ON AN EFFECTIVE CRITERION OF SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATION OF n-TH ORDER

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Abstract. New sufficient conditions for the existence of a solution of the boundary value problem for an ordinary differential equation of \( n \)-th order with certain functional boundary conditions are constructed by a method of a priori estimates.

Introduction

In this paper we give new sufficient conditions for the existence of a solution of the ordinary differential equation

\( u^{(n)}(t) = f(t, u(t), \ldots, u^{(n-1)}(t)) \)

with the boundary conditions

\[
\Phi_{0i}\left(u^{(i-1)}\right) = \varphi_i(u) \quad i = 1, \ldots, n,
\]

resp.

\[
l_i\left(u, u', \ldots, u^{(k_0-1)}\right) = 0 \quad i = 1, \ldots, k_0,
\]

\[
\Phi_{0i}\left(u^{(i-1)}\right) = \varphi_i\left(u^{(k_0)}\right) \quad i = k_0 + 1, \ldots, n,
\]

where \( f : [a, b] \times \mathbb{R}^n \to \mathbb{R} \) satisfies the local Carathéodory conditions, \( n \geq 2 \), and \( 1 \leq k_0 \leq n - 2 \).

For each index \( i \), the functional \( \Phi_{0i} \) in the conditions (2), resp. (3_2), is supposed to be linear, nondecreasing, nontrivial, continuous on \( C([a, b]) \), and concentrated on \([a_i, b_i] \subseteq [a, b] \) (i.e., the value of functional \( \Phi_{0i} \) depends only on a function restricted to \([a_i, b_i]\) and this segment can be degenerated to a point). In general \( \Phi_{0i}(1) \in \mathbb{R} \), without loss of generality we can suppose that \( \Phi_{0i}(1) = 1 \), which simplifies the notation.

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In the condition (3), the functionals \( l_i : [C([a,b])]^{k_0} \to R \) \((i = 1, \ldots, k_0)\) are linear and continuous.

For each index \( i \) \((i = 1, \ldots, n)\), the functional \( \varphi_i : C^{n-1}(a,b) \to R \) in the conditions (2) is continuous and satisfies

\[
\xi_i(\rho) = \frac{1}{\rho} \sup \left\{ |\varphi_i(\rho v)| : \|v\|_{C^{n-1}(a,b)} \leq 1 \right\} \to 0 \quad \text{as} \quad \rho \to +\infty.
\]

For each index \( i \) \((i = k_0 + 1, \ldots, n)\), the functional \( \varphi_i : C^{n-1-k_0}(a,b) \to R \) in the conditions (3) is continuous and satisfies

\[
\delta_i(\rho) = \frac{1}{\rho} \sup \left\{ |\varphi_i(\rho v)| : \|v\|_{C^{n-1-k_0}(a,b)} \leq 1 \right\} \to 0 \quad \text{as} \quad \rho \to +\infty.
\]

The special cases of boundary conditions (2) are

\[
u^{(i-1)}(t_i) = \varphi_i(u) \quad \text{for} \quad i = 1, \ldots, n,
\]

where \( a \leq a_i \leq t_i \leq b \) \((i = 1, \ldots, n)\) or

\[
\int_{a_i}^{b_i} u^{(i-1)}(t) \, d\sigma_i(t) = \varphi_i(u) \quad \text{for} \quad i = 1, \ldots, n.
\]

The integral is understood in the Lebesgue–Stieltjes sense, where \( \sigma_i \) is nondecreasing in \([a_i, b_i]\) and \( \sigma(b_i) - \sigma(a_i) > 0 \) \((i = 1, \ldots, n)\). We know that the problem (1), (5) was studied by B. Půža in the paper [4], so in this paper we will receive more general results than in [4].

Problem (1), (3) was studied by Nguyen Anh Tuan in the paper [5] and by Gegelia G. T. in the paper [1]. In this paper, however, we will give new sufficient conditions for the existence of a solution of the problem (1), (3).

**Main results**

We adopt the following notation:

- \([a, b]\) - a segment, \(-\infty < a \leq a_i \leq b_i < +\infty \) \((i = 1, \ldots, n)\).
- \(R^n\) - \(n\)-dimensional real space with elements \( x = (x_i)_{i=1}^n \) normed by \( \|x\| = \sum_{i=1}^n |x_i| \).
- \(R^n_+ = \{x \in R^n : x_i \geq 0, i = 1, \ldots, n\}\), \((0, +\infty) = R_+ - \{0\}\).
- \(C^{n-1}(a,b)\) - the space of functions continuous together with their derivatives up to the order \((n-1)\) on \([a,b]\) with the norm

\[
\|u\|_{C^{n-1}(a,b)} = \max \left\{ \sum_{i=1}^n |u^{(i-1)}(t)| : a \leq t \leq b \right\}.
\]

- \(AC^{n-1}(a,b)\) - the set of all functions absolutely continuous together with their derivatives up to the order \((n-1)\) on \([a,b]\).
$L^p([a, b])$ - the space of functions Lebesgue integrable on $[a, b]$ in the $p$-th power with the norm

$$
\|u\|_{L^p([a, b])} = \begin{cases} 
\left( \frac{1}{a} \int_a^b |u(t)|^p \, dt \right)^{\frac{1}{p}} & \text{if } 1 \leq p < +\infty, \\
\text{ess sup} \{ |u(t)| : a \leq t \leq b \} & \text{if } p = +\infty.
\end{cases}
$$

$L^p([a, b], R_+) = \{ u \in L^p([a, b]) : u(t) \geq 0 \text{ for a. a. } a \leq t \leq b \}$.

Let $x = (x_i(t))^n_{i=1}$, $y = (y_i(t))^n_{i=1} \in [C([a, b])]^n$. We will say that $x \leq y$ if $x_i(t) \leq y_i(t)$ for all $t \in [a, b]$ and $i = 1, \ldots, n$.

A functional $\Phi : [C([a, b])]^n \to R$ is said to be nondecreasing if $\Phi(x) \leq \Phi(y)$ for all $x, y \in [C([a, b])]^n$, $x \leq y$, and positively homogeneous if $\Phi(\lambda x) = \lambda \Phi(x)$ for all $\lambda \in (0, +\infty)$ and $x \in [C([a, b])]^n$.

Let us consider the problems (1), (2) and (1), (3). Under a solution of the problem (1), (2), resp. (1), (3), we understand a function $u \in AC^{n-1}([a, b])$ which satisfies the equation (1) almost everywhere on $[a, b]$ and fulfils the boundary conditions (2), resp. (3).

**Theorem 1.** Let the inequalities

\begin{equation}
(6)_1
f(t, x_1, x_2, \ldots, x_n) \text{ sign } x_n \leq \omega(|x_n|) \sum_{i=1}^{n-1} \sum_{j=1}^{m} g_{ij}(t) h_{ij}(x_i) |x_{i+1}|^{\frac{1}{q_{ij}}},
\end{equation}

for $t \in [a_n, b_n], (x_i)^n_{i=1} \in R^n$

\begin{equation}
(6)_2
f(t, x_1, x_2, \ldots, x_n) \text{ sign } x_n \geq -\omega(|x_n|) \sum_{i=1}^{n-1} \sum_{j=1}^{m} g_{ij}(t) h_{ij}(x_i) |x_{i+1}|^{\frac{1}{q_{ij}}},
\end{equation}

\[\text{for } t \in [a, b_n], (x_i)^n_{i=1} \in R^n\]

hold, where $g_{ij} \in L^{p_{ij}}([a, b], R_+)$, $p_{ij}, q_{ij} \geq 1$, $1/p_{ij} + 1/q_{ij} = 1 (i = 1, \ldots, n-1; j = 1, \ldots, m)$, $\omega : R_+ \to (0, +\infty)$ and $h_{ij} : R \to R_+ (i = 1, \ldots, n-1; j = 1, \ldots, m)$ are continuous nondecreasing functions satisfying

\begin{equation}
(7)
\Omega(\rho) = \int_0^\rho \frac{ds}{\omega(s)} \to +\infty \text{ as } \rho \to +\infty
\end{equation}

and

\begin{equation}
(8)
\lim_{\rho \to +\infty} \frac{\Omega(\rho \xi_n(\rho))}{\Omega(\rho)} = 0 = \lim_{\rho \to +\infty} \frac{\|h_{ij}\|_{L_{\rho \xi_n(\rho)}^{q_{ij}}}}{\Omega(\rho)}
\end{equation}

$i = 1, \ldots, n-1; j = 1, \ldots, m$.

Then the problem (1), (2) has at least one solution.

To prove Theorem 1 we need the following

**Lemma 1.** Let the functions $\omega$, $\Omega$, $g_{ij}$, $h_{ij}$ and the numbers $p_{ij}$, $q_{ij} (i = 1, \ldots, n-1; j = 1, \ldots, m)$ be given as in Theorem 1, and let $\eta_i : R_+ \to R_+ (i = 1, \ldots, n)$ be
nondecreasing functions satisfying

\[
\lim_{\rho \to +\infty} \frac{\Omega(\eta_i(\rho))}{\Omega(\rho)} = 0 = \lim_{\rho \to +\infty} \frac{\eta_i(\rho)}{\rho} \quad i = 1, \ldots, n.
\]

Then there exists a constant \( \rho_0 > 0 \) such that the estimate

\[
\|u\|_{C^{n-1}([a,b])} \leq \rho_0
\]

holds for each solution \( u \in AC^{n-1}([a,b]) \) of the differential inequalities

\[
(11)_1 \quad u^{(n)}(t) \text{sign} u^{(n-1)}(t) \\
\leq \omega(|u^{(n-1)}(t)|) \sum_{i=1}^{n-1} \sum_{j=1}^{m} q_{ij}(t) h_{ij}(u^{(i-1)}(t)) |u^{(i)}(t)|^\frac{1}{n_i} \\
\quad \text{for } t \in [a, b]
\]

\[
(11)_2 \quad u^{(n)}(t) \text{sign} u^{(n-1)}(t) \\
\geq -\omega(|u^{(n-1)}(t)|) \sum_{i=1}^{n-1} \sum_{j=1}^{m} q_{ij}(t) h_{ij}(u^{(i-1)}(t)) |u^{(i)}(t)|^\frac{1}{n_i} \\
\quad \text{for } t \in [a, b_n]
\]

with the boundary condition

\[
(12) \quad \min \{|u^{(i-1)}(t)| : a_i \leq t \leq b_i\} \leq \eta_i \left( \|u\|_{C^{n-1}([a,b])} \right) \quad i = 1, \ldots, n.
\]

**Proof.** Put

\[
\mu = \sum_{i=1}^{n} (b - a)^{n-1} \quad \text{and} \quad \epsilon = \left[ 2\mu(n-1) \right]^{-1}.
\]

Then according to (9) there exists a number \( r_0 > 0 \) such that

\[
(13) \quad \eta_i(\rho) \leq \epsilon \rho \quad \text{for} \quad \rho > r_0 \quad i = 1, \ldots, n.
\]

We suppose that the estimate (10) does not hold. Then for arbitrary \( \rho_1 \geq r_0 \) there exists a solution \( u \) of the problem (11), (12) such that

\[
(14) \quad \|u\|_{C^{n-1}([a,b])} > \rho_1.
\]

We put

\[
(15) \quad \rho = \max \{|u^{(n-1)}(t)| : a \leq t \leq b\}
\]

and choose \( \tau_i \in [a_i, b_i] \) (\( i = 1, \ldots, n \)) such that

\[
|u^{(i-1)}(\tau_i)| = \min \{|u^{(i-1)}(t)| : a_i \leq t \leq b_i\}.
\]

Then from (12) we have

\[
(16) \quad |u^{(i-1)}(\tau_i)| \leq \eta_i \left( \|u\|_{C^{n-1}([a,b])} \right) \quad i = 1, \ldots, n.
\]
Using (15), (16) we have

$$|u^{(n-2)}(t)| \leq \left| \int_{\tau_{n-1}}^{t} |u^{(n-1)}(\tau)| \, d\tau \right| + |u^{(n-2)}(\tau_{n-1})|$$

(17)

$$\leq (b-a)\rho + \eta_{n-1} \left( \|u\|_{C_{[a,b]}^{n-1}} \right) \quad \text{for} \quad t \in [a,b].$$

Integrating $u^{(n-2)}$ from $\tau_{n-2}$ to $t$ and using (16) and (17) again we get

$$|u^{(n-3)}(t)| \leq \left| \int_{\tau_{n-2}}^{t} |u^{(n-2)}(\tau)| \, d\tau \right| + |u^{(n-3)}(\tau_{n-2})|$$

$$\leq (b-a)^2 \rho + (b-a)\eta_{n-1} \left( \|u\|_{C_{[a,b]}^{n-1}} \right) + \eta_{n-2} \left( \|u\|_{C_{[a,b]}^{n-1}} \right)$$

for $t \in [a,b]$. Applying this procedure $(n-1)$-times we obtain

$$\|u\|_{C_{[a,b]}^{n-1}} \leq \mu \left( \rho + \sum_{i=1}^{n-1} \eta_i \left( \|u\|_{C_{[a,b]}^{n-1}} \right) \right).$$

Using (13) and (14) we get

$$\|u\|_{C_{[a,b]}^{n-1}} \leq \mu \left( \rho + (n-1)\epsilon \|u\|_{C_{[a,b]}^{n-1}} \right) = \mu \rho + \frac{1}{2} \|u\|_{C_{[a,b]}^{n-1}}.$$

Therefore we have

(18)

$$\|u\|_{C_{[a,b]}^{n-1}} \leq 2\mu \rho.$$

We choose a point $\tau^* \in [a,b]$ such that $\tau^* \neq \tau_n$ and

$$|u^{(n-1)}(\tau^*)| = \max \left\{ |u^{(n-1)}(t)| : a \leq t \leq b \right\}.$$

Then either $\tau_n < \tau^*$ or $\tau^* < \tau_n$.

If $\tau_n < \tau^*$, then the integration of (11.1) from $\tau_n$ to $\tau^*$, in view of (18) and using Hölder’s inequality, we get

$$\int_{\tau_n}^{\tau^*} \frac{u^{(n)}(t) \text{ sign } u^{(n-1)}(t)}{\omega(|u^{(n-1)}(t)|)} \, dt \leq \int_{\tau_n}^{\tau^*} \sum_{i=1}^{n-1} \sum_{j=1}^{m} g_{ij} (t) h_{ij} \left( u^{(i-1)}(t) \right) |u^{(i)}(t)| \frac{v_{ij}}{\omega} \, dt$$

(19)

$$\leq \sum_{i=1}^{n-1} \sum_{j=1}^{m} \|g_{ij}\|_{L_{[a,b]}^{n}} \|h_{ij}\|_{L_{[-2\mu \rho,2\mu \rho]}^{n}}.$$

Applying (15), (16), (18), and the definition of $\Omega$ in (19), we get

$$\Omega(\rho) \leq \Omega(\eta_{n}(2\mu \rho)) + \sum_{i=1}^{n-1} \sum_{j=1}^{m} \|g_{ij}\|_{L_{[a,b]}^{n}} \|h_{ij}\|_{L_{[-2\mu \rho,2\mu \rho]}^{n}}.$$

Now, in view of (8), (9), (14), and (18), since $\rho_1$ was chosen arbitrarily, we get

$$\lim_{\rho \to +\infty} \frac{\Omega(\rho)}{\Omega(2\mu \rho)} = 0.$$
On the other hand, in view of (7) and the facts that \(2\mu > 1\) and \(\omega\) is a nondecreasing function, we have
\[
\liminf_{\rho \to +\infty} \frac{\Omega(\rho)}{\Omega(2\mu \rho)} > 0,
\]
a contradiction.

If \(\tau^* = \tau_n\), then the integration of (112) from \(\tau^*\) to \(\tau_n\) yields the same contradiction in analogous way. \(\square\)

**Proof of Theorem 1.** Let \(\rho_0\) be the constant from Lemma 1. Put
\[
\chi(s) = \begin{cases} 
1 & \text{if } |s| \leq \rho_0 \\
2 - \frac{|s|}{\rho_0} & \text{if } \rho_0 < |s| < 2\rho_0, \\
0 & \text{if } |s| \geq 2\rho_0
\end{cases}
\]
and consider the problem
\[
\begin{align*}
\tilde{f}(t, x_1, \ldots, x_n) &= \chi(\|x\|) f(t, x_1, \ldots, x_n) \quad \text{for } a \leq t \leq b, \quad (x_i)_{i=1}^n \in \mathbb{R}^n, \\
\tilde{\varphi}_i(u) &= \chi(\|u\|_{C_{\|u\|_C}^{n-1}}) \varphi_i(u) \quad \text{for } u \in C^{n-1}([a, b]), \quad i = 1, \ldots, n
\end{align*}
\]
From (20) it immediately follows that \(\tilde{f} : [a, b] \times \mathbb{R}^n \to \mathbb{R}\) satisfies the local Carathéodory conditions, \(\tilde{\varphi}_i : C^{n-1}([a, b]) \to \mathbb{R}\) \((i = 1, \ldots, n)\) are continuous functionals and
\[
\begin{align*}
&\sup \{|\tilde{f}(\cdot, x_1, \ldots, x_n)| : (x_i)_{i=1}^n \in \mathbb{R}^n\} \in L([a, b]), \\
&\sup \{|\tilde{\varphi}_i(u)| : u \in C^{n-1}([a, b])\} < +\infty \quad i = 1, \ldots, n.
\end{align*}
\]
Now we will show that the homogeneous problem
\[
\begin{align*}
(21_0) & \quad v^{(n)}(t) = 0, \\
(22_0) & \quad \Phi_{0i}(v^{(i-1)}) = 0 \quad i = 1, \ldots, n
\end{align*}
\]
has only the trivial solution.

Let \(v\) be an arbitrary solution of this problem. Integrating (210) we get
\[
v^{(n-1)}(t) = \text{const} \quad \text{for } \ a \leq t \leq b.
\]
According to (220) we have
\[
v^{(n-1)}(a)\Phi_{0i}(1) = 0.
\]
However, since \(\Phi_{0i}(1) = 1\), we have \(v^{(n-1)}(t) = 0\) for \(a \leq t \leq b\). Referring to (220) and \(\Phi_{0i}(1) = 1 \ (i = 1, \ldots, n - 1)\), we come to the conclusion that \(v(t) \equiv 0\). Using Theorem 2.1 from [3], in view of (23) and the uniqueness of the trivial solution of the problem \((21_0), (22_0)\), we get the existence of a solution of the problem \((21), (22)\).
Let \( u \) be a solution of the problem (21), (22). Then, using (6), we get
\[
\begin{align*}
u^{(n)}(t) \text{ sign } u^{(n-1)}(t) &= \tilde{f}(t, u(t), \ldots, u^{(n-1)}(t)) \text{ sign } u^{(n-1)}(t) \\
&= \chi \left( \sum_{j=1}^{n} \left| u^{(j-1)}(t) \right| \right) f(t, u(t), \ldots, u^{(n-1)}(t)) \text{ sign } u^{(n-1)}(t) \\
&\leq \omega \left( \left| u^{(n-1)}(t) \right| \right) \sum_{i=1}^{n-1} \sum_{j=1}^{m} g_{ij}(t) h_{ij}(u^{(i-1)}(t)) |u^{(i)}(t)| \frac{1}{\eta_{ij}}
\end{align*}
\]

for \( t \in [a, b] \), and
\[
\begin{align*}
u^{(n)}(t) \text{ sign } u^{(n-1)}(t) &= \tilde{f}(t, u(t), \ldots, u^{(n-1)}(t)) \text{ sign } u^{(n-1)}(t) \\
&= \chi \left( \sum_{j=1}^{n} \left| u^{(j-1)}(t) \right| \right) f(t, u(t), \ldots, u^{(n-1)}(t)) \text{ sign } u^{(n-1)}(t) \\
&\geq -\omega \left( \left| u^{(n-1)}(t) \right| \right) \sum_{i=1}^{n-1} \sum_{j=1}^{m} g_{ij}(t) h_{ij}(u^{(i-1)}(t)) |u^{(i)}(t)| \frac{1}{\eta_{ij}}
\end{align*}
\]

for \( t \in [a, b_n] \). Put
\[
\eta_{i}(\rho) = \sup \left\{ \left| \tilde{\varphi}_{i}(v) \right| : \|v\|_{C^{n-1}_{([a,b])}} \leq \rho \right\} \quad i = 1, \ldots, n .
\]

From (41) and (8), it immediately follows that the functions \( \eta_{i} \) \( i = 1, \ldots, n \) satisfy (9) and
\[
\begin{align*}
\min \left\{ \left| u^{(i-1)}(t) \right| : a_i \leq t \leq b_i \right\} &= \Phi_{0i} \left( \min \left\{ \left| u^{(i-1)}(t) \right| : a_i \leq t \leq b_i \right\} \right) \\
&\leq \left| \Phi_{0i}(u^{(i-1)}) \right| = \left| \tilde{\varphi}_{i}(u) \right| \leq \eta_{i}(\|u\|_{C^{n-1}_{([a,b])}}) \\
&i = 1, \ldots, n .
\end{align*}
\]

Therefore, by Lemma 1 we get
\[
\|u\|_{C^{n-1}_{([a,b])}} \leq \rho_{0} .
\]

Consequently,
\[
\chi \left( \sum_{i=1}^{n} \left| u^{(i-1)}(t) \right| \right) = 1 \quad \text{for } a \leq t \leq b
\]
and
\[
\chi \left( \|u\|_{C^{n-1}_{([a,b])}} \right) = 1 .
\]

Using these equalities in (20), we obtain that \( u \) is a solution of the problem (1), (2).

**Remark 1.** If \( \Phi_{0i}(u^{(i-1)}) = u^{(i-1)}(t_i) \), \( a_i \leq a_i \leq t_i \leq b_i \leq b \) \( i = 1, \ldots, n \), then Theorem 1 is Theorem in [4].
Now we give new sufficient conditions guaranteeing the existence of a solution of the problem (1), (3) provided that the equation

\[ u^{(k_0)} = 0 \]

with the boundary conditions (31) has only the trivial solution.

**Theorem 2.** Let the problem (24), (31) have only the trivial solution and let the inequalities

\[
 f(t, x_1, \ldots, x_n) \sign x_n \leq \omega(|x_n|) \sum_{i=k_0+1}^{n-1} \sum_{j=1}^{m} g_{ij}(t) h_{ij}(x_i)|x_{i+1}|^{\frac{1}{q_{ij}}}
\]

for \( t \in [a_n, b], (x_i)_{i=1}^n \in \mathbb{R}^n \)

\[
 f(t, x_1, \ldots, x_n) \sign x_n \geq -\omega(|x_n|) \sum_{i=k_0+1}^{n-1} \sum_{j=1}^{m} g_{ij}(t) h_{ij}(x_i)|x_{i+1}|^{\frac{1}{q_{ij}}}
\]

for \( t \in [a, b_n], (x_i)_{i=1}^n \in \mathbb{R}^n \)

hold, where \( g_{ij} \in L^{p_{ij}}([a, b], R_+), p_{ij}, q_{ij} \geq 1, 1/p_{ij} + 1/q_{ij} = 1 \) \( (i = k_0 + 1, \ldots, n - 1; j = 1, \ldots, m) \), \( \omega : R_+ \to (0, +\infty) \) and \( h_{ij} : R \to R_+ \) \( (i = k_0 + 1, \ldots, n - 1; j = 1, \ldots, m) \) are continuous nondecreasing functions satisfying (7) and

\[
 \lim_{\rho \to +\infty} \frac{\Omega(\rho \delta_{k_0}(\rho))}{\Omega(\rho)} = 0
\]

\[
 \lim_{\rho \to +\infty} \frac{\|h_{ij}\|_{L^{q_{ij}}([a, b], R_+)}}{\Omega(\rho)} = 0 \quad i = k_0 + 1, \ldots, n - 1; j = 1, \ldots, m.
\]

Then the problem (1), (3) has at least one solution.

To prove Theorem 2 we need the following

**Lemma 2.** Let the problem (24), (31) have only the trivial solution and let the functions \( \omega, \Omega, g_{ij}, h_{ij} \) and the numbers \( p_{ij}, q_{ij} \) \( (i = k_0 + 1, \ldots, n - 1; j = 1, \ldots, m) \) be given as in Theorem 2, and let \( \eta_k : R_+ \to R_+ \) \( (i = k_0 + 1, \ldots, n) \) be nondecreasing functions satisfying

\[
 \lim_{\rho \to +\infty} \frac{\Omega(\rho \delta_k(\rho))}{\Omega(\rho)} = 0 = \lim_{\rho \to +\infty} \frac{\eta_k(\rho)}{\rho} \quad i = k_0 + 1, \ldots, n.
\]

Then there exists a constant \( \rho_0 > 0 \) such that the estimate (10) holds for each solution \( u \in AC^{n-1}([a, b]) \) of the differential inequalities

\[
 u^{(n)}(t) \sign u^{(n-1)}(t) \leq \omega(|u^{(n-1)}(t)|) \sum_{i=k_0+1}^{n-1} \sum_{j=1}^{m} g_{ij}(t) h_{ij}(u^{(i-1)}(t)|u^{(i)}(t)|^{\frac{1}{q_{ij}}}
\]

for \( t \in [a_n, b] \).
\[ v^{(n)}(t) \text{ sign } u^{(n-1)}(t) \geq -\omega \left( |u^{(n-1)}(t)| \right) \sum_{i=k_0+1}^{n-1} \sum_{j=1}^{m} g_{ij}(t) h_{ij} |u^{(i)}(t)| \frac{1}{\omega_{ij}} \] (27.2)

for \( t \in [a, b] \)

with the boundary conditions (3.1) and

\[ \min \left\{ |u^{(i-1)}(t)| : a_i \leq t \leq b_i \right\} \leq \eta_i (\| u^{(k_0)} \|_{C^{n-k_0-1}}) \quad i = k_0 + 1, \ldots, n. \] (28)

**Proof.** Let \( u \) be an arbitrary solution of the problem (27), (3.1), (28). Put

\[ v(t) = u^{(k_0)}(t). \] (29)

Then the formulas (27) and (28) imply that

\[ v^{(n-k_0)}(t) \text{ sign } v^{(n-k_0-1)}(t) \leq \omega \left( |v^{(n-k_0-1)}(t)| \right) \]

for \( t \in [a, b] \),

\[ v^{(n-k_0)}(t) \text{ sign } v^{(n-k_0-1)}(t) \geq -\omega \left( |v^{(n-k_0-1)}(t)| \right) \]

for \( t \in [a, b] \),

and

\[ \min \left\{ |v^{(i-1)}(t)| : a_i \leq t \leq b_i \right\} \leq \eta_i (\| v \|_{C^{n-k_0-1}}) \quad i = 1, \ldots, n - k_0. \]

Consequently, according to Lemma 1 there exists \( \rho_1 > 0 \) such that

\[ \| v \|_{C^{n-k_0-1}} \leq \rho_1. \] (30)

By virtue of the assumption that the problem (24), (3.1) has only the trivial solution, there exists a Green function \( G(t, s) \) such that

\[ u^{(i-1)}(t) = \int_{a}^{b} \frac{\partial^{i-1} G(t, s)}{\partial t^{i-1}} v(s) \, ds \quad \text{for} \quad t \in [a, b] \quad i = 1, \ldots, k_0 \] (31)

(see e.g., [2]).

Put

\[ \rho_2 = \max_{a \leq t \leq b} \sum_{i=1}^{k_0} \left| \frac{\partial^{i-1} G(t, s)}{\partial t^{i-1}} \right| \, ds. \]

According to (30) and (31) we have

\[ \| u \|_{C^{k_0-1}} \leq \rho_1 \rho_2. \]

Therefore we obtain (10), where \( \rho_0 = \rho_1 + \rho_2 \rho_1. \)
Theorem 2 can be proved analogously to Theorem 1 using Lemma 2.

REFERENCES


