PROLONGATION OF PAIRS OF CONNECTIONS INTO CONNECTIONS ON VERTICAL BUNDLES

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Abstract. Let $F$ be a natural bundle. We introduce the geometrical construction transforming two general connections into a general connection on the $F$-vertical bundle. Then we determine all natural operators of this type and we generalize the result by I. Kolár and the second author on the prolongation of connections to $F$-vertical bundles. We also present some examples and applications.

Introduction

Let $\mathcal{M}_{fm}$ be the category of $m$-dimensional manifolds and local diffeomorphisms, $\mathcal{FM}$ be the category of fibered manifolds and fiber respecting mappings and $\mathcal{FM}_{m,n}$ be the category of fibered manifolds with $m$-dimensional bases and $n$-dimensional fibers and locally invertible fiber respecting mappings.

Consider an arbitrary bundle functor $F$ on the category $\mathcal{M}_{fm}$ and denote by $V^F$ its vertical modification. Our starting point is the paper [9] by I. Kolár and the second author, who studied the prolongation of a connection $\Gamma$ on an arbitrary fibered manifold $Y \to M$ with respect to an $F$-vertical functor $V^F$. In particular, they have introduced an $F$-vertical prolongation $V^F\Gamma$ of a connection $\Gamma$ and have proved that $V^F$ is the only natural operator of finite order transforming connections on $Y \to M$ into connections on $V^FY \to M$. They have also described some conditions under which every natural operator of such a type has finite order. Further, in the case of the vertical Weil functor $V^A$ they have proved that the operator transforming a connection $\Gamma$ on $Y \to M$ into its vertical prolongation $V^A\Gamma$ is the only natural one.

The aim of this paper is to study the prolongation of a pair of connections $\Gamma_1$ and $\Gamma_2$ on $Y \to M$ into a connection on $V^FY \to M$. Our main result is Theorem 1 which describes all such natural operators. As a direct consequence we prove the generalization of a result by I. Kolár and the second author. In particular, we show that $V^F$ is the only natural operator transforming connections on $Y \to M$ into connections on $V^FY \to M$ (without any additional assumption.

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on the finite order). In Section 1 we discuss the prolongation of connections on $Y \to M$ into connections on $GY \to M$, where $G$ is a bundle functor on $\mathcal{FM}_{m,n}$. Section 2 is devoted to the construction of a connection on $V^A Y \to M$ by means of a pair $\Gamma_1, \Gamma_2$ of connections on $Y \to M$. This geometrical construction will be based on linear natural operators transforming vector fields on $n$-manifolds $N$ into vector fields on $FN$. In Section 3 we introduce some examples and applications. We also show, that in the case of a vertical Weil functor $V^A$ the connection on $V^A Y \to M$ depending on a pair $\Gamma_1, \Gamma_2$ can be constructed by means of the vertical prolongation of the deviation $\delta(\Gamma_1, \Gamma_2)$ of $\Gamma_1$ and $\Gamma_2$. Finally, the whole Section 4 is devoted to the proof of Theorem 1.

In what follows $Y \to M$ stands for $\mathcal{FM}_{m,n}$-objects and $N$ stands for $\mathcal{MF}_{n}$-objects. All manifolds and maps are assumed to be of the class $C^\infty$. Unless otherwise specified, we use the terminology and notation from the book [7].

1. Prolongation of connections to $GY \to M$

Recently it has been clarified that the order of bundle functors on $\mathcal{FM}$ is characterized by three integers $(r, s, q)$, $s \geq r \leq q$ and is based on the concept of $(r, s, q)$-jet, [7]. Consider two fibered manifolds $p : Y \to M$ and $\overline{p} : \overline{Y} \to \overline{M}$ and let $r, s \geq r, q \geq r$ be integers. We recall that two $\mathcal{FM}$-morphisms $f, g : Y \to \overline{Y}$ with the base maps $f, g : M \to \overline{M}$ determine the same $(r, s, q)$-jet $j^{r,s,q}_y f = j^{r,s,q}_y g$ at $y \in Y, p(y) = x$. If

$$j^r_y f = j^r_y g, \quad j^s_y (f|_{Y_x}) = j^s_y (g|_{Y_x}), \quad j^q_y f = j^q_y g.$$ 

The space of all such $(r, s, q)$-jets will be denoted by $J^{r,s,q}(Y, \overline{Y})$. By 12.19 in [7], the composition of $\mathcal{FM}$-morphisms induces the composition of $(r, s, q)$-jets.

**Definition 1** ([9]). A bundle functor $G$ on $\mathcal{FM}_{m,n}$ is said to be of order $(r, s, q)$, if $j^{r,s,q}_y f = j^{r,s,q}_y g$ implies $Gf|_{GY} = Gg|_{GY}$.

Then the integer $q$ is called the base order, $s$ is called the fiber order and $r$ is called the total order of $G$.

If $X : N \to TN$ is a vector field and $F$ is a bundle functor on $\mathcal{MF}_n$, then we can define the flow prolongation $\mathcal{FX} : FN \to TFN$ of $X$ with respect to $F$ by

$$\mathcal{FX} = \frac{\partial}{\partial t} \big|_0 F(\exp tX)$$

where $\exp tX$ denotes the flow of $X$, [7]. Quite analogously, a projectable vector field on a fibered manifold $Y \to M$ is an $\mathcal{FM}$-morphism $Z : Y \to TY$ over the underlying vector field $M \to TM$, and its flow $\exp tZ$ is formed by local $\mathcal{FM}_{m,n}$-morphisms. Further, if $G$ is a bundle functor on $\mathcal{FM}_{m,n}$, the flow prolongation of $Z$ with respect to $G$ is defined by

$$GZ = \frac{\partial}{\partial t} \big|_0 G(\exp tZ).$$

By [9], this map is $\mathbb{R}$-linear and preserves bracket.
Proposition 1 ([9]). If $G$ is of order $(r, s, q)$, then the value of $G Z$ at each point of $G Y$ depends on $j^r_s q Z$ only.

Thus the construction of the flow prolongation of projectable vector fields can be interpreted as a map

$$G_Y : G Y \times_Y J^{r,s,q} Y \to T G Y,$$

where $J^{r,s,q} Y$ denotes the space of all $(r, s, q)$-jets of projectable vector fields on $Y$. Since the flow prolongation is $\mathbb{R}$-linear, $G_Y$ is linear in the second factor.

Now let $\Gamma : Y \to J^1 Y$ be a general connection on $p : Y \to M$. In [7] and [9] it is clarified, that if the functor $G$ on $\mathcal{F} \mathcal{M}_{m,n}$ has the base order $q$, then one can construct the induced connection $G(\Gamma, \Delta)$ on $G Y \to M$ by means of an auxiliary linear $q$-th order connection $\Delta$ on the base manifold $M$. The geometrical construction of the connection $G(\Gamma, \Delta)$ is the following. Let $X$ be a vector field on $M$ with the coordinate components $X^i(x)$ and let

$$dy^p = \Gamma^p_i(x, y) dx^i$$

be the coordinate expression of $\Gamma$. Then the $\Gamma$-lift of $X$ is a vector field $\Gamma X$ on $Y$, whose coordinate form is

$$X^i(x) \frac{\partial}{\partial x^i} + \Gamma^p_i(x, y) X^i(x) \frac{\partial}{\partial y^p}.$$

By Proposition 1, the flow prolongation $G(\Gamma X)$ depends on the $q$-jets of $X$ only. So we obtain a map

$$G(\Gamma) : G Y \times M J^q T M \to T G Y,$$

which is linear in the second factor. Further, let $\Delta : T M \to J^q T M$ be a linear $q$-th order connection on $M$. By linearity, the composition

$$G(\Gamma, \Delta) := G(\Gamma) \circ (id_G \times id_M) \Delta : G Y \times M T M \to T G Y$$

is the lifting map of a connection on $G Y \to M$. Clearly, if the base order of $G$ is zero, then (2) is a connection on $G Y \to M$ and we need no auxiliary linear connection $\Delta$. This is the case of a vertical functor $V^F$, which is defined as follows. Let $F$ be a bundle functor on $\mathcal{M}_{f, n}$ of order $s$. Its vertical modification $V^F$ is a bundle functor on $\mathcal{F} \mathcal{M}_{m,n}$ defined by

$$V^F Y = \bigcup_{x \in M} F(Y_x), \quad V^F f = \bigcup_{x \in M} F(f_x),$$

where $f_x$ is the restriction and corestriction of $f : Y \to \overline{Y}$ over $f : M \to \overline{M}$ to the fibers $Y_x$ and $\text{Graph}(f_x)$, [9]. Obviously, the order of the functor $V^F$ is $(0, s, 0)$. Since the base order of $V^F$ is zero, the map (2) defines a connection $V^F \Gamma$ for every connection $\Gamma$ on $Y \to M$. 


Definition 2 ([9]). The connection $V^F \Gamma$ is called the $F$-vertical prolongation of $\Gamma$.

If $F = T^A$ is a Weil functor, then $V^T^A$ is the vertical Weil functor on $\mathcal{F}M_{m,n}$, which will be denoted by $V^A$. This functor induces the vertical $A$-prolongation $V^A \Gamma$. In particular, for $F = T$ we obtain the classical vertical bundle, which will be denoted by $V^T$. I. Kolár [5] has proved that $V^T$ is the only natural operator transforming connections on $Y \to M$ into connections on $VY \to M$, see also [7], p. 255. Moreover, the following naturality property of the $F$-vertical prolongation $V^F \Gamma$ is an interesting generalization of the well known result concerning the classical vertical prolongation $V^T$ to an arbitrary bundle functor $F$ on $M_{f\cdot n}$.

Proposition 2 ([9]). $V^F$ is the only natural operator of finite order transforming connections on $Y \to M$ into connections on $V^F Y \to M$.

Proposition 3 ([9]). If the standard fiber $F_0(\mathbb{R}^n)$ of $F$ is compact or if $F_0(\mathbb{R}^n)$ contains a point $z_0$ such that $F(bid_{\mathbb{R}^n})(z) \to z_0$ if $b \to 0$ for any $z \in F_0(\mathbb{R}^n)$, then every natural operator $D$ transforming connections on $Y \to M$ into connections on $V^F Y \to M$ has finite order.

For example, the assumption of Proposition 3 is satisfied in the case $F$ is a Weil functor $T^A$. On the other hand, this assumption is not satisfied in the case $F$ is a cotangent bundle functor $T^*$. 

Remark 1. It is well known, that there is no natural operator transforming connections on $Y \to M$ into connections on $J^1 Y \to M$, see [5] and [7]. Quite analogously, I. Kolár and the first author have proved that there is no first order natural operator transforming connections on $Y \to M$ into connections on $TY \to M$, [2]. The second author has recently proved the following general result, [13]: If $G$ is a bundle functor on $\mathcal{F}M_{m,n}$ such that $G^1 : M_{f\cdot n} \to \mathcal{F}M$, $G^1 M = G(M \times \mathbb{R}^n)$, $G^1(\phi) = G(\phi \times id_{\mathbb{R}^n})$ is not of order zero, then there is no natural operator transforming connections on $Y \to M$ into connections on $GY \to M$. This means that in this case, the use of an auxiliary linear connection $\Delta$ on the base manifold $M$ in the construction (3) is unavoidable. We remark that all natural operators transforming a connection $\Gamma$ on $Y \to M$ and a linear connection $\Delta : TM \to J^1 TM$ into a connection on $J^1 Y \to M$ are determined in [5].

2. Prolongation of pairs of connections into connections on vertical bundles

Let $F : M_{f\cdot n} \to \mathcal{F}M$ be a natural bundle of order $s$ and $V^F : \mathcal{F}M_{m,n} \to \mathcal{F}M$ be the corresponding vertical modification. Suppose we have a natural linear operator

$L : T \to TF$

transforming vector fields on $N$ into vector fields on $FN$. Let $\Gamma_1, \Gamma_2 : Y \times_M TM \to TY$ be connections on an $\mathcal{F}M_{m,n}$-object $Y \to M$. We are going to construct
a connection $\mathcal{V}^{F,L}(\Gamma_1, \Gamma_2)$ on $V^F Y \to M$ depending canonically on $\Gamma_1$ and $\Gamma_2$. Clearly, such a connection can be written in the form

$$\mathcal{V}^{F,L}(\Gamma_1, \Gamma_2) : V^F Y \times_M TM \to TV^F Y.$$ 

Firstly, we define a fiber linear map

$$(4) \quad (\Gamma_1, \Gamma_2)^{F,L} : V^F Y \times_M TM \to V(V^F Y)$$

covering the identity on $V^F Y$ as follows. Let $(u, v) \in (V^F Y \times_M TM)_x$, $x \in M$ and let $v^{\Gamma_1}$, $v^{\Gamma_2}$ (defined on $Y_x$) be the horizontal lifts of $v$ with respect to $\Gamma_1$ and $\Gamma_2$ respectively. The difference $v^{\Gamma_1, \Gamma_2} := (v^{\Gamma_1} - v^{\Gamma_2})$ is vertical, so it can be considered as the vector field on $Y_x$, $v^{\Gamma_1, \Gamma_2} : Y_x \to T(Y_x) = (VY)_x$. Using the linear operator $L$, we have the vector field $L(v^{\Gamma_1, \Gamma_2}) : F(Y_x) = (V^F Y)_x \to T(F(Y_x)) = (V(V^F Y))_x$ which can be considered as (defined on $(V^F Y)_x$) vertical vector field $L(v^{\Gamma_1, \Gamma_2}) : V^F Y \to V(V^F Y)$. We put

$$(\Gamma_1, \Gamma_2)^{F,L}(u, v) = L(v^{\Gamma_1, \Gamma_2})(u).$$

Since $L$ is a linear operator, the map $(\Gamma_1, \Gamma_2)^{F,L}$ is linear in the second factor. Further,

$$\mathcal{V}^{F,L}(\Gamma_1, \Gamma_2) := \mathcal{V}^{F, \Gamma_1} + (\Gamma_1, \Gamma_2)^{F,L} : V^F Y \times_M TM \to TV^F Y$$

is a connection on $V^F Y \to M$ canonically dependent on $\Gamma_1$ and $\Gamma_2$.

**Definition 3.** The connection $\mathcal{V}^{F,L}(\Gamma_1, \Gamma_2)$ is called the $(F, L)$-vertical prolongation of $(\Gamma_1, \Gamma_2)$.

From the geometrical construction of $(\Gamma_1, \Gamma_2)^{F,L}$ it follows directly

**Lemma 1.** We have

(i) $(\Gamma_1, \Gamma_2)^{F,L} = -(\Gamma_2, \Gamma_1)^{F,L}$,

(ii) $(\Gamma_1, \Gamma_2)^{F,c_1 L_1 + c_2 L_2} = c_1(\Gamma_1, \Gamma_2)^{F,L_1} + c_2(\Gamma_1, \Gamma_2)^{F,L_2}$, $c_1, c_2 \in \mathbb{R}$,

(iii) $\mathcal{V}^{F,L}(\Gamma, \Gamma) = \mathcal{V}^{F, \Gamma}.$

The main result of the present paper is the following classification theorem.

**Theorem 1.** $\mathcal{V}^{F,L}$ are the only natural operators transforming pairs of connections on $Y \to M$ into connections on $V^F Y \to M$.

We have the following corollary of Theorem 1.
Corollary 1. $\hat{\nabla}^F(\Gamma_1, \Gamma_2) := \frac{1}{2}(\nabla^F \Gamma_1 + \nabla^F \Gamma_2)$ is the only natural symmetric operator transforming pairs of connections on $Y \to M$ into connections on $V^F Y \to M$.

Proof of Corollary 1. Let $D$ be such an operator. By Theorem 1, $D(\Gamma_1, \Gamma_2) = \nabla^F \Gamma_1 + (\Gamma_1, \Gamma_2)^{F,L}$. By the symmetry of $D$ we get $\nabla^F \Gamma_1 + (\Gamma_1, \Gamma_2)^{F,L} = \nabla^F \Gamma_2 - (\Gamma_1, \Gamma_2)^{F,L}$ because $(\Gamma_2, \Gamma_1)^{F,L} = -(\Gamma_1, \Gamma_2)^{F,L}$. Then $(\Gamma_1, \Gamma_2)^{F,L} = \frac{1}{2}(\nabla^F \Gamma_2 - \nabla^F \Gamma_1)$ and $D(\Gamma_1, \Gamma_2) = \frac{1}{2}(\nabla^F \Gamma_1 + \nabla^F \Gamma_2)$ as well. \[\Box\]

Now we show that one can omit the finite order assumption in Proposition 2. In this way we obtain the following generalization of this result:

Proposition 2'. $\nabla^F$ is the only natural operator transforming connections on $Y \to M$ into connections on $V^F Y \to M$.

Proof. Write $\Gamma_1 = \Gamma_2 = \Gamma$ in Corollary 1. Then we obtain $\hat{\nabla}^F(\Gamma, \Gamma) = \nabla^F \Gamma$. \[\Box\]

Remark 2. The $(F,L)$-prolongation is a geometrical construction, which transforms two connections $\Gamma_1$ and $\Gamma_2$ on $Y \to M$ into a connection $\nabla^{F,L}(\Gamma_1, \Gamma_2)$ on $V^F Y \to M$. Another example of a geometrical construction defined on pairs of connections is the mixed curvature, which is defined as the Frölicher-Nijenhuis bracket $[\Gamma_1, \Gamma_2]$. We remark that the mixed curvature is a section $Y \to VY \otimes \otimes^2 T^\ast M$, see 27.4 in [7].

By Theorem 1, natural operators transforming pairs of connections on $Y \to M$ into a connection on $V^F Y \to M$ depend on linear natural operators $L: T \to TF$ on vector fields. Now we show that it suffices to find the basis of such linear operators.

Proposition 4. Let $L_1, \ldots, L_k$ be the basis of linear natural operators $T \to TF$ transforming vector fields on $n$-manifolds $N$ into vector fields on $FN$. Then all natural operators transforming pairs of connections on $Y \to M$ into a connection on $V^F Y \to M$ are of the form

$$(\Gamma_1, \Gamma_2) \mapsto \nabla^F \Gamma_1 + c_1(\Gamma_1, \Gamma_2)^{F,L_1} + \cdots + c_k(\Gamma_1, \Gamma_2)^{F,L_k}, \quad c_i \in \mathbb{R}.$$ 

Proof. An arbitrary linear operator $L: T \to TF$ is of the form $L = c_1L_1 + \cdots + c_kL_k$, $c_i \in \mathbb{R}$. Then the assertion follows from Theorem 1 and from Lemma 1. \[\Box\]

3. Applications

Clearly, the flow prolongation (1) is a natural linear operator $T \to TF$. So for an arbitrary natural bundle $F$ on $\mathcal{M}f_n$ there exists a natural operator transforming pairs of connections $\Gamma_1, \Gamma_2$ on $Y \to M$ into a connection $\nabla^{F,A}(\Gamma_1, \Gamma_2)$ on $V^F Y \to M$. Now let $F = T^A$ be a Weil functor determined by a Weil algebra $A$. By [7], all product preserving functors on $\mathcal{M}f$ are of this type. We have the following action

$$A \times TT^A N \to TT^A N$$ 

of the elements of $A$ on the tangent vectors on $T^A N$. Indeed, the multiplication of the tangent vectors of $N$ by reals is a map $m: \mathbb{R} \times TN \to TN$. Applying the
functor $T^A$ and using the fact that $T^A R = A$ we obtain a map $T^A m : A \times T^A T^A N \to T^A T^A N$. Finally, the canonical identification $T^A T^A N \cong T^A T^A N$ yields the action (5). So for an arbitrary $a \in A$ we have a natural affinor on $T^A N$ of the form

$$a f(a)_N : T^A T^A N \to T^A T^A N.$$ 

By [7], all natural linear operators transforming vector fields on $N$ into vector fields on $T^A N$ are of the form

$$a f(a) \circ T^A$$

for all $a \in A$, where $T^A$ means the flow operator. Thus, we have

**Proposition 5.** All natural operators transforming pairs of connections on $Y \to M$ into a connection on $V^A Y \to M$ are of the form

$$(\Gamma_1, \Gamma_2) \mapsto \nu^{T^A}, a f(a) \circ T^A (\Gamma_1, \Gamma_2)$$

for all $a \in A$.

It is well known that $J^1 Y \to Y$ is an affine bundle with the associated vector bundle $V Y \otimes T^* M$. So the difference of two connections $\Gamma_1, \Gamma_2 : Y \to J^1 Y$ is a map $\delta(\Gamma_1, \Gamma_2) : Y \to V Y \otimes T^* M$, which is called the deviation of $\Gamma_1$ and $\Gamma_2$. Clearly, this map can be written as

$$(\delta(\Gamma_1, \Gamma_2))(y, z) = x$$

for all $x \in A$. A. Cabras and I. Kolář [1] have constructed the vertical $A$-prolongation of (6) with respect to the first factor

$$(\nu^{V^A} \delta(\Gamma_1, \Gamma_2)) : V^A Y \times_M T^A TM \to V V^A Y$$

fiberwise in the following way. Denoting by $q : T M \to M$ the bundle projection, we can write $\delta_z : Y \to (V Y)_z$ for the map $y \mapsto \delta(\Gamma_1, \Gamma_2)(y, z)$, $y \in Y$, $z \in T M$, $q(z) = x$. Applying $T^A$ we obtain a map

$$(V^A \delta)_z := T^A(\delta_z) : V^A Y \to T^A(\nu^{V^A} Y)_z = T^A(\nu^{V^A} Y)_x$$

which yields a map $V^A \delta : V^A Y \times_M T^A TM \to V^A V Y$. Further, the canonical exchange diffeomorphism of Weil functors $i^A_B : T^B(T^A N) \to T^A(T^B N)$ from [7] induces the exchange diffeomorphism $i_Y : V^A V Y \to V V^A Y$, [1]. Then the map (7) can be defined by

$$(\nu^{V^A} \delta(\Gamma_1, \Gamma_2))(y, z) = i_Y \circ (V^A \delta).$$

On the other hand, we can construct the vertical $A$-prolongations $\nu^{V^A} \Gamma_1, V^A \Gamma_2 : V^A Y \times_M T M \to T V^A Y$ of $\Gamma_1$ and $\Gamma_2$. The deviation of the connections $\nu^{V^A} \Gamma_1$ and $V^A \Gamma_2$ is a map

$$(\delta(\nu^{V^A} \Gamma_1, V^A \Gamma_2)) : V^A Y \times_M T M \to V^A V Y.$$

A. Cabras and I. Kolář have proved the formula

$$(\delta(\nu^{V^A} \Gamma_1, V^A \Gamma_2)) = \nu^{V^A} \delta(\Gamma_1, \Gamma_2).$$

Consider now a linear map (4), where we put $F = T^A$ and $L = T^A, (\Gamma_1, \Gamma_2)T^A : V^A Y \times_M T M \to V(V^A Y)$. We have
Proposition 6. Let $T^A$ be the flow operator. Then we have

$$ (\Gamma_1, \Gamma_2)^{T^A, T^A} = \mathcal{V}^1 A \delta(\Gamma_1, \Gamma_2). $$

Proof. Denoting by $\delta := \delta(\Gamma_1, \Gamma_2): (TM)_x \to (VY)_x$, we have $\delta(v) = \Gamma_1 v - \Gamma_2 v$ for $v \in (TM)_x$. Since $\delta(v)$ is vertical, it can be considered as a vector field $Y_x \to T(Y_x)$. Applying the flow operator $T^A$ we obtain a vector field $T^A \delta(v): T^A(Y_x) = (V^A Y)_x \to T((V^A Y)_x) = (V(V^A Y))_x$, which can be considered as a vertical vector field on $V^A Y$. This defines the map

$$ (\Gamma_1, \Gamma_2)^{T^A, T^A}: V^A Y \times_M TM \to V(V^A Y), \quad (\Gamma_1, \Gamma_2)^{T^A, T^A}(u, v) = T^A \delta(v)(u). $$

In general, given a vector field $\xi: N \to TN$, the flow prolongation $T^A \xi$ can be also constructed as the composition $T^A \xi = i^{A, B}_N \circ T^A \xi$, where $i^{A, B}_N: T^A TN \to TT^A N$ is the canonical exchange diffeomorphism and $D$ is the Weil algebra of dual numbers corresponding to the tangent bundle $T$. By (8) and (12) we have $T^A \delta = V^1 A \delta$. $\Box$

Remark 3. It is interesting to pose a question whether the formulas (10) and (11) can be generalized for an arbitrary natural bundle $F$ on $M_f^n$. Given any connections $\Gamma_1$ and $\Gamma_2$ on $Y \to M$, one can construct their $F$-vertical prolongations $V^F \Gamma_1, V^F \Gamma_2: V^F Y \times_M TM \to T(V^F Y)$ and then the deviation

$$ \delta(V^F \Gamma_1, V^F \Gamma_2): V^F Y \times_M TM \to V(V^F Y). $$

Further, for any linear natural operator $L: T \to TF$ we have the map (4). From Theorem 1 it follows that

$$ \delta(V^F \Gamma_1, V^F \Gamma_2) = (\Gamma_1, \Gamma_2)^{F, L}, $$

for some linear natural operator $L$. By (10) and (11), if $F = T^A$, then $L = T^A$. From the proof of Theorem 1 (see the construction (14) of $L^D$) it follows that even in the general case of an arbitrary natural bundle $F$ we have $L = F$, where $F$ is the flow operator (1). We remark that the construction of the vertical prolongation (7) and the proof of (11) essentially depend on the existence of the exchange diffeomorphism $i_Y: V^A Y \to VV^A Y$. We recall that the bundle functor $F$ is said to have the point property, if $F(\text{pt}) = \text{pt}$, where $\text{pt}$ denote the one-point manifold. From Theorem 39.2 in [7] it follows directly that if $F$ has the point property, then there exists a natural equivalence $i^F_Y: V^F Y \to VV^F Y$ if and only if $F$ is a Weil functor $T^A$. In this case, $i^F_Y$ coincides with $i_Y$.

Let $T^{r}*N = J^r(N, R)_0$ be the space of all $r$-jets from an $n$-manifold $N$ into reals with target 0. Since $R$ is a vector space, $T^{r}*N$ has a canonical structure of the vector bundle over $N$. $T^{r}*N$ is called the $r$-th order cotangent bundle and the dual vector bundle

$$ T^{(r)}N = (T^{r}*N)^* $$

is called the $r$-th order tangent bundle. For every map $f: N \to N_1$ the jet composition $A \mapsto A \circ (j^r_x f)$, $x \in N$, $A \in (T^{r}*N_1)_{f(x)}$ defines a linear map
by Vnations (with real coefficients) of the flow operator \( T \), which is defined on the whole category \( Mf \) of all smooth manifolds and all smooth maps. Clearly, for \( r = 1 \) we obtain the classical tangent functor \( T \) and for \( r > 1 \) the functor \( T^{(r)} \) does not preserve products. Obviously, we have the canonical inclusion \( TN \subset T^{(r)}N \). Using fiber translations on \( T^{(r)}N \), we can extend every section \( X : N \rightarrow TN \) into a vector field \( V(X) \) on \( T^{(r)}N \). This defines a linear natural operator \( V : T \rightsquigarrow TT^{(r)} \). The second author has in [10] determined all natural operators \( T \rightsquigarrow TT^{(r)} \). From this result we obtain directly that all linear natural operators \( T \rightsquigarrow TT^{(r)} \) transforming vector fields on \( N \) into vector fields on \( T^{(r)}N \) are of the form \( c_1T^{(r)} + c_2V \), \( c_i \in \mathbb{R} \). Using Proposition 4 we have

**Proposition 7.** All natural operators transforming pairs of connections on \( Y \rightarrow M \) into a connection on \( V^{T^{(r)}Y} \rightarrow M \) are of the form

\[
(\Gamma_1, \Gamma_2) \mapsto \mathcal{V}T^{(r)} \Gamma_1 + c_1(\Gamma_1, \Gamma_2)T^{(r)}\mathcal{J} \Gamma_2 + c_2(\Gamma_1, \Gamma_2)T^{(r)}V, \quad c_i \in \mathbb{R}.
\]

By Corollary 4.1 in [11], all linear natural operators \( T \rightsquigarrow TT^* \) are linear combinations (with real coefficients) of the flow operator \( T^* \) and the operator \( V \) defined by \( V(X)_x = \langle \omega, X_x \rangle \cdot C_\omega \), where \( C \) is the Liouville vector field of the cotangent bundle and \( X \in \mathfrak{X}(N) \), \( \omega \in T^*_xN \), \( x \in N \). Thus, we have

**Proposition 8.** All natural operators transforming pairs of connections on \( Y \rightarrow M \) into a connection on \( V^{T^*Y} \rightarrow M \) are of the form

\[
(\Gamma_1, \Gamma_2) \mapsto \mathcal{V}T^* \Gamma_1 + c_1(\Gamma_1, \Gamma_2)T^*\mathcal{J} \Gamma_2 + c_2(\Gamma_1, \Gamma_2)T^*V, \quad c_i \in \mathbb{R}.
\]

Using [11], we can generalize this result in the following way. First, we have \( r \) linear natural operators \( E_1, \ldots, E_r : T \rightsquigarrow TT^{*r} \) defined by

\[
E_k(X)(j^r_x\gamma) = \langle X(x), j^r_x\gamma \rangle - \frac{d}{dt}j^r_x\gamma \bigg|_{t=0} + tj^r_x(\gamma)^k, \quad k = 1, \ldots, r
\]

where \( X \in \mathfrak{X}(N) \) is a vector field on \( N \), \( j^r_x\gamma \in T^{(r)*}_xN \) and \( (\gamma)^k \) is the \( k \)-th power of the map \( \gamma : N \rightarrow \mathbb{R} \). Further, if we interpret \( X \) as the differentiation, then \( (X \gamma - X\gamma(x))(\gamma)^{s-1} \) is a function on \( N \) which maps the point \( x \in N \) into zero. So we can define linear natural operators \( F_2, \ldots, F_r : T \rightsquigarrow TT^{*r} \) by

\[
F_s(X)(j^r_x\gamma) = \frac{d}{dt}\bigg|_{t=0} \left[ j^r_x\gamma + tj^r_x((X \gamma - X\gamma(x))(\gamma)^{s-1}) \right], \quad s = 2, \ldots, r.
\]

By [11], the flow operator \( T^{*r} \) and the operators \( E_1, \ldots, E_r, F_2, \ldots, F_r \) form the basis over \( \mathbb{R} \) of the vector space of all linear natural operators \( T \rightsquigarrow TT^{*r} \). By Proposition 4 we have
Proposition 9. All natural operators transforming pairs of connections on $Y \to M$ into a connection on $V^TFY \to M$ are of the form

$$(\Gamma_1, \Gamma_2) \mapsto V^TF \Gamma_1 + c_0(\Gamma_1, \Gamma_2)T^*E_1 + \cdots + c_r(\Gamma_1, \Gamma_2)T^*E_r + d_2(\Gamma_1, \Gamma_2)T^*E_2 + \cdots + d_r(\Gamma_1, \Gamma_2)T^*E_r,$$

$c_i, d_i \in \mathbb{R}$.

We remark that there are many papers which classify all natural operators $T \mapsto TF$ for particular natural bundles $F$, see e.g. [4], [6], [10]-[12], [14] and [15]. For example, P. Kobak [4] has determined all natural operators $T \mapsto TT^*$ and J. Tomáš [14] has classified all natural operators $T \mapsto TT^*T^k$, where $T^k_0N = J^k_0(R, N)$ is the bundle of $k$-dimensional velocities of order $r$. If we restrict ourselves only to linear natural operators, we can easily determine all natural operators transforming pairs of connections on $Y \to M$ into a connection on $V^FY \to M$.

4. Proof of Theorem 1

From now on $R^{m,n}$ is the trivial bundle $R^m \times R^n$ over $R^m$. The usual coordinates on $R^{m,n}$ will be denoted by $x^1, \ldots, x^m, y^1, \ldots, y^n$. If $D$ is a natural operator of our type, then for given connections $\Gamma_1$ and $\Gamma_2$ on an $F\mathcal{M}_{m,n}$-object $Y \to M$ the difference

$$\Delta(\Gamma_1, \Gamma_2) = \tilde{D}(\Gamma_1, \Gamma_2) - V^F \Gamma_1 : V^FY \times_M TM \to V(V^FY)$$

is a fiber linear map covering the identity on $V^FY$. So it remains to describe all natural operators of the type as $\Delta$. Consider a natural operator $D$ of the type as $\Delta$. We prove some auxiliary lemmas.

Lemma 2. Suppose that

$$D\left(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^m \sum_{j=1}^n \sum_{|\alpha| + |\beta| \leq K} \Gamma_{1\alpha\beta}^j x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j}, \right.$$

$$\left. \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^m \sum_{j=1}^n \sum_{|\alpha| + |\beta| \leq K} \Gamma_{2\alpha\beta}^j x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j}\right) (u,v) = 0$$

for any $K \in \mathbb{N}$, any $(u,v) \in (V^F R^{m,n})_0 \times T_0 R^m$, any $\Gamma_{1\alpha\beta}^j$ and any $\Gamma_{2\alpha\beta}^j$ for $i,j,\alpha,\beta$ as indicated. Then $D = 0$.

Proof. It follows from a corollary of non-linear Peetre theorem (Corollary 19.8 in [7]).

 Lemma 3. Suppose that

$$D\left(\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + y^\beta dx^i \otimes \frac{\partial}{\partial y^\beta}, \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}\right) (u,v) = 0$$
and
\[ D\left( \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + y^\beta dx^\alpha \otimes \frac{\partial}{\partial y^\beta} \right)(u, v) = 0 \]

for any \((u, v) \in (V^F R^{m,n})_j \times T_0 R^m\), any \(n\)-tuple \(\beta\) and any \(i_0 = 1, \ldots, m\) and \(j_0 = 1, \ldots, n\). Then \(D = 0\).

**Proof.** Using the invariance of \(D\) with respect to the base homotheties \(t \text{id}_{R^m} \times \text{id}_{R^n}\) for \(t > 0\) we get the homogeneity condition

\[ D\left( \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{|\alpha|+|\beta| \leq K} t^{|\alpha|+1} \Gamma^j_{110\alpha\beta} x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j} \right), \]

\[ \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{|\alpha|+|\beta| \leq K} \Gamma^j_{12\alpha\beta} x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j} \right)(u, v) \]

\[ = \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{|\alpha|+|\beta| \leq K} \Gamma^j_{12\alpha\beta} x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j} \right)(u, v). \]

By the homogeneous function theorem, this type of homogeneity gives that

\[ D\left( \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{|\alpha|+|\beta| \leq K} \Gamma^j_{11\alpha\beta} x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j} \right), \]

\[ \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{|\alpha|+|\beta| \leq K} \Gamma^j_{21\alpha\beta} x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^j} \right)(u, v) \]

depends linearly on \(\Gamma^j_{11(0)\beta}\) and \(\Gamma^j_{21(0)\beta}\) and is independent of \(\Gamma^j_{11\alpha\beta}\) and \(\Gamma^j_{21\alpha\beta}\) for \(|\alpha| > 0\). So, the assumptions of the lemma imply the assumption of Lemma 2, which completes the proof. \(\square\)

**Lemma 4.** Suppose that

\[ D\left( \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^\alpha \otimes Y, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} \right)(u, v) = 0 \]

and

\[ D\left( \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^\alpha \otimes Y \right)(u, v) = 0 \]

for any \((u, v) \in (V^F R^{m,n})_0 \times T_0 R^m\), any \(i_0 = 1, \ldots, m\) and any vector field \(Y\) on \(R^n\). Then \(D = 0\).

**Proof.** Obviously, the assumptions of the lemma imply the assumptions of Lemma 3, which completes the proof. \(\square\)
Lemma 5. Suppose that
\[
D\left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes \frac{\partial}{\partial y^1} + \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}\right)(u, v) = 0
\]
and
\[
D\left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes \frac{\partial}{\partial y^1}\right)(u, v) = 0
\]
for any \((u, v) \in (V^F R^{m,n})_0 \times T_0 R^m\). Then \(D = 0\).

Proof. Any non-vanishing vector field \(Y\) on \(R^n\) is locally \(\frac{\partial}{\partial y^1}\) modulo a local diffeomorphism \(\varphi : R^m \rightarrow R^n\). There exists a diffeomorphism \(\psi : R^m \rightarrow R^n\) sending \(x^1\) into \(x^1\). Using the invariance of \(D\) with respect to \(\mathcal{F}\mathcal{M}_{m,n}\)-map \(\psi \times \varphi\) we can see that the assumptions of the lemma imply the assumptions of Lemma 4 with non-vanishing \(Y\). Then the regularity of \(D\) implies the assumptions of Lemma 4, which completes the proof. \(\square\)

Lemma 6. Suppose that
\[
D\left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes \frac{\partial}{\partial x^i} \otimes \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}\right)(u, v) = 0
\]
for any \((u, v) \in (V^F R^{m,n})_0 \times T_0 R^m\), and any vector field \(Y\) on \(R^n\). Then \(D = 0\).

Proof. The assumption of the lemma implies the first assumption of Lemma 5. Further, using the invariance of \(D\) with respect to \(\mathcal{F}\mathcal{M}_{m,n}\)-map \((x^1, \ldots, x^m, -y^1 + x^1, y^2, \ldots, y^n)\) we obtain the second assumption of Lemma 5. Finally, Lemma 5 completes the proof. \(\square\)

Lemma 7. Suppose that
\[
D\left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}\right)(u, \frac{\partial}{\partial x^1}(0)) = 0
\]
for any \(u \in (V^F R^{m,n})_0\), and any vector field \(Y\) on \(R^n\). Then \(D = 0\).

Proof. Any vector \(v \in T_0 R^m\) with \(d_0 x^1(v) \neq 0\) is proportional to \(\frac{\partial}{\partial x^1}(0)\) modulo a diffeomorphism \(\psi : R^m \rightarrow R^n\) preserving \(x^1\). Using the invariance of \(D\) with respect to \(\mathcal{F}\mathcal{M}_{m,n}\)-map \(\psi \times \text{id}_{R^n}\) we see that the assumption of the lemma implies the assumption of Lemma 6 with \(d_0 x^1(v) \neq 0\). Then using the regularity of \(D\) we obtain the assumption of Lemma 6, which completes the proof. \(\square\)

Let \(Y\) be a vector field on an \(n\)-manifold \(N\). Define a vector field \(L^D(Y)\) on \(F(N)\) by
\[
L^D(Y)(u) = D\left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes \frac{\partial}{\partial x^i}, \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}\right)(u, \frac{\partial}{\partial x^1}(0)) \in T_u F(N)
\]
for any \(u \in (V^F (R^m \times N))_0 = F(N)\), where we use the obvious identification \(V_u (V^F (R^m \times N)) = T_u F(N)\).
Lemma 8. The $M_{fn}$-natural operator $L^D : T \rightarrow TF$ is linear.

Proof. The $M_{fn}$-naturalness is a simple consequence of the invariance of $D$ with respect to $FM_{m,n}$-maps of the form $id_{Rm} \times id_{Rn}$. Further, by the invariance of $D$ with respect to the base homotheties $t id_{Rm} \times id_{Rn}$ for $t > 0$ we get the homogeneity condition $D(tY)(u) = tD(Y)(u)$. So, the linearity is an immediate consequence of the homogeneous function theorem. □

Lemma 9. We have

$$D\left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes Y \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}\right)\left(u, \frac{\partial}{\partial x^1}(0)\right)$$

$$= \left(\sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes Y \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i}\right)^{F,L_D}(u, \frac{\partial}{\partial x^1}(0))$$

for any $u \in (VF_{m,n})_0$ and $Y \in \mathcal{X}(R^n)$, where $(\Gamma_1, \Gamma_2)^{F,L}$ was defined in Section 2.

Proof. Observe that $v_F = v + Y$ if $\Gamma = \sum_{i=1}^{m} dx^i \otimes \frac{\partial}{\partial x^i} + dx^1 \otimes Y$ and $v = \frac{\partial}{\partial x^1}(0)$. □

Now, using Lemma 7 we see that $D(\Gamma_1, \Gamma_2) = (\Gamma_1, \Gamma_2)^{F,L_D}$. Therefore $\tilde{D} = V^{F,L_D}$ and the proof of Theorem 1 is complete. □

References


