ON THE LIMIT POINTS OF THE FRACTIONAL PARTS OF POWERS OF PISOT NUMBERS

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Abstract. We consider the sequence of fractional parts \( \{\xi\alpha^n\} \), \( n = 1, 2, 3, \ldots \), where \( \alpha > 1 \) is a Pisot number and \( \xi \in \mathbb{Q}(\alpha) \) is a positive number. We find the set of limit points of this sequence and describe all cases when it has a unique limit point. The case, where \( \xi = 1 \) and the unique limit point is zero, was earlier described by the author and Luca, independently.

1. Introduction

Suppose that \( \alpha > 1 \) is an arbitrary algebraic number, and suppose that \( \xi \) is an arbitrary positive number that lies outside the field \( \mathbb{Q}(\alpha) \) if \( \alpha \) is a Pisot number or a Salem number. For such pairs \( \xi, \alpha \), in [6] we proved a lower bound (in terms of \( \alpha \) only) for the distance between the largest and the smallest limit points of the sequence of fractional parts \( \{\xi\alpha^n\} \). More precisely, we showed that the distance between the largest and the smallest limit points of this sequence is at least \( 1/\inf L(PG) \), where \( P(z) = a_d z^d + \cdots + a_1 z + a_0 \in \mathbb{Z}[z] \) is the minimal polynomial of \( \alpha \) and where \( G \) runs through polynomials with real coefficients having either leading or constant coefficient 1. (Here, \( L \) stands for the length of a polynomial.) For this result, we showed first that with the above conditions the sequence

\[
s_n := a_d[\xi\alpha^{n+d}] + \cdots + a_1[\xi\alpha^{n+1}] + a_0[\xi\alpha^n]
\]

is not ultimately periodic. Recall that \( s_n, n = 0, 1, 2, \ldots \), is called ultimately periodic if there is \( t \in \mathbb{N} \) such that \( s_{n+t} = s_n \) for all sufficiently large \( n \). (In contrast, \( s_n, n = 0, 1, 2, \ldots \), is called purely periodic if there is \( t \in \mathbb{N} \) such that \( s_{n+t} = s_n \) for all \( n \geq 0 \).) For rational \( \alpha = p/q > 1 \), our result in [6] recovers the result of Flatto, Lagarias and Pollington [7]: the difference between the largest and the smallest limit points of the sequence \( \{\xi(p/q)^n\} \) is at least \( 1/p \). (See also [1].)

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Moreover, the results of [6] imply that we always have
\[ \limsup_{n \to \infty} \{ \xi \alpha^n \} - \liminf_{n \to \infty} \{ \xi \alpha^n \} \geq 1/L(P), \]
unless \( s_n, n = 1, 2, \ldots \) is ultimately periodic with period of length 1. However, for some Pisot and Salem numbers \( \alpha \) and for some \( \xi \in \mathbb{Q}(\alpha) \), this can happen. As a result, no bound for the difference between the largest and the smallest limit points of the sequence \( \{ \xi \alpha^n \}_{n=1,2,3,\ldots} \) can be obtained in terms of \( \alpha \) only. More precisely, for Salem numbers \( \alpha \) such that \( \alpha - 1 \) is not a unit, Zaimi [11] showed that for every \( \varepsilon > 0 \) there exist positive numbers \( \xi \in \mathbb{Q}(\alpha) \) such that all fractional parts \( \{ \xi \alpha^n \}_{n=1,2,3,\ldots} \) belong to an interval of length \( \varepsilon \). In this context, the only pairs that remain to be considered are of the form \( \xi, \alpha \), where \( \alpha \) is a Pisot number and \( \xi \in \mathbb{Q}(\alpha) \). The aim of this paper is to consider such pairs.

Recall that \( \alpha > 1 \) is a Pisot number if it is an algebraic integer (i.e. \( a_d = 1 \)) and if all its conjugates over \( \mathbb{Q} \) different from \( \alpha \) itself lie in the open unit disc. The problem of finding all such pairs \( \xi > 0, \alpha > 1 \), where \( \alpha \) is a Pisot number and \( \xi \in \mathbb{Q}(\alpha) \), for which the sequence \( \{ \xi \alpha^n \}_{n=1,2,3,\ldots} \) has a unique limit point is also of interest in connection with the papers [3], [8] and [9]. In [8] Kuba asked whether there are algebraic numbers \( \alpha > 1 \) other than integers satisfying \( \lim_{n \to \infty} \{ \alpha^n \} = 0 \). This was answered by the author [3] and by Luca [9] independently: the answer is ‘no’.

## 2. Results

From now on, suppose that \( \alpha = \alpha_1 > 1 \) is a Pisot number with minimal polynomial
\[ P(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_0 = (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_d) \in \mathbb{Z}[z]. \]
Since \( \xi \in \mathbb{Q}(\alpha) \), we can write \( \xi = f(\alpha) > 0 \), where \( f \) is a non-zero polynomial of degree at most \( d - 1 \) with rational coefficients
\[
f(z) = (b_0 + b_1z + \cdots + b_{d-1}z^{d-1})/b,
\]
Here \( b_0, b_1, \ldots, b_{d-1} \in \mathbb{Z} \) and \( b \) is the smallest positive integer for which \( bf(z) \in \mathbb{Z}[z] \). Set \( S_n := \alpha_1^n + \alpha_2^n + \cdots + \alpha_d^n \) (which is a rational integer for each non-negative integer \( n \)) and
\[
Y_n := b_0S_n + b_1S_{n+1} + \cdots + b_{d-1}S_{n+d-1}.
\]
Then \( Y_n = b \text{Trace}(f(\alpha)\alpha^n) \). By Newton’s formulae, we have \( S_{n+d} + a_{d-1}S_{n+d-1} + \cdots + a_0S_n = 0 \) for every \( n \geq 0 \). It is easy to see that the sequence \( Y_0, Y_1, Y_2, \ldots \) satisfies the same linear recurrence
\[ Y_{n+d} + a_{d-1}Y_{n+d-1} + \cdots + a_0Y_n = 0 \]
for every non-negative integer \( n \). By Lemma 2 of [4], the sequence \( Y_n, n = 0, 1, 2, \ldots \), modulo \( b \) is ultimately periodic. Moreover, in case if \( \gcd(b, a_0) = 1 \), by Lemma 2 of [5], the sequence \( Y_n, n = 0, 1, 2, \ldots \), modulo \( b \) is purely periodic. (These statements both can be proved directly. Firstly, there are at most \( b^d \) different vectors for \( (Y_{n+1}, \ldots, Y_n) \) modulo \( b \) to occur, which implies the first
statement by (2). Secondly, if\( \gcd(b,a_0) = 1 \), then \( Y_n \) modulo \( b \) is uniquely determined by \( Y_{n+d}, \ldots, Y_{n+1} \) modulo \( b \). This shows that a respective sequence is purely periodic.

Suppose that \( B_1B_2 \ldots B_k \), where \( 0 \leq B_j \leq b - 1 \), is the period of \( Y_0, Y_1, Y_2, \ldots \) modulo \( b \). Some of \( B_j \) may be equal. Let \( B \) be the set \( \{B_1, \ldots, B_k\} \). In other words, \( B = B_{\xi,\alpha} \) is the set of residues of the sequence \( Y_n, n = 0, 1, 2, \ldots \), modulo \( b \) which occur infinitely often. We can now state our results.

**Theorem 1.** Let \( \alpha > 1 \) be a Pisot number and let \( f(z) \) be a polynomial given in (1). Then \( t \in (0, 1) \) is a limit point of the sequence \( \{f(\alpha)\alpha^n\}_{n=1,2,3,\ldots} \) if and only if there is \( c \in B \) such that \( t = c/b \). Furthermore, at least one of the numbers 0 and 1 is a limit point of \( \{f(\alpha)\alpha^n\}_{n=1,2,3,\ldots} \) if and only if \( 0 \notin B \).

Without loss of generality we can assume that the conjugates of \( \alpha \) are labelled so that \( \alpha = \alpha_1 > 1 > |\alpha_2| > |\alpha_3| > \cdots > |\alpha_d| \). Then \( \alpha \) is called a strong Pisot number if \( d \geq 2 \) and \( \alpha_2 \) is positive [3]. By a result of Smyth [10] claiming that each circle \( |z| = r \) contains at most two conjugates of a Pisot number \( \alpha \), the inequality \( \alpha_2 > |\alpha_3| \) holds for every strong Pisot number \( \alpha \). Recall that a result of Pisot and Vijayaraghavan (see, e.g., [2]) implies that if the sequence \( \{\xi \alpha^n\}_{n=1,2,3,\ldots} \), where \( \alpha > 1 \) is algebraic and \( \xi > 0 \) is real, has a unique limit point, then \( \alpha \) is a Pisot number and \( \xi \in \mathbb{Q}(\alpha) \). So our next result characterizes all possible cases when the sequence \( \{\xi \alpha^n\}_{n=1,2,3,\ldots} \) has a unique limit point and completes the results of the author [3] and of Luca [9].

**Theorem 2.** Let \( \alpha > 1 \) be a Pisot number and let \( f(z) \) be a polynomial given in (1). Then

(i) \( \lim_{n \to \infty} \{f(\alpha)\alpha^n\} = t \), where \( t \neq 0,1 \), if and only if \( B = \{c\}, c > 0, t = c/b \).

(ii) \( \lim_{n \to \infty} \{f(\alpha)\alpha^n\} = 0 \) if and only if \( B = \{0\} \) and \( \alpha \) is either an integer or a strong Pisot number and \( f(\alpha_2) < 0 \).

(iii) \( \lim_{n \to \infty} \{f(\alpha)\alpha^n\} = 1 \) if and only if \( B = \{0\} \), \( \alpha \) is a strong Pisot number and \( f(\alpha_2) > 0 \).

The following theorem gives a simple practical criterion of determining whether the sequence \( \{f(\alpha)\alpha^n\}_{n=1,2,3,\ldots} \) has one or more than one limit point.

**Theorem 3.** If \( B = \{c\}, c > 0 \), then there is an integer \( r \), where \( 1 \leq r \leq |P(1)|-1 \), such that \( c/b = r/|P(1)| \). Furthermore, if \( \gcd(b,a_0) = 1 \) then \( B = \{c\} \) is equivalent to \( b \mid cP(1) \) and \( b \mid (Y_n-c) \) for every \( n = 0, 1, \ldots, d-1 \).

Theorems 2 and 3 imply the following corollary.

**Corollary.** Let \( \xi \) be an arbitrary positive number, and let \( \alpha \) be a Pisot number which is not an integer or a strong Pisot number. If \( P(1) = -1 \), then \( \{\xi \alpha^n\}_{n=1,2,3,\ldots} \) has more than one limit point.

Since \( P(z) \) is the minimal polynomial of a Pisot number \( \alpha \), we have \( P(1) < 0 \) and \( P'(\alpha) > 0 \). Note that the condition \( P(1) = -1 \) is equivalent to the fact that
Suppose that $\alpha > 1$ is a unit. Our final theorem describes all algebraic numbers $\alpha > 1$ for which there is a positive number $\xi$ such that the sequence $\{\xi \alpha^n\}_{n=1,2,...}$ tends to a limit.

**Theorem 4.** Suppose that $\alpha > 1$ is an algebraic number. Then there is a real number $\xi > 0$ such that the sequence $\{\xi \alpha^n\}_{n=1,2,3,...}$ tends to a limit if and only if $\alpha$ is either a strong Pisot number, or $\alpha = 2$, or $\alpha$ is a Pisot number whose minimal polynomial $P$ satisfies $P(1) \leq -2$.

In fact, we will show that if $\alpha$ is strong Pisot number or $\alpha = 2$ we can take $\xi = 1$, whereas in the third case of Theorem 4 we can take $\xi = 1/(P'(\alpha)(\alpha - 1))$. Some examples will be given in Section 4.

3. **Proofs of the theorems**

**Proof of Theorem 1.** Consider the trace of $f(\alpha)\alpha^n$:

$$f(\alpha_1)\alpha_1^n + f(\alpha_2)\alpha_2^n + \cdots + f(\alpha_d)\alpha_d^n = Y_n/b.$$

Setting

$$L_n := f(\alpha_2)\alpha_2^n + \cdots + f(\alpha_d)\alpha_d^n$$

(which is a real number), we have

$$\{f(\alpha)\alpha^n\} = Y_n/b - L_n - \lfloor f(\alpha)\alpha^n\rfloor.$$

Assume that $\mathcal{B}$ contains a non-zero integer $c$. Then $b \geq 2$. Since $1 \leq c \leq b - 1$ and all $f(\alpha_j)\alpha_j^n$, where $j \geq 2$, tend to zero as $n \to \infty$, we get that $L_n \to 0$ as $n \to \infty$ and so $\{f(\alpha)\alpha^n\} = c/b - L_n$ for infinitely many $n$. Hence $c/b$ is the limit point of $\{f(\alpha)\alpha^n\}_{n=1,2,...}$ for each non-zero $c \in \mathcal{B}$. Suppose now that $t \in (0, 1)$ is a limit point of $\{f(\alpha)\alpha^n\}_{n=1,2,...}$. Since $L_n \to 0$ as $n \to \infty$, equality (4) implies that $t$ is a limit point of $\{f(\alpha)\alpha^n\}_{n=1,2,...}$ only if $t = c/b$, where $c \in \mathcal{B}$. This proves the first part of Theorem 1. The second part follows from (3) and (4) by a similar argument.

**Proof of Theorem 2.** We begin with (i). As above, since $L_n \to 0$ as $n \to \infty$, (4) shows that the sequence $\{f(\alpha)\alpha^n\}_{n=1,2,...}$ has a unique limit point only if $Y_n$ modulo $b$ is ultimately periodic with period of length 1. Since the unique limit point is neither 0 nor 1, it follows that $\mathcal{B} = \{c\}$, where $c > 0$. For the converse, suppose that $\mathcal{B} = \{c\}$, where $c$ is non-zero. Then $b \geq 2$ and $1 \leq c \leq b - 1$. Furthermore, $Y_n$ modulo $b$ is $c$ for each sufficiently large $n$. With these conditions, (4) implies that $\lim_{n \to \infty} \{f(\alpha)\alpha^n\} = c/b$. This proves (i).

If $\alpha$ is an integer, say $\alpha = g$, then $\{(b_0/b)g^n\} \to 0$ as $n \to \infty$ precisely when each prime divisor of $b$ divides $g$, i.e. $\mathcal{B} = \{0\}$, because $Y_n = b_0 g^n$. This proves the subcase of (ii) corresponding to integer $\alpha$. Suppose now that $\alpha$ is irrational. If $\mathcal{B} = \{0\}$, $\alpha$ is a strong Pisot number and $f(\alpha_2) < 0$, then $L_n$ defined by (3) is negative for all sufficiently large $n$. So (4) implies that $\lim_{n \to \infty} \{f(\alpha)\alpha^n\} = 0$.

For the converse, suppose that $\lim_{n \to \infty} \{f(\alpha)\alpha^n\} = 0$. Then (4) shows immediately that $\mathcal{B} = \{0\}$, as otherwise the sequence of fractional parts has other limit points. We already know that one case when $\mathcal{B} = \{0\}$ and $\lim_{n \to \infty} \{f(\alpha)\alpha^n\} = 0$
both occur is when $\alpha$ is an integer. Suppose it is not. Then, since 1 is not the limit point of $\{f(\alpha)\alpha^n\}_{n=1,2,\ldots}$, the sum $L_n$ defined by (3) must be negative for all sufficiently large $n$. Recall that $\alpha_1 > |\alpha_2| > \cdots > |\alpha_d|$.

We will consider three cases corresponding to $\alpha_2$ being complex, negative and positive. By the above mentioned result of Smyth [10], if $\alpha_2$ is complex, then $\alpha_2$ and $\alpha_3$ is the only complex conjugate pair on the circle $|z| = |\alpha_2|$. Since $\alpha_3 = \overline{\alpha_2}$, for each $n$ sufficiently large, the sign of $L_n$ is determined by the sign of $f(\alpha_2)\alpha_2^n + f(\alpha_3)\alpha_3^n$. Clearly, $f(\alpha_2) \neq 0$, because deg $f < d$. Writing $\alpha_2 = \varphi e^{i\phi}$ and $f(\alpha_2) = g e^{i\phi}$, where $g, g' > 0$ and $i = \sqrt{-1}$, we see that $\alpha_3 = \varphi^{-1}e^{-i\phi}$, $f(\alpha_3) = g'e^{-i\phi}$. Hence $L_n < 0$ (for $n$ sufficiently large) precisely when $\cos(n\varphi + \phi) < 0$. Note that $\varphi/\pi$ is irrational, as otherwise there is a positive integer $v$ such that $\alpha_2^v = \alpha_3^v$. Mapping $\alpha_2$ to $\alpha_1$ we get a contradiction, because $\alpha_1$ is the only conjugate of $\alpha$ lying outside the unit circle. Hence, as the sequence $n\varphi/\pi + \phi/\pi$ modulo 1 has each point in $[0, 1]$ as its limit point, $\cos(n\varphi + \phi)$ will be both positive and negative for infinitely many $n$. This rules out the case of $\alpha_2$ being complex. Similarly, if $\alpha_2$ is negative then $L_n$ is both positive and negative infinitely often, because so is $f(\alpha_2)\alpha_2^n$. This implies that $\alpha_2$ must be positive, namely, $\alpha$ must be a strong Pisot number. Then $L_n < 0$ implies that $f(\alpha_2) < 0$. This proves (ii).

The case (iii) can be proved by the same argument as (ii). Indeed, if $\alpha$ is a strong Pisot number, $f(\alpha_2) > 0$, and $B = \{0\}$, then (4) implies that $\lim_{n \to \infty} \{f(\alpha)\alpha^n\} = 1$. For the converse, assume that $\lim_{n \to \infty} \{f(\alpha)\alpha^n\} = 1$. It is easy to see that then $B = \{0\}$. Furthermore, $\alpha$ cannot be a rational integer. Now, (4) shows that $L_n$ must be positive for all sufficiently large $n$. We already proved that this is impossible, unless $\alpha$ is a strong Pisot number. In case it is, (3) shows that $f(\alpha_2)$ must be positive too. This completes the proof of Theorem 2.

**Proof of Theorem 3.** Suppose that $B = \{c\}$, $c > 0$. Then (2) shows that $b$ divides $c(1 + a_{d-1} + \cdots + a_0) = cP(1)$, where $P(1) < 0$. It follows that there is $r \in \mathbb{N}$ such that $br = c|P(1)|$, giving $c/b = r/|P(1)|$. This proves the first statement of Theorem 3.

Now, let $\gcd(b, a_0) = 1$ and suppose again that $B = \{c\}$, where $c$ can be equal to zero. The above argument implies that $b \mid cP(1)$. Evidently, $B = \{c\}$ is equivalent to the fact that $Y_n$ modulo $b$ is equal to $c$ for every sufficiently large $n$. Suppose that there are $k \geq 0$ for which $Y_k$ modulo $b$ is different from $c$. Take the largest such $k$. Let $Y_k$ modulo $b$ be $c'$, where $c' \neq c$. Then (2) with $n = k$ shows that $Y_{k+d} + \cdots + a_1Y_{k+1} + a_0Y_k$ modulo $b$ is $cP(1) + a_0(c' - c)$ which is divisible by $b$. Since $b \mid cP(1)$, we have that $b \mid a_0(c' - c)$. Since $\gcd(a_0, b) = 1$, we conclude that $c' = c$, a contradiction.

For the converse, suppose that $Y_0, Y_1, \ldots, Y_{d-1}$ are all equal to $c$ modulo $b$, and $b \mid cP(1)$. Evidently, (2) with $n = 0$ shows that $Y_d + a_{d-1}Y_{d-1} + \cdots + a_0Y_0$ modulo $b$ is zero. But it is equal to $Y_d + c(P(1) - 1) = Y_d - c + cP(1)$ modulo $b$. Since $b \mid cP(1)$, we obtain that $Y_d$ is $c$ modulo $b$. In the same manner (setting $n = 1$ into (2) and so on) we can see that $Y_n$ is equal to $c$ modulo $b$ for every $n \geq 0$. Therefore, $B = \{c\}$. Note that we were not using the condition $\gcd(a_0, b) = 1$ for this part of the proof.
Proof of the Corollary. For $\xi \notin \mathbb{Q}(\alpha)$, the sequence $\{\xi \alpha^n\}_{n=1,2,...}$ has more than one limit point by the above mentioned result of Pisot and Vijayaraghavan (and by the results of [6] mentioned in Section 1 too). So suppose that $\xi \in \mathbb{Q}(\alpha)$, where $\alpha$ satisfies the conditions of the corollary. If $\{\xi \alpha^n\}_{n=1,2,...}$ has a unique limit point, then Theorem 2 implies that $B = \{c\}$. Clearly, by the first part of Theorem 3, $|P(1)| = 1$ yields $c = 0$. Now, parts (ii) and (iii) of Theorem 2 show that $\alpha$ is either a rational integer or a strong Pisot number, a contradiction. \qed

Proof of Theorem 4. Suppose that $\xi > 0$ and an algebraic number $\alpha > 1$ are such that $\{\xi \alpha^n\}_{n=1,2,...}$ has a unique limit point. Then (again by the theorem of Pisot and Vijayaraghavan) $\alpha$ is a Pisot number. The corollary shows that $\alpha$ must be either an integer, or a strong Pisot number, or a Pisot number whose minimal polynomial $P$ satisfies $P(1) \leq -2$. Since all rational integers, except for $\alpha = 2$, are covered by the case $P(1) \leq -2$, the theorem is proved in one direction.

Now, if $\alpha$ is a strong Pisot number, then, with $\xi = 1$, we have $\lim_{n \to \infty} \{\alpha^n\} = 1$. (See, for instance, Theorem 2 (iii) with $b = 1$ and $f(z) = 1$.) If $\alpha$ is a rational integer, greater than or equal to 2, then, with $\xi = 1$, $\lim_{n \to \infty} \{\alpha^n\} = 0$.

Finally, suppose that $\alpha$ is a Pisot number of degree $d \geq 2$ whose minimal polynomial $P$ satisfies $P(1) \leq -2$. Let us take $\xi = 1/(P'(\alpha)(\alpha - 1)) > 0$. We will show that then $\lim_{n \to \infty} \{\alpha^n\} = 1/P(1)$. Note that, for each $k = 0, 1, \ldots, d - 1$,\begin{equation}
\frac{z^k}{P(z)} = \sum_{j=1}^{d} \frac{\alpha_j^k}{P'(\alpha_j)(z - \alpha_j)}.
\end{equation}

Indeed, for each non-negative integer $k < d$, (5) is the identity, because multiplying both sides of (5) by $P(z)$ we obtain two polynomials, both of degree smaller than $d$, which are equal at $d$ distinct points $z = \alpha_j$, $j = 1, 2, \ldots, d$. Setting $z = 1$ into (5)), we deduce that the trace of $\alpha^k/(P'(\alpha)(\alpha - 1))$ is equal to $-1/P(1) = 1/|P(1)| < 1$ for every $k = 0, 1, \ldots, d - 1$. Of course, we can write $\xi = 1/(P'(\alpha)(\alpha - 1)) = f(\alpha)$ for some polynomial $f$ of the form (1). Then, as in the proof of Theorem 3, we will get that $Y_n$, $n = 0, 1, \ldots, d - 1$, modulo $b$ are all equal to $c$, where $b = c|P(1)|$. Hence, as in the second part of the proof of Theorem 3 we obtain that $Y_n$ modulo $b$ is equal to $c$ for every non-negative integer $n$. Consequently, $B = \{c\}$, where $c/b = 1/|P(1)|$. Now, Theorem 2 (i) implies that\[
\lim_{n \to \infty} \{\alpha^n/(P'(\alpha)(\alpha - 1))\} = 1/|P(1)|
\]
provided that $\alpha$ is a Pisot number whose minimal polynomial $P$ satisfies $P(1) \leq -2$. (This result trivially holds for integer $\alpha \geq 3$ too.) The proof of Theorem 4 is completed. \qed

4. Examples

We remark that the condition $\gcd(b, a_0) = 1$ of Theorem 3 cannot be removed. Take, for example, $\alpha = 3 + \sqrt{5}$. It is a strong Pisot number with other conjugate being $\alpha_2 = 3 - \sqrt{5}$. Its minimal polynomial is $P(z) = z^2 - 6z + 4$. Set $f(z) = (1 + z)/4$. Here, $b = 4$ and $a_0 = 4$. Note that $S_0 = 2$, $S_1 = 6$, $S_2 = 28$, $S_3 = 144$,
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and so on. All \( S_n \), \( n = 2, 3, \ldots \), are divisible by 4. Hence \( Y_n = S_n + S_{n+1} \) modulo 4 is equal to 2 for \( n = 1 \) and to zero for all non-negative \( n \neq 1 \).

Suppose that \( \theta > 1 \) solves \( z^3 - z - 1 = 0 \). Then \( \theta \) is a Pisot number having a pair complex conjugates inside the unit circle. Clearly, \( P(1) = -1 \). The corollary implies that there are no \( \xi > 0 \) (algebraic or transcendental) such that the sequence \( \{\xi^{\theta^n}\}_{n=1,2,\ldots} \) tends to a limit with \( n \) tending to infinity.

Set, for instance, \( f(z) = \frac{2 + z}{3} \). Let us find the set of limit points of \( \{2^{n/3} + 3^{n/3}\}_{n=1,2,\ldots} \). Taking, for example, \( f(z) = \frac{2 + z}{3} \), we deduce that \( Y_n = 2S_n + 3S_{n+1} \) modulo 4 is ultimately periodic, with \( B = \{1\} \).

Finally, if, say, \( \alpha > 1 \) solves \( z^2 - 7z + 2 = 0 \) then \( S_0, S_1, S_2, \ldots \) modulo 4 is 2, 3, 1, 1, 1, \ldots. Taking, for example, \( f(z) = \frac{2 + 3z}{4} \), we deduce that \( Y_n = 2S_n + 3S_{n+1} \) modulo 4 is ultimately periodic, with \( B = \{1\} \). Consequently, \( \lim_{n \to \infty} \{\frac{2}{3} + \frac{2}{3} \alpha^n\} = 1/4 \).

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