ON 4-DIMENSIONAL LOCALLY CONFORMALLY FLAT ALMOST KÄHLER MANIFOLDS

WIESŁAW KRÓLIKOWSKI

Abstract. Using the fundamental notions of the quaternionic analysis we show that there are no 4-dimensional almost Kähler manifolds which are locally conformally flat with a metric of a special form.

I. Basic notions and the aim of the paper

Let $M^{2n}$ be a real $C^\infty$-manifold of dimension $2n$ endowed with an almost complex structure $J$ and a Riemannian metric $g$. If the metric $g$ is invariant by the almost complex structure $J$, i.e.

$$g(JX, JY) = g(X, Y)$$

for any vector fields $X$ and $Y$ on $M^{2n}$, then $(M^{2n}, J, g)$ is called almost Hermitian manifold.

Define the fundamental 2-form $\Omega$ by

$$\Omega(X, Y) := g(X, JY).$$

An almost Hermitian manifold $(M^{2n}, J, g, \Omega)$ is said to be almost Kähler if $\Omega$ is a closed form, i.e.

$$d\Omega = 0.$$ 

Suppose that

$$n = 2.$$ 

The aim of the paper is to prove the following:


Key words and phrases: almost Kähler manifold, quaternionic analysis, regular function in the sense of Fueter.

Received December 7, 2004.
Theorem I. If \((M^4, J, g, \Omega)\) is a 4-dimensional almost Kähler manifold which is locally conformally flat, i.e. in a neighbourhood of every point \(p_0 \in M^4\) there exists a system of local coordinates \((U_{p_0}; w, x, y, z)\) such that the metric \(g\) is expressed by

\[
g = g_0(p)[dw^2 + dx^2 + dy^2 + dz^2], \quad p \in U_{p_0},
\]

where \(g_0(p)\) is a real positive \(C^\infty\)-function defined around \(p_0\), then \(g_0\) is a modulus of some quaternionic function left (right) regular in the sense of Fueter [1] uniquely determined by \(J\) and \(\Omega\).

II. PROOF OF THEOREM

Let us denote by the same letters the matrices of \(g, J\) and \(\Omega\) with respect to the coordinate basis. These matrices satisfy the equality:

\[
g \cdot J = \Omega.
\]

The metric \(g\), by the assumption, is proportional to the identity, so it has the form

\[
g = g_0 \cdot I = g_0 \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

An almost complex structure \(J\) satisfies the condition:

\[
J^2 = -I.
\]

Since \(\Omega\) is skew-symmetric then \(J\) is a skew-symmetric and orthogonal 4×4-matrix.

It is easy to check that \(J\) is of the form

\[
\begin{align*}
(1) \quad a) & \quad \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ -b & -c & 0 & a \\ -c & b & -a & 0 \end{pmatrix} \quad \text{or} \quad b) & \quad \begin{pmatrix} 0 & a & b & c \\ -a & 0 & -c & b \\ -b & c & 0 & -a \\ -c & -b & a & 0 \end{pmatrix}
\end{align*}
\]

with

\[
a^2 + b^2 + c^2 = 1.
\]

Suppose that \(J\) is of the form (1a). Then the matrix \(\Omega\) looks as follows:

\[
\Omega = g_0 \cdot \begin{pmatrix} 0 & a & -b & c \\ -a & 0 & c & b \\ b & -c & 0 & a \\ -c & -b & -a & 0 \end{pmatrix} = \begin{pmatrix} 0 & A & -B & C \\ -A & 0 & C & B \\ B & -C & 0 & A \\ -C & -B & -A & 0 \end{pmatrix}.
\]

Since

\[
\left(\frac{A}{g_0}\right)^2 + \left(\frac{B}{g_0}\right)^2 + \left(\frac{C}{g_0}\right)^2 = a^2 + b^2 + c^2 = 1
\]
then we get

\[(2) \quad A^2 + B^2 + C^2 = g_0^2.\]

By the assumption

\[d\Omega = 0.\]

Using the following formula (see e.g. [4], p.36):

\[
d\Omega(X, Y, Z) = \frac{1}{3} \left( X\Omega(Y, Z) + Y\Omega(Z, X) + Z\Omega(X, Y) \right.
\]

\[
- \Omega([X, Y], Z) - \Omega([Z, X], Y) - \Omega([Y, Z], X) \bigg),
\]

the condition \(d\Omega = 0\) can be written in the form:

\[
0 = 3d\Omega(\partial_x, \partial_y, \partial_z) = A_x + B_y + C_z, \]

\[
0 = 3d\Omega(\partial_x, \partial_y, \partial_w) = B_x - A_y + C_w, \]

\[
0 = 3d\Omega(\partial_x, \partial_z, \partial_w) = C_x - A_z - B_w, \]

\[
0 = 3d\Omega(\partial_y, \partial_z, \partial_w) = C_y - B_z + A_w.
\]

Then the components \(A, B\) and \(C\) of \(\Omega\) satisfy the following system of first order partial differential equations:

\[
(3) \quad A_x + B_y + C_z = 0,
\]

\[
B_x - A_y + C_w = 0,
\]

\[
C_x - A_z - B_w = 0,
\]

\[
C_y - B_z + A_w = 0.
\]

and the condition (2).

The above system (3), although overdetermined, does have solutions. We will show that the system (3) has a nice interpretation in the quaternionic analysis.

**III. Fueter’s regular functions**

Denote by \(\mathbf{H}\) the field of quaternions. \(\mathbf{H}\) is a 4-dimensional division algebra over \(\mathbb{R}\) with basis \(\{1, i, j, k\}\) and the quaternionic units \(i, j, k\) satisfy:

\[
i^2 = j^2 = k^2 = ijk = -1, \]

\[
i j = -ji = k.
\]

A typical element \(q\) of \(\mathbf{H}\) can be written as

\[
q = w + ix + jy + kz, \quad w, x, y, z \in \mathbb{R}.
\]

The conjugate of \(q\) is defined by

\[
\overline{q} := w - ix - jy - kz.
\]
and the modulus $\|q\|$ by

$$\|q\|^2 = q\overline{q} = \overline{q}q = w^2 + x^2 + y^2 + z^2.$$  

We will need the following relation (which is easy to check)

$$\overline{q_1 q_2} = \overline{q_2} \overline{q_1}.$$  

A function $F : \mathbb{H} \to \mathbb{H}$ of the quaternionic variable $q$ can be written as

$$F = F_o + iF_1 + jF_2 + kF_3.$$  

$F_o$ is called the real part of $F$ and $iF_1 + jF_2 + kF_3$ the imaginary part of $F$.

In [1] Fueter introduced the following operators:

$$\overline{\partial}_{\text{left}} := \frac{1}{4} \left( \frac{\partial}{\partial w} + i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right),$$  

$$\overline{\partial}_{\text{right}} := \frac{1}{4} \left( \frac{\partial}{\partial w} + \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right),$$  

analogous to $\overline{\partial} = \frac{1}{2} (\overline{i} + \overline{j})$ in the complex analysis, to generalize the Cauchy-Riemann equations.

A quaternionic function $F$ is said to be left regular (respectively, right regular) (in the sense of Fueter) if it is differentiable in the real variable sense and

$$\overline{\partial}_{\text{left}} F = 0 \quad \text{(resp. } \overline{\partial}_{\text{right}} F = 0).$$  

Note that the condition (4) is equivalent to the following system of equations:

$$\partial_w F_o - \partial_x F_1 - \partial_y F_2 - \partial_z F_3 = 0,$$

$$\partial_w F_1 + \partial_x F_o + \partial_y F_3 - \partial_z F_2 = 0,$$

$$\partial_w F_2 - \partial_x F_3 + \partial_y F_o + \partial_z F_1 = 0,$$

$$\partial_w F_3 + \partial_x F_2 - \partial_y F_1 + \partial_z F_o = 0.$$  

There are many examples of left and right regular functions in the sense of Fueter. Many papers have been devoted studying the properties of those functions (e.g. [3]). One has found the quaternionic generalizations of the Cauchy theorem, the Cauchy integral formula, Taylor series in terms of special polynomials etc.

Now we need an important result of [5]. It can be described as follows.

Let $\nu$ be an unordered set of $n$ integers $\{i_1, \ldots, i_n\}$ with $1 \leq i_r \leq 3$; $\nu$ is determined by three integers $n_1$, $n_2$ and $n_3$ with $n_1 + n_2 + n_3 = n$, where $n_1$ is the number of 1’s in $\nu$, $n_2$ - the number of 2’s and $n_3$ - the number of 3’s.

There are $\frac{1}{2}(n+1)(n+2)$ such sets $\nu$ and we denote the set of all of them by $\sigma_n$. 
Let $e_i$ and $x_i$ denote $i, j, k$ and $x, y, z$ according as $i_r$ is 1, 2 or 3, respectively. Then one defines the following polynomials

$$P_\nu(q) := \frac{1}{n!} \sum (we_{i_1} - x_{i_1}) \cdots (we_{i_n} - x_{i_n}),$$

where the sum is taken over all $n! \cdot n_1! \cdot n_2! \cdot n_3!$ different orderings of $n_1$ 1’s, $n_2$ 2’s and $n_3$ 3’s; when $n = 0$, so $\nu = \emptyset$, we take $P_\emptyset(q) = 1$.

For example we present the explicit forms of the polynomials $P_\nu$ of the first and second degrees. Thus we have

$$P_1 = wi - x, \quad P_2 = wj - y, \quad P_3 = wk - z,$$

$$P_{11} = \frac{1}{2} (x^2 - w^2) - xwi, \quad P_{12} = xy - wyi - wxj,$$
$$P_{13} = xz - wzi - wxk,$$

$$P_{22} = \frac{1}{2} (y^2 - w^2) - ywj,$$
$$P_{23} = yz - wzj - wyk,$$
$$P_{33} = \frac{1}{2} (z^2 - w^2) - zwk.$$

In [5] Sudbery proved the following

**Proposition.** Suppose $F$ is left regular in a neighbourhood of the origin $0 \in H$. Then there is a ball $B = B(0, r)$ with center $0$ in which $F(q)$ is represented by a uniformly convergent series

$$F(q) = \sum_{n=0}^{\infty} \sum_{\nu \in \sigma_n} P_\nu(q)a_\nu, \quad a_\nu \in H.$$ 

**IV. The end of the proof**

Let us denote

$$F_{ABC}(q) := Ai + Bj + Ck,$$

where we have identified $q \in H$ with $(w, x, y, z) \in \mathbb{R}^4$. Then (3) is nothing but the condition that $F_{ABC}$ is left regular in the sense of Fueter. Then, by (2), we have

$$\|F_{ABC}\| = g_0. \quad \square$$
V. Conclusions

Let $F$ satisfy the assumptions of Proposition. Then

$$ F(q) = a_0 + \sum_{i=1}^{3} P_i a_i + \sum_{i \leq j} P_{ij} a_{ij} + \sum_{i \leq j \leq k} P_{ijk} a_{ijk} + \ldots $$

and

$$ F(q) = a_0 + \sum_{i=1}^{3} \bar{a}_i \bar{T}_i + \sum_{i \leq j} \bar{a}_{ij} \bar{T}_{ij} + \sum_{i \leq j \leq k} \bar{a}_{ijk} \bar{T}_{ijk} + \ldots $$

Multiplying the above expressions we get

$$ ||F(q)||^2 = ||a_0||^2 + \sum_{i=1}^{3} (P_i a_i \bar{a}_0 + a_0 \bar{a}_i \bar{T}_i) $$
$$ + \sum_{i \leq j} (P_{ij} a_{ij} \bar{a}_0 + a_0 \bar{a}_{ij} \bar{T}_{ij}) + \sum_{i, j} P_i a_i \bar{T}_i $$
$$ + \sum_{i \leq j} (P_{ijk} a_{ijk} \bar{a}_0 + a_0 \bar{a}_{ijk} \bar{T}_{ijk}) $$
$$ + \sum_{m=1}^{3} \sum_{i \leq j} (P_{m,i} a_m \bar{a}_{ij} \bar{T}_{ij} + P_{ij} a_{ij} \bar{a}_m \bar{T}_m) + \ldots $$

(5)

Example 1. Let

$$ g_0(w, x, y, z) = \frac{1}{1 + r} , \quad r^2 = w^2 + x^2 + y^2 + z^2 , $$

then

$$ g_0^2 = \frac{1}{(1 + r)^2} = 1 - 2r + 3r^2 - 4r^3 + \ldots + (-1)^n (n + 1) r^n + \ldots $$

(6)

Comparing the right sides of (5) and (6) we see that

$$ a_0 \neq 0 , $$
$$ -2r = \sum_{i=1}^{3} (P_i a_i \bar{a}_0 + a_0 \bar{a}_i \bar{T}_i) $$

but the second equality is impossible.
Example 2. Take
\[ g_0(w, x, y, z) = \frac{1}{\sqrt{1 + r^2}}, \quad r^2 = w^2 + x^2 + y^2 + z^2, \]
then
\[ g_0^2 = \frac{1}{1 + r^2} = 1 - r^3 + r^6 - r^9 + \ldots + (-1)^k r^{3k} + \ldots. \]  
(7)
Comparing the right sides of (5) and (7) we get
\[ a_0 \neq 0, \quad a_i = 0, \quad a_{ij} = 0 \]
and
\[ -r^3 = \sum_{i \leq j \leq k} (P_{ijk}a_{ijk}w_0 + a_0a_{ijk}P_{ijk}) \]
but the last equality is impossible.

Example 3. Let
\[ g_0(w, x, y, z) = \frac{1}{\sqrt{1 - r^2}}, \quad r^2 = w^2 + x^2 + y^2 + z^2, \]
then
\[ g_0^2 = \frac{1}{1 - r^2} = 1 + r^2 + \frac{4}{3}r^3 + \ldots \]  
(8)
Comparing the right sides of (5) and (8) we have
\[ a_0 \neq 0, \quad a_i = 0 \]
and
\[ r^2 = \sum_{i \leq j} (P_{ij}a_{ij}w_0 + a_0a_{ij}P_{ij}). \]  
(9)
Set
\[ d_{ij} := a_{ij}w_0 := d^0_{ij} + d^1_{ij}i + d^2_{ij}j + d^3_{ij}k \]
(i, j, k denote the quaternionic units) and rewrite (9) in the form
\[ w^2 + x^2 + y^2 + z^2 = 2 \sum_{i \leq j} Re (P_{ij}d_{ij}) \]
then we get
\[
\begin{align*}
w^2 + x^2 + y^2 + z^2 &= 2Re \left( \left[ \frac{1}{2}(x^2 - w^2) - xwi \right]d_{11} +\right. \\
&\left. + 2Re \left[ \frac{1}{2}(y^2 - w^2) - ywj \right]d_{22} +\right. \\
&\left. + 2Re \left[ \frac{1}{2}(z^2 - w^2) - zwk \right]d_{33} + \ldots \right) \\
&= (x^2 - w^2)d_{11}^0 + (y^2 - w^2)d_{22}^0 + (z^2 - w^2)d_{33}^0.
\end{align*}
\]
Comparing the terms in \(x^2\), \(y^2\) and \(z^2\) we get
\[
d_{11}^0 = d_{22}^0 = d_{33}^0 = 1
\]
but then
\[
w^2 = -3w^2
\]
and this is impossible.

**Example 4.** Let
\[
g_0(w, x, y, z) = \frac{1}{(1 - r^2)^2}, \quad r^2 = w^2 + x^2 + y^2 + z^2,
\]
then
\[
g_0^2 = \frac{1}{(1 - r^2)^4} = 1 + 4r^2 + \ldots \tag{10}
\]
Comparing the right sides of (5) and (10) we obtain
\[
a_0 \neq 0, \quad a_i = 0
\]
and
\[
4r^2 = \sum_{i \leq j} (P_{ij}a_{ij} + a_0\overline{a}_{ij}\overline{P}_{ij}).
\]
Analogously, like in the Example 3, we have
\[
2w^2 + 2x^2 + 2y^2 + 2z^2 = \sum_{i \leq j} Re (P_{ij}d_{ij}).
\]
This time, comparing the terms in \(x^2\), \(y^2\) and \(z^2\), we get
\[
a_0 \neq 0, \quad a_i = 0,
\]
\[
d_{11}^0 = d_{22}^0 = d_{33}^0 = 4
\]
but then
\[
-6w^2 = 2w^2.
\]
This is again impossible.
VI. General conclusion

There is no 4-dimensional almost Kähler manifold \((M^4, J, g, \Omega)\) which is locally conformally flat with the metric

\[ g = g_0(p)[dw^2 + dx^2 + dy^2 + dz^2], \]

where \(g_0\) is expressed by the formulae (6), (7), (8) and (10). In particular the Poincaré model, i.e. the unit ball \(B^4\) in \(\mathbb{R}^4\) with the metric

\[ g := \frac{4}{(1 - r^2)^2}[dw^2 + dx^2 + dy^2 + dz^2], \quad r^2 := w^2 + x^2 + y^2 + z^2, \]

is not an almost Kähler manifold.

**Remark.** If \(J\) is of the form (1b) then the proof of Theorem is similar. One has to replace the left regular quaternionic function with the right one (see [3], p.10).

**References**


MOTYŁOWA 4/27, 91-360 Łódź, Poland