THE NATURAL AFFINORS ON SOME FIBER PRODUCT PRESERVING GAUGE BUNDLE FUNCTORS OF VECTOR BUNDLES

JAN KUREK AND WŁODZIMIERZ M. MIKULSKI

Dedicated to Professor Ivan Kolár on the occasion of his 70th birthday with respect and gratitude

ABSTRACT. We classify all natural affinors on vertical fiber product preserving gauge bundle functors $F$ on vector bundles. We explain this result for some more known such $F$. We present some applications. We remark a similar classification of all natural affinors on the gauge bundle functor $F^*$ dual to $F$ as above. We study also a similar problem for some (not all) not vertical fiber product preserving gauge bundle functors on vector bundles.

INTRODUCTION

Let $m, n$ be fixed positive integers.

The category of vector bundles with $m$-dimensional bases and vector bundle maps with embeddings as base maps will be denoted by $\mathcal{VB}_m$.

The category of vector bundles with $m$-dimensional bases and $n$-dimensional fibers and vector bundle embeddings will be denoted by $\mathcal{VB}_{m,n}$.

Let $F : \mathcal{VB}_m \to \mathcal{FM}$ be a covariant functor. Let $B_{\mathcal{FM}} : \mathcal{FM} \to \mathcal{MF}$ and $B_{\mathcal{VB}_m} : \mathcal{VB}_m \to \mathcal{MF}$ be the base functors.

A gauge bundle functor on $\mathcal{VB}_m$ is a functor $F$ as above satisfying:

(i) (Base preservation) $B_{\mathcal{FM}} \circ F = B_{\mathcal{VB}_m}$. Hence the induced projections form a functor transformation $\pi : F \to B_{\mathcal{VB}_m}$.

(ii) (Localization) For every inclusion of an open vector subbundle $i_{E|U} : E|U \to E$, $F(E|U)$ is the restriction $\pi^{-1}(U)$ of $\pi : F(E) \to B_{\mathcal{VB}_m}(E)$ to $U$ and $Fi_{E|U}$ is the inclusion $\pi^{-1}(U) \to F(E)$.

(iii) (Regularity) $F$ transforms smoothly parametrized systems of $\mathcal{VB}_m$-morphisms into smoothly parametrized families of $\mathcal{FM}$-morphisms.

2000 Mathematics Subject Classification: 58A05, 58A20.

Key words and phrases: gauge bundle functors, natural operators, natural transformations, natural affinors, jets.

Received September 29, 2004.
A gauge bundle functor $F : \mathcal{VB}_m \to \mathcal{F}M$ is of finite order $r$ if from $j^r_x f = j^r_x g$ it follows $F_x f = F_x g$ for any $\mathcal{VB}_m$-objects $E_1 \to M$, $E_2 \to M$, any $\mathcal{VB}_m$-maps $f, g : E_1 \to E_2$ and any $x \in M_1$.

A gauge bundle functor $F$ on $\mathcal{VB}_m$ is fiber product preserving if for any fiber product projections

$$E_1 \xrightarrow{\text{pr}_1} E_1 \times_M E_2 \xrightarrow{\text{pr}_2} E_2$$

in the category $\mathcal{VB}_m$,

$$FE_1 \xrightarrow{F\text{pr}_1} F(E_1 \times_M E_2) \xrightarrow{F\text{pr}_2} FE_2$$

are fiber product projections in the category $\mathcal{F}M$. In other words we have $F(E_1 \times_M E_2) = F(E_1) \times_M F(E_2)$.

A gauge bundle functor $F$ on $\mathcal{VB}_m$ is called vertical if for any $\mathcal{VB}_m$-objects $E \to M$ and $E_1 \to M$ with the same basis, any $x \in M$ and any $\mathcal{VB}_m$-map $f : E \to E_1$ covering the identity of $M$ the fiber restriction $F_x f : F_x E \to F_x E_1$ depends only on $f_x : E_x \to (E_1)_x$.

From now on we are interested in vertical fiber product preserving gauge bundle functors on $\mathcal{VB}_m$.

The most known example of vertical fiber product preserving gauge bundle functor $F$ on $\mathcal{VB}_m$ is the so-called vertical $r$-jet prolongation functor $J^r_\nu : \mathcal{VB}_m \to \mathcal{F}M$, where for a $\mathcal{VB}_m$-object $p : E \to M$ we have a vector bundle $J^r_\nu E = \{ j^r_\nu \gamma \mid \gamma \}$ is a local map $M \to E_x$, $x \in M \}$ and for a $\mathcal{VB}_m$-map $f : E_1 \to E_2$ covering $f : M_1 \to M_2$ we have a vector bundle map $J^r_{\nu f} : J^r_\nu E_1 \to J^r_\nu E_2$, where $J^r_{\nu f}(j^r_\nu \gamma) = j^r_{\nu f(x)}(f \circ \gamma \circ \overleftarrow{i}^{-1})$ for $j^r_\nu \gamma \in J^r_\nu E_1$.

Another example is the vertical Weil functor $V^A$ on $\mathcal{VB}_m$ corresponding to a Weil algebra $A$, where for a $\mathcal{VB}_m$-object $p : E \to M$ we have $V^A E = \cup_{x \in M} T^A(E_x)$ and for a $\mathcal{VB}_m$-map $f : E_1 \to E_2$ we have $V^A f = \cup_{x \in M} T^A(f_x) : V^A E_1 \to V^A E_2$. The functor $V^A$ is equivalent to $E \otimes A$.

The fiber product $F_1 \times_{\mathcal{VB}_m} F_2 : \mathcal{VB}_m \to \mathcal{F}M$ of vertical fiber product preserving gauge bundle functors $F_1, F_2 : \mathcal{VB}_m \to \mathcal{F}M$ is again a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$.

In [8], we proved that every fiber product preserving gauge bundle functor $F$ on $\mathcal{VB}_m$ has values in $\mathcal{VB}_m$. (More precisely, the fiber sum map $+ : E \times_M E \to E$, the fiber scalar multiplication $\lambda_t : E \to E$ for $t \in \mathbb{R}$ and the zero map $0 : E \to E$ are $\mathcal{VB}_m$-map and we can apply $F$. We obtain $F(+) : FE \times_M FE \to FE$, $F(\lambda_t) : FE \to FE$ and $F(0) : FE \to FE$. Then $(F(+), F(\lambda_t), F(0))$ is a vector bundle structure on $FE$.) Then we can compose such functors. The composition of vertical fiber product preserving gauge bundle functors on $\mathcal{VB}_m$ is again a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$.

If $F$ is a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$, then $(F^*)^* : \mathcal{VB}_m \to \mathcal{F}M$, $(F^*)^*(E) = (FE^*)^*$, $(F^*)^*(f) = (Ff)^*$ is a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$ ($E^*$ denote the dual vector bundle of $E$).
In [8], we classified all fiber product preserving gauge bundle functors $F$ on $\mathcal{VB}_m$ of finite order $r$ in terms of triples $(V, H, t)$, where $V$ is a finite-dimensional vector space over $\mathbb{R}$, $H : G^r_m \to GL(V)$ is a smooth group homomorphism from $G^r_m = \text{inv}J_0^r(\mathbb{R}^m, \mathbb{R}^m)_0$ into $GL(V)$ and $t : D^r_m \to gl(V)$ is a $G^r_m$-equivariant unity preserving associative algebra homomorphism from $D^r_m = J_0^r(\mathbb{R}^m, \mathbb{R})$ into $gl(V)$. Moreover, we proved that all fiber product preserving gauge bundle functors $F$ on $\mathcal{VB}_m$ are of finite order. Analyzing the construction on $(V, H, t)$ one can easily seen that the triple $(V, H, t)$ corresponding to a vertical $F$ in question has trivial $t : D^r_m \to gl(V)$, $t(j^r_0 \gamma) = \gamma(0) \text{Id}$, $j^r_0 \gamma \in D^r_m$. Then by Fact 5 and Theorem 2 in [8] it follows that all vertical fiber product preserving gauge bundle functors on $\mathcal{VB}_m$ can be constructed (up to $\mathcal{VB}_m$-equivalence) as follows.

Let $V : \mathcal{M}f_m \to \mathcal{VB}$ be a vector natural bundle. For any $\mathcal{VB}_m$-object $p : E \to M$ we put $F^V E = E \otimes_M VM$ and for any $\mathcal{VB}_m$-map $f : E_1 \to E_2$ covering $f : M_1 \to E_2$ we put $F^V f = f \otimes \lambda$, $F^V f : F^V E_1 \to F^V E_2$. The correspondence $F^V : \mathcal{VB}_m \to \mathcal{FM}$ is a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$. (For example, if $V : \mathcal{M}f_m \to \mathcal{VB}$ is the natural vector bundle corresponding to the standard $G^r_m$-space $D^r_m$, then $F^V$ is equivalent with $J^r_0$. If $V : \mathcal{M}f_m \to \mathcal{VB}$ is the trivial vector natural bundle with the standard fiber $A$, then $F^V$ is equivalent to $V^A$.)

Let $F$ be a gauge bundle functor on $\mathcal{VB}_m$. A $\mathcal{VB}_m$-$n$-natural affinor $B$ on $F$ is a system of $\mathcal{VB}_m,n$-invariant affinors $B : TFE \to TFE$ on $FE$ for any $\mathcal{VB}_m,n$-object $E$. The invariance means that $B \circ TFf = TFf \circ B$ for any $\mathcal{VB}_m,n$-map $f$.

In the present paper we describe all $\mathcal{VB}_m,n$-natural affinors $B$ on vertical fiber product preserving gauge bundle functors $F$ on $\mathcal{VB}_m$. We prove that $B : TFE \to TFE$ is of the form

$$B = \lambda \text{Id} + \text{Mod}(A)$$

for a real number $\lambda$ and a fiber bilinear $\mathcal{VB}_m,n$-natural transformation $A : TM \times_M FE \to FE$, where $\text{Mod}(A)$ is the $\mathcal{VB}_m,n$-natural affinor corresponding to $A$ (see Example 2) and $\text{Id}$ is the identity affinor.

In Section 3, we explain this main result for some more known vertical fiber product preserving gauge bundle functors on $\mathcal{VB}_m$. Thus for $J^r_0$ we reobtain the result from [15] saying that the vector space of all $\mathcal{VB}_m,n$-natural affinors on $J^r_0$ is 2-dimensional.

In Section 4, we remark a similar classification of $\mathcal{VB}_m,n$-natural affinors on a gauge bundle functor $F^*$ dual to a vertical fiber product preserving gauge bundle functor $F$ on $\mathcal{VB}_m$.

In Section 5, we remark a similar classification of $\mathcal{VB}_m,n$-natural affinors for some (not all) not vertical fiber product preserving gauge bundle functors $F$ on $\mathcal{VB}_m$ (as the r-jet prolongation gauge bundle functor $J^r$ on $\mathcal{VB}_m$ and the vector r-tangent gauge bundle functor $T^{(r)}B$ on $\mathcal{VB}_m$). Thus a similar result as the main result for not necessarily vertical $F$ is very very probably.

Natural affinors can be used to study torsions of connections, see [5]. That is
why they have been classified in many papers, [1] – [4], [6], [8] – [16], etc.

The trivial vector bundle $\mathbb{R}^m \times \mathbb{R}^n$ over $\mathbb{R}^m$ with standard fiber $\mathbb{R}^n$ will be denoted by $\mathbb{R}^{m,n}$. The coordinates on $\mathbb{R}^m$ will be denoted by $x^1, \ldots, x^m$. The fiber coordinates on $\mathbb{R}^{m,n}$ will be denoted by $y^1, \ldots, y^n$.

All manifolds are assumed to be finite dimensional and smooth. Maps are assumed to be smooth, i.e. of class $C^\infty$.

1. The main result

Let $F$ be a fiber product preserving gauge bundle functor on $\mathcal{VB}_m$. We are going to present examples of $\mathcal{VB}_{m,n}$-natural affinors on $F$.

**Example 1 (The identity affinor).** For any $\mathcal{VB}_{m,n}$-object $E$ we have the identity map $\text{Id} : TFE \to TFE$. The family $\text{Id}$ is a $\mathcal{VB}_{m,n}$-natural affinor on $FE$.

**Example 2.** Suppose we have a family $A$ of fiber bilinear maps $A : T M \times_F E \to E$ covering the identity of $M$ for any $\mathcal{VB}_{m,n}$-object $E \to M$ such that $F f \circ A = A \circ (T f \times_f F f)$ for any $\mathcal{VB}_{m,n}$-map $f : E_1 \to E_2$ covering $f : M_1 \to M_2$, i.e. we have a fiber bilinear $\mathcal{VB}_{m,n}$-natural transformation $A : T M \times_M FE \to FE$, where $TM$ is the tangent bundle of $M$ and $FE$ is the vector bundle as is explained in Introduction. For any $\mathcal{VB}_{m,n}$-object $p : E \to M$ we define $\text{Mod}(A) : TFE \to TFE$ by

$$\text{Mod}(A)(v) = \frac{d}{dt} (y + tA(T\pi(v), y)) \in T_y FY, \quad v \in T_y FE, \quad y \in FE,$$

where $\pi : FE \to M$ is the bundle projection. Then $\text{Mod}(A)$ is a $\mathcal{VB}_{m,n}$-natural affinor on $F$. We call $\text{Mod}(A)$ the $\mathcal{VB}_{m,n}$-natural affinor on $F$ corresponding to $A$ (the modification of $A$).

For example, in the case of $F = J^r_v$ we have a fiber bilinear $\mathcal{VB}_{m,n}$-natural transformation $A^r_v : TM \times_J^r_v E \to J^r_v E, \ A^r_v(w, j^r_v x \sigma) = j^r_v(w \sigma), \ w \in T_x M, \ x \in M, \ \sigma : M \to E_x, \ w \sigma \in E_x$ is the differential of $\sigma$ with respect to $w$ and $w \sigma : M \to E_x$ is the constant map.

The main result of the present paper is the following classification theorem.

**Theorem 1.** Let $F$ be a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$. Any $\mathcal{VB}_{m,n}$-natural affinor $B$ on $F$ is the form

$$B = \lambda \text{Id} + \text{Mod}(A)$$

for some real number $\lambda$ and some fiber bilinear $\mathcal{VB}_{m,n}$-natural transformation $A : TM \times_M FE \to FE$.

Thus for $F = J^r_v$ we reobtain the result from [15] saying that any $\mathcal{VB}_{m,n}$-natural affinor on $J^r_v$ is a linear combination with real coefficients of the identity affinor and $\text{Mod}(A^r_v)$ (see Corollary 5 below).

We end this section by the following observation.
Let $F$ be of the form $F^V$ for some natural vector bundle $V : \mathcal{MF}_m \to VB$ (see Introduction). Let $C : TM \times_M VM \to VM$ be an $\mathcal{MF}_m$-natural fiber bilinear transformation. Then we have a $VB_{m,n}$-natural fiber bilinear transformation $A^C : TM \times_M F^V E \to F^V E$,

\[
A^C(v, e \otimes y) = e \otimes C(v, y),
\]

\(y \in V_x M, \ e \in E_x, \ v \in T_x M, \ x \in M.\)

**Proposition 1.** Let $V : \mathcal{MF}_m \to VB$ be a natural vector bundle. Any $VB_{m,n}$-natural fiber bilinear transformation $A : TM \times_M F^V E \to F^V E$ is of the form $A^C$ for some $\mathcal{MF}_m$-natural fiber bilinear transformation $C : TM \times_M VM \to VM$.

**Proof of Proposition 1.** By the $VB_{m,n}$-invariance, $A$ is determined by the $\mathcal{MF}_m$-natural fiber bilinear transformation

\[
TM \times_M VM \ni (v, y) \to \langle A(v, e_1(\pi^T(v)) \otimes y), e_1^*(\pi^T(v)) \rangle \in VM,
\]

where $e_1, \ldots, e_n$ is the usual basis of sections of the trivial vector bundle $M \times \mathbb{R}^n$ and $e_1^*, \ldots, e_n^*$ is the dual basis, and $\pi^T : TM \to M$ is the tangent bundle projection.

### 2. Proof of Theorem 1

We fix a basis in the vector space $F_0 \mathbb{R}^{m,n}$.

**Step 1.** Consider

\[
T \pi \circ B : (T \mathbb{R}^{m,n})_0 \cong \mathbb{R}^m \times F_0 \mathbb{R}^{m,n} \times F_0 \mathbb{R}^{m,n} \to T_0 \mathbb{R}^m,
\]

where $\pi : FE \to M$ is the bundle projection. Using the invariance of $B$ with respect to the fiber homotheties we deduce that $T \pi \circ B(a, u, v) = T \pi \circ B(a, tu, tv)$ for any $u, v \in F_0 \mathbb{R}^{m,n}, \ a \in \mathbb{R}^m, \ t \neq 0$. Then $T \pi \circ B(a, u, v) = T \pi \circ B(a, 0, 0)$ for $u, v, a$ as above. Then using the invariance of $B$ with respect to $C \times \text{id} \mathbb{R}^n$ for linear isomorphisms $C$ of $\mathbb{R}^n$ we deduce that $T \pi \circ B(a, 0, 0) = \lambda a$ for some real number $\lambda$. Then replacing $B$ by $B - \lambda \text{id}$ we have $T \pi \circ B(a, u, v) = 0$ for any $a, u, v$ as above. Then $B$ is of vertical type.

**Step 2.** Consider

\[
pr_2 \circ B : (T \mathbb{R}^{m,n})_0 \cong \mathbb{R}^m \times F_0 \mathbb{R}^{m,n} \times F_0 \mathbb{R}^{m,n} \to F_0 \mathbb{R}^{m,n},
\]

where $(V \mathbb{R}^{m,n})_0 \cong F_0 \mathbb{R}^{m,n} \times F_0 \mathbb{R}^{m,n} \to F_0 \mathbb{R}^{m,n}$ is the projection onto the second (essential) factor. Using the invariance of $B$ with respect to the fiber homotheties we deduce that $pr_2 \circ B(a, tu, tv) = t pr_2 \circ B(a, u, v)$ for $a, u, v$ as in Step 1. Then $pr_2 \circ B(a, u, v)$ is a system of linear combinations of the coefficients of $u$ and $v$ with coefficients being smooth maps in $a$ because of the homogeneous function theorem. On the other hand, since $B$ is an affinor, $pr_2 \circ B(a, u, v)$ is a
system of linear combinations of the coefficients of \( a \) and \( v \) with coefficients being smooth functions in \( u \). Then

\[(*) \quad pr_2 \circ B(a, u, v) = G(a, u) + H(v)\]

for some bilinear map \( G \) and some linear map \( H \).

Let \( \Phi : \mathbb{R}^{m,n} \to \mathbb{R}^{p,r} \) be a \( \mathcal{VB}_{m,n} \)-map such that \( \Phi(x, v) = (x, e^x v) \), \((x, y) \in \mathbb{R}^{m,n} \). Then \( \Phi \) sends \( \frac{\partial}{\partial x} \) into \( \frac{\partial}{\partial y} + L \), where \( L \) is the Liouville vector field on \( \mathbb{R}^{m,n} \). Then using the invariance of \( \mathcal{B} \) with respect to \( \Phi \) we obtain

\[F \Phi(G(e_1, F \Phi^{-1}(v))) = G(e_1, v) + H(v),\]

where \( e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^m \). Since \( F \) is vertical, \( F \Phi = \text{id} \). Hence \( H(v) = 0 \), and

\[pr_2 \circ B(a, u, v) = G(a, u).\]

Then by the \( \mathcal{VB}_{m,n} \)-invariance of \( \mathcal{B} \) we obtain the equivariant condition

\[F_0 f (G(a, u)) = G(T_0 f(a), F_0 f(u))\]

for any \( a, u \) as above and any \( \mathcal{VB}_{m,n} \)-map \( f : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n} \) preserving \( 0 \in \mathbb{R}^m \). Hence there is a \( \mathcal{VB}_{m,n} \)-natural fiber bilinear transformation \( A : TM \times_M FE \to FE \) corresponding to \( G \). It is easy to see that \( B = \text{Mod}(A) \). \( \square \)

3. Applications

Let \( T^{(p,q)} = \otimes^p T^* \otimes \otimes^q T : M_f \to \mathcal{V} \mathcal{M} \) be the natural vector bundle of tensor fields of type \((p,q)\) over \( m \)-manifolds. Let \( F^{(p,r)} = F^{T^{(p,r)}} : \mathcal{VB}_m \to \mathcal{F} \mathcal{M} \), \( F^{(p,r)} E = E \otimes M T^{(p,r)} M \), \( F^{(p,q)} f = f \otimes f T^{(p,q)} f \) be the corresponding vertical fiber product preserving gauge bundle functor (see Introduction).

Suppose that \( C : TM \times_M T^{(p,r)} M \to T^{(p,q)} M \) is a fiber bilinear \( M_f \)-natural transformation. Using the invariance of \( C \) with respect to base homotheties on \( \mathbb{R}^{m,n} \) one can easily deduce that \( C = 0 \). Thus we have the following corollary

**Corollary 1.** Any \( \mathcal{VB}_{m,n} \)-natural affinor on \( F^{(p,q)} \) as above is a constant multiple of the identity affinor.

Similarly, any \( M_f \)-natural fiber bilinear transformation \( C : TM \times_M M \to M \), where \( M \) is treated as the zero vector bundle over \( M \), is zero. Thus we have

**Corollary 2.** Any \( \mathcal{VB}_{m,n} \)-natural affinor on the vertical Weil bundle \( V^A \) is a constant multiple of the identity affinor.

Let \( T^{(r)} = (J^r(\cdot, \mathbb{R})_0)^* : M_f \to \mathcal{V} \mathcal{B} \) be the linear \( r \)-tangent bundle functor. Let \( F^{(r)} = F^{T^{(r)}} : \mathcal{VB}_m \to \mathcal{F} \mathcal{M} \) be the corresponding vertical fiber product preserving gauge bundle functor.

Suppose that \( C : TM \times_M T^{(r)} M \to T^{(r)} M \) is a \( M_f \)-natural fiber bilinear transformation. By the rank theorem, \( C \) is determined by the contraction \((C, j^{(r)}_0 x^1) : T^{(r)}_0 \mathbb{R}^m \to \mathbb{R} \). Then using the invariance of \( C \) with respect to the base homotheties one can easily show that this contraction is zero. Then \( C = 0 \). Thus we have
Corollary 3. Any $\mathcal{VB}_{m,n}$-natural affinor on $F(r)$ as above is a constant multiple of the identity one.

Let $T^* = J^*(\cdot, R)_0 : \mathcal{M}_m \to \mathcal{V}B$ be the $r$-cotangent bundle functor. Let $F^* = F^{T^*} : \mathcal{VB}_m \to \mathcal{F}M$ be the corresponding vertical fiber product preserving gauge bundle functor.

Suppose that $C : TM \times_M T^*M \to T^*M$ is a $\mathcal{M}_m$-natural fiber bilinear transformation. By the rank theorem, $C$ is determined by the evaluations $C(v, j^*_0 x^1) \in T_0^* R^m$, where $v \in T_0 R^m$. Then using the invariance of $C$ with respect to the base homotheties one can easily show that these evaluations are zero. Then $C = 0$. Thus we have

Corollary 4. Any $\mathcal{VB}_{m,n}$-natural affinor on $F^*$ as above is a constant multiple of the identity one.

Let $E^* = J^*(\cdot, R) : \mathcal{M}_m \to \mathcal{V}B$ be the extended $r$-cotangent bundle functor. As we know the corresponding vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$ is equivalent to the vertical $r$-jet functor $J^r$ (see Introduction).

Suppose that $C : TM \times_M E^*M \to E^*M$ is a $\mathcal{M}_m$-natural fiber bilinear transformation. By the rank theorem, $C$ is determined by the evaluations $C(\frac{\partial}{\partial x^0}, j^0_0) \in E^r_0 R^m$ and $C(\frac{\partial}{\partial x^1}, j^0_0 x^1) \in E^r_0 R^m$. Then using the invariance of $C$ with respect to the base homotheties one can easily show that the second evaluation is a constant multiple of $j^0_0$ and the first one is zero. Then the vector space of all $C$ in question is of dimension less or equal to 1. Thus we reobtain

Corollary 5 ([15]). Any $\mathcal{VB}_{m,n}$-natural affinor on $J^r$ is a linear combination with real coefficients of the identity affinor and the affinor $\text{Mod}(A^r)$.

Corollary 6. Let $F$ be a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$. Any $\mathcal{VB}_{m,n}$-natural 1-form $\omega$ on $F$ is zero.

Proof. Let $L$ be the Liouville vector field on the vector bundle $FE$. Then $\omega \otimes L$ is a $\mathcal{VB}_{m,n}$-natural affinor. Since it is not isomorphic, it is of the form $\omega \otimes L = \text{Mod}(A)$ for some bilinear $\mathcal{VB}_{m,n}$-natural transformation $A : TM \times_M FE \to FE$. Then $A$ is of the form $A(v, y) = \lambda(v)y$ for some uniquely (and then $\mathcal{M}_m$-natural) 1-form $\lambda : TM \to R$ on $M$. But any such 1-form is zero. Then $A = 0$. Then $\omega = 0$.

Corollary 7. Let $F$ be a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$. There is no $\mathcal{VB}_{m,n}$-natural symplectic structure $\omega$ on $F$.

Proof. Suppose that such $\omega$ exists. Then $\omega(L, \cdot)$ is a $\mathcal{VB}_{m,n}$-natural 1-form on $F$. Then $\omega(L, \cdot) = 0$ because of Corollary 6. Then $\omega$ is degenerate. Contradiction.

Quite similarly one can prove

Corollary 8. Let $F$ be a vertical fiber product preserving gauge bundle functor on $\mathcal{VB}_m$. Then there is no $\mathcal{VB}_{m,n}$-natural non-degenerate Riemannian tensor field $g$ on $F$. 

4. A DUAL VERSION OF THE MAIN RESULT

Let \( F \) be a vertical fiber product preserving gauge bundle functor on \( \mathcal{VB}_m \). Let \( F^* \) be the dual gauge bundle functor on \( \mathcal{VB}_{m,n} \), \( F^* E = (FE)^* \) and \( F^* f = (Ff^{-1})^* \). Replacing in the proof of Theorem 1 \( F \) by \( F^* \) we obtain

**Theorem 1’.** Let \( F \) be a vertical fiber product preserving gauge bundle functor on \( \mathcal{VB}_m \). Let \( F^* \) be the dual gauge bundle functor. Any \( \mathcal{VB}_{m,n} \)-natural affinor \( B \) on \( F^* \) is of the form

\[
B = \lambda \text{Id} + \text{Mod}(A^*)
\]

for some \( \lambda \in \mathbb{R} \) and some \( \mathcal{VB}_{m,n} \)-natural fiber bilinear transformation \( A : TM \times_M FM \to FM \), where \( A^* : TM \times_M F^* E \to F^* E \) is the \( \mathcal{VB}_{m,n} \)-natural fiber bilinear transformation given by \( A^*(v, \cdot) = (A(v, \cdot))^* \) for any \( v \in TM \).

5. THE NOT NECESSARILY VERTICAL CASE

In our opinion, it is very probably that Theorem 1 holds for (not necessarily vertical) fiber product preserving gauge bundle functors on \( \mathcal{VB}_m \). For example, in [15] we proved.

**Fact 1** ([15]). Any \( \mathcal{VB}_{m,n} \)-natural affinor on the \( r \)-jet prolongation functor \( J^r \), which is a not vertical fiber product preserving gauge bundle functor on \( \mathcal{VB}_m \), is a constant multiple of the identity affinor.

The crucial property of \( J^r \) which we used to prove Fact 1 is that any \( \mathcal{VB}_{m,n} \)-natural linear operator lifting linear vector fields from \( E \) to vector fields on \( J^r E \) is a constant multiple of the flow operator.

Replacing in [15] \( J^r \) be an arbitrary fiber product preserving gauge bundle functor \( F \) on \( \mathcal{VB}_m \) we can obtain

**Proposition 2.** Let \( F \) be a (not necessarily vertical) fiber product preserving gauge bundle functor on \( \mathcal{VB}_m \) such that any \( \mathcal{VB}_{m,n} \)-natural linear operator lifting linear vector fields from \( E \) into vector fields on \( FE \) is a constant multiple of the flow operator \( F \). Then any \( \mathcal{VB}_{m,n} \)-natural affinor \( B \) on \( F \) is a constant multiple of the identity affinor.

**Proof.** Clearly, \( B \circ F \) is a \( \mathcal{VB}_{m,n} \)-natural linear operator lifting linear vector fields to \( F \). By the assumption, there is \( \lambda \in \mathbb{R} \) such that \( B \circ F = \lambda F \). Next we use the same proof as the one of Theorem 1 up to the formula (*) Obviously, after Step 1, \( B \) satisfies \( B(\mathcal{F}X) = 0 \) for any linear vector field on \( \mathbb{R}^{m,n} \). Putting in (*) \( X = v \frac{\partial}{\partial \gamma} \) (i.e. \( (a, u, v) = (a, u, 0) \)) we get \( G(a, u) = 0 \). Putting \( X = L, \) the Liouville vector field on \( \mathbb{R}^{m,n} \) (i.e. \( (a, u, v) = (0, v, v) \)) we get \( H(v) = 0 \).

In [7], we proved that the assumption of Proposition 1 is satisfied for the vector \( r \)-tangent gauge bundle functor \( T^{(r)\|} \) on \( \mathcal{VB}_m \) defined as follows. Given a \( \mathcal{VB}_m \)-object \( p : E \to M, T^{(r)\|} E = (J^r_0(E, \mathbb{R})_0)^* \) is the vector bundle over \( M \) dual to \( J^r_0 E = \{ j^r_x \gamma : \gamma : E \to \mathbb{R} \text{ is fiber linear, } \gamma_x = 0, x \in M \} \). For every \( \mathcal{VB}_m \)-map \( f : E_1 \to E_2 \) covering \( f : M_1 \to M_2, T^{(r)\|} f : T^{(r)\|} E_1 \to T^{(r)\|} E_2 \)
is a vector bundle map covering $f$ such that $\langle T^{(r)}f(\omega), j^r_{\xi}(\xi \circ f) \rangle = \langle \omega, j^r_{\xi}(\xi \circ f) \rangle$, $\omega \in T^{(r)}_x E_1$, $j^r_{\xi}(\xi \circ f) \in J^r_0(E_2, R)_0$, $x \in M$. (The correspondence $T^{(r)}_x$ is a not vertical fiber product preserving gauge bundle functor on $\text{VB}_m$.) Thus we have

**Fact 2.** Any $\text{VB}_{m,n}$-natural affinor on $T^{(r)}_x$ is a constant multiple of the identity affinor.

**References**


**Institute of Mathematics, Maria Curie-Skłodowska University**

**Lublin, PL**

**Maria Curie-Skłodowska University**

E-mail kurek@goem.umcs.lublin.pl

**Institute of Mathematics, Jagiellonian University**

**Kraków, Réymonta 4**

**Poland**

E-mail mikulski@im.uj.edu.pl