{a_i} : i = 1, 2, \ldots, s \}$ is a finite cover of $K$. Thus $\bigcup \{ Cl_K(V_{a_i}) : i = 1, 2, \ldots, s \} = K$, and so $\bigcup \{ f(Cl_K(V_{a_i})) : i = 1, 2, \ldots, s \} = f(K) = C$. For each $i = 1, 2, \ldots, s$, put $C_i = f(Cl_K(V_{a_i}))$. Since $Cl_K(V_{a_i})$ is compact in $K$, $C_i$ is compact in $C$, so $C_i$ is closed in $C$, and $C = \bigcup \{ C_i : i = 1, 2, \ldots, s \}$. For each $i = 1, 2, \ldots, s$, $C_i = f(Cl_K(V_{a_i})) \subset f(U_{a_i} \cap K) \subset f(U_{a_i})$, and $f(U_{a_i}) \subset P_{(a_i)_{m}}$ from Lemma 2.6, so $C_i \subset P_{(a_i)_{m}}$. Note that $(a_i)_{m} \in \Lambda_m$, so $P_{(a_i)_{m}} \subset \mathcal{P}_m$. This proves that $\mathcal{P}_m$ is a cf$p$-cover of $X$. 

By viewing the above theorem, we ask: in a Ponomarev-system $(f, M, X, \{ \mathcal{P}_n \})$, what is the sufficient and necessary condition such that $f$ is a sequence-covering mapping? We give an answer to this question.

**Theorem 2.8.** Let $(f, M, X, \{ \mathcal{P}_n \})$ be a Ponomarev-system. Then $f$ is a sequence-covering mapping if and only if each $\mathcal{P}_n$ is an $fcs$-cover of $X$.

**Proof.** Suficiency: Let each $\mathcal{P}_n$ be an $fcs$-cover of $X$, and let $S = \{ x_n \}$ be a sequence converging to $x$ in $X$. For each $n \in \mathbb{N}$, since $\mathcal{P}_n$ is an $fcs$-cover, there exists a finite subfamily $\mathcal{F}_n$ of $(\mathcal{P}_n)_x$ such that $S$ is eventually in $\bigcup \mathcal{F}_n$. 

Note that $S - \bigcup \mathcal{F}_n$ is finite. There exists a finite subfamily $\mathcal{G}_n$ of $\mathcal{P}_n$ such that $S - \bigcup \mathcal{F}_n \subset \bigcup \mathcal{G}_n$. Put $\mathcal{F}_n \cup \mathcal{G}_n = \{P_{\beta_n} : \beta_n \in \Gamma_n\}$, where $\Gamma_n$ is a finite subset of $\Lambda_n$. For each $\beta_n \in \Gamma_n$, if $P_{\beta_n} \in \mathcal{F}_n$, put $S_{\beta_n} = (S \cap P_{\beta_n}) \cup \{x\}$, otherwise, put $S_{\beta_n} = (S - \bigcup \mathcal{F}_n) \cap P_{\beta_n}$. It is easy to see that $S = \bigcup_{\beta_n \in \Gamma_n} S_{\beta_n}$ and $\{S_{\beta_n} : \beta_n \in \Gamma_n\}$ is a family of compact subsets of $X$.

Put $K = \{(\beta_n) : \beta_n \in \Pi_{n \in \mathbb{N}} \Gamma_n : \bigcap_{n \in \mathbb{N}} S_{\beta_n} \neq \emptyset\}$. Then

**Claim 1:** $K \subset M$ and $f(K) \subset S$.

Let $b = (\beta_n) \in K$, then $\bigcap_{n \in \mathbb{N}} S_{\beta_n} \neq \emptyset$. Pick $y \in \bigcap_{n \in \mathbb{N}} S_{\beta_n}$, then $y \in \bigcap_{n \in \mathbb{N}} P_{\beta_n}$. Note that $\{P_{\beta_n} : n \in \mathbb{N}\}$ forms a network at $y$ in $X$ if and only if $y \in \bigcap_{n \in \mathbb{N}} P_{\beta_n}$. So $b \in M$ and $f(b) = y \in S$. This proves that $K \subset M$ and $f(K) \subset S$.

**Claim 2:** $S \subset f(K)$.

Let $y \in S$. For each $n \in \mathbb{N}$, pick $\beta_n \in \Gamma_n$ such that $y \in S_{\beta_n}$. Put $b = (\beta_n)$, then $b \in K$ and $f(b) = y$. This proves that $S \subset f(K)$.

**Claim 3:** $K$ is a compact subset of $M$.

Since $K \subset M$ and $\Pi_{n \in \mathbb{N}} \Gamma_n$ is a compact subset of $\prod_{n \in \mathbb{N}} \Lambda_n$. We only need to prove that $K$ is a closed subset of $\Pi_{n \in \mathbb{N}} \Gamma_n$. It is clear that $K \subset \Pi_{n \in \mathbb{N}} \Gamma_n$. Let $b = (\beta_n) \in \Pi_{n \in \mathbb{N}} \Gamma_n - K$. Then $\bigcap_{n \in \mathbb{N}} S_{\beta_n} = \emptyset$. There exists $n_0 \in \mathbb{N}$ such that $\bigcap_{n \leq n_0} S_{\beta_n} = \emptyset$. Put $W = \{(\gamma_n) : \gamma_n = \beta_n \text{ for } n \leq n_0\}$. Then $W$ is open in $\Pi_{n \in \mathbb{N}} \Gamma_n$, and $b \in W$. It is easy to see that $W \cap K = \emptyset$. So $K$ is a closed subset of $\Pi_{n \in \mathbb{N}} \Gamma_n$.

By the above three claims, $f$ is a sequence-covering mapping.

**Necessity:** Let $f$ be a sequence-covering mapping and let $m \in \mathbb{N}$. Whenever $\{x_n\}$ is a sequence converging to $x$ in $X$, there exists a compact subset $K$ of $M$ such that $f(K) = \{x_n : n \in \mathbb{N}\} \cup \{x\}$. Since $f^{-1}(x) \cap K$ is a compact subset of $M$, there exists a finite subset $\{a_i : i = 1, 2, \ldots, s\}$ of $f^{-1}(x) \cap K$ and a finite open cover $\{U_i : i = 1, 2, \ldots, s\}$ of $f^{-1}(x) \cap K$, where for each $i = 1, 2, \ldots, s$, $U_i = ((\Pi_{i<m} \Lambda_n) \times \{(a_i)_m\}) \times (\Pi_{i>n} \Lambda_n)) \cap M$ is an open neighbourhood of $a_i$, and $(a_i)_m \in \Lambda_m$ is the $m$-th coordinate of $a_i$. By Lemma 2.6, $x = f(a_i) \in f(U_i \subset P_{(a_i)_m}) \in (\mathcal{P}_m)_x$ for each $i = 1, 2, \ldots, s$. We only need to prove that sequence $\{x_n\}$ converging to $x$ is eventually in $\bigcup\{P_{(a_i)_m} : i = 1, 2, \ldots, s\}$. If not, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \notin \bigcup\{P_{(a_i)_m} : i = 1, 2, \ldots, s\}$ for each $k \in \mathbb{N}$. That is, for each $k \in \mathbb{N}$ and each $i = 1, 2, \ldots, s$, $x_{n_k} \notin P_{(a_i)_m}$. For each $k \in \mathbb{N}$, we pick $b_k \in K$ such that $f(b_k) = x_{n_k}$. If for some $k \in \mathbb{N}$ and some $i = 1, 2, \ldots, s$, $b_k \in U_i$, then $x_{n_k} = f(b_k) \in f(U_i) \subset P_{(a_i)_m}$, from Lemma 2.6. This is a contradiction. So $b_k \notin U_i$ for each $k \in \mathbb{N}$ and each $i = 1, 2, \ldots, s$. Thus $\{b_k : k \in \mathbb{N}\} \subset K - \bigcup\{U_i : i = 1, 2, \ldots, s\}$. Note that $K - \bigcup\{U_i : i = 1, 2, \ldots, s\}$ is a compact metric subspace, there exists a sequence $\{b_{k_j}\}$ converging to a point $b \in K - \bigcup\{U_i : i = 1, 2, \ldots, s\}$. Thus $b \notin f^{-1}(x)$, so $f(b) \neq x$. On the other hand, $(f(b_{k_j}))$ converges to $f(b)$ by the continuity of $f$ and $\{f(b_{k_j})\} = \{x_{n_k}\}$ converges to $x$, so $f(b) = x$. This is a contradiction. So sequence $\{x_n\}$ converging to $x$ is eventually in $\bigcup\{P_{(a_i)_m} : i = 1, 2, \ldots, s\}$.

□
3. Some consequences

cs*-cover and fcs-cover are not equivalent in general, but there exist some relations between cs*-cover and fcs-cover.

**Proposition 3.1.** Let \( \mathcal{P} \) be a cover of a space \( X \). Then the following hold.

1. If \( \mathcal{P} \) is an fcs-cover of \( X \), then \( \mathcal{P} \) is a cs*-cover of \( X \).
2. If \( \mathcal{P} \) is a point-countable cs*-cover of \( X \), then \( \mathcal{P} \) is an fcs-cover of \( X \).

**Proof.** (1) holds from Definition 2.5. We only need to prove (2).

Let \( \mathcal{P} \) be a point-countable cs*-cover of \( X \). Let \( S = \{x_n\} \) be a sequence converging to \( x \) in \( X \). Since \( \mathcal{P} \) is point-countable, put \((\mathcal{P})_x = \{P_n : n \in \mathbb{N}\}\). Then \( S \) is eventually in \( \bigcup_{n \leq k} P_n \) for some \( k \in \mathbb{N} \). If not, then for any \( k \in \mathbb{N} \), \( S \) is not eventually in \( \bigcup_{n \leq k} P_n \). So, for each \( k \in \mathbb{N} \), there exists \( x_{n_k} \in S - \bigcup_{n \leq k} P_n \). We may assume \( n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots \). Put \( S' = \{x_{n_k} : k \in \mathbb{N}\} \), then \( S' \) is a sequence converging to \( x \). Since \( \mathcal{P} \) is a cs*-cover, there exists \( m \in \mathbb{N} \) and a subsequence \( S'' \) of \( S' \) such that \( S'' \) is eventually in \( P_m \). Note that \( P_m \in (\mathcal{P})_x \).

This contradicts the construction of \( S' \).

**Corollary 3.2.** Let \((f, M, X, \{\mathcal{P}_n\})\) be a Ponomarev-system. Then the following are equivalent.

1. \( f \) is a sequentially-quotient, s-mapping;
2. \( f \) is a sequence-covering, s-mapping.

**Proof.** Consider the following conditions.

3. \( \mathcal{P}_n \) is a point-countable cs*-cover of \( X \) for each \( n \in \mathbb{N} \);
4. \( \mathcal{P}_n \) is a point-countable fcs-cover of \( X \) for each \( n \in \mathbb{N} \).

Then (1)\(\iff\) (3) and (2)\(\iff\) (4) from Theorem 2.7 and Theorem 2.8 respectively.

(3)\(\iff\) (4) from Proposition 3.1. So (1)\(\iff\) (2).

Can “s-” in Corollary 3.2 be omitted? We give a negative answer for this question. We call a family \( D \) of subsets of a set \( D \) is an almost disjoint family if \( A \cap B = \emptyset \) finite whenever \( A, B \in D \), \( A \neq B \).

**Example 3.3.** There exists a space \( X \), which has a point-star network \( \{\mathcal{P}_n\} \) consisting of cs*-covers of \( X \), but \( \mathcal{P}_n \) is not an fcs-cover of \( X \) for each \( n \in \mathbb{N} \).

**Proof.** Let \( X = \{0\} \cup \{1/n : n \in \mathbb{N}\} \) endowed with usual subspace topology of real line \( \mathbb{R} \). Let \( n \in \mathbb{N} \), we construct \( \mathcal{P}_n \) as follows.

Put \( A_n = \{1/k : k > n\} \). Using Zorn’s Lemma, there exists a family \( \mathcal{A}_n \) of infinite subsets of \( A_n \) such that \( \mathcal{A}_n \) is an almost disjoint family and maximal with respect to these properties. Then \( \mathcal{A}_n \) must be infinite (in fact, \( \mathcal{A}_n \) must be uncountable) and denote it by \( \{P_\beta : \beta \in \Lambda_n\} \). Put \( \mathcal{B}_n = \{P_\beta \cup \{0\} : \beta \in \Lambda_n\} \), and put \( \mathcal{P}_n = \mathcal{B}_n \cup \{1/k : k = 1, 2, \ldots, n\} \). Thus \( \mathcal{P}_n \) is constructed. We only need to prove the following three claims.

**Claim 1.** \( \{\mathcal{P}_n\} \) is a point-star network of \( X \).

Let \( x \in U \) with \( U \) open in \( X \). If \( x = 0 \), then there exists \( m \in \mathbb{N} \) such that \( A_m \subset U \). It is easy to check that \( st(0, \mathcal{P}_n) = A_m \cup \{0\} \). So \( 0 \in st(0, \mathcal{P}_n) \subset U \). If
This proves that \( \{P_n\} \) is a point-star network of \( X \).

**Claim 2:** For each \( n \in \mathbb{N} \), \( P_n \) is a \( cs^* \)-cover of \( X \).

Let \( n \in \mathbb{N} \) and let \( S = \{x_k\} \) be a sequence converging to \( x \) in \( X \). Without loss of generalization, we can assume \( S \) is nontrivial, that is, the set \( L = \{x_k : k \in \mathbb{N}\} \cap A_n \) is an infinite subset of \( A_n \) and the limit point \( x = 0 \). If \( L \in A_n \), it is clear that \( S \) has a subsequence is eventually in \( \bigcup\{0\} \in B_n \subset P_n \). If \( L \not\in A_n \), then there exists \( \beta \in A_n \) such that \( L \cap P_\beta \) is infinite. Otherwise, \( L \in A_n \) by maximality of \( A_n \). Thus \( S \) has a subsequence is eventually in \( P_\beta \cup \{0\} \in B_n \subset P_n \). So \( P_n \) is a \( cs^* \)-cover of \( X \).

**Claim 3:** For each \( n \in \mathbb{N} \), \( P_n \) is not an \( fcs \)-cover of \( X \).

Let \( n \in \mathbb{N} \). If \( P_n \) is an \( fcs \)-cover of \( X \), then, for sequence \( \{1/k\} \) converging to \( 0 \) in \( X \), there exist \( P_{\beta_1}, P_{\beta_2}, \ldots, P_{\beta_k} \in A_n \) and some \( m \in \mathbb{N} \) such that \( A_m = \{1/k : k > m\} \subset \bigcup\{P_\beta : i = 1, 2, \ldots, s\} \). Since \( A_n \) is infinite, pick \( \beta \in A_n \) such that \( \beta = 1, 2, \ldots, s \). Then \( A_m \cap P_\beta \) is infinite, and \( A_m \cap P_\beta \subset \bigcup\{P_\beta : i = 1, 2, \ldots, s\} \). So there exists \( i \in \{1, 2, \ldots, s\} \) such that \( A_m \cap P_\beta \cap P_{\beta_i} \) is infinite. Thus \( P_\beta \cap P_{\beta_i} \) is infinite. This contradicts that \( A_n \) is almost disjoint. So \( P_n \) is not an \( fcs \)-cover of \( X \).

Thus we complete the proof of this example.

**Remark 3.4.** Let \( X \) and \( \{P_n\} \) be given as in Example 3.3. Then, for Ponomarev-system \( (f, M, X, \{P_n\}) \), \( f \) is sequentially-quotient from Theorem 2.7 and Claim 2 in Example 3.3 (note: \( f \) is also a \( \pi \)-mapping from Remark 2.(1)), and \( f \) is not sequence-covering from Theorem 2.8 and Claim 3 in Example 3.3. So “\( s^\ast \)” in Corollary 3.2 can not be omitted.

**Remark 3.5.** Recently, Lin proved that each sequentially-quotient, compact mapping from a metric space is sequence-covering, which answers [6, Question 3.4.8] (also, [2, Question 2.6]). Naturally, we ask: is each sequentially-quotient, \( \pi \)-mapping from a metric space sequence-covering? The answer is negative. In fact, let \( f \) be a mapping in Remark 3.4. Then \( f \) is a sequentially-quotient, \( \pi \)-mapping from a metric space \( M \), but it is not sequence-covering.

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**References**


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