THE SYMMETRY OF UNIT IDEAL STABLE RANGE CONDITIONS

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Abstract. In this paper, we prove that unit ideal-stable range condition is right and left symmetric.

Let $I$ be an ideal of a ring $R$. Following the first author (see [1]), $(a_{11}, a_{12})$ is an $(I)$-unimodular row in case there exists some invertible matrix $A = (a_{ij})_{2 \times 2} \in \text{GL}_2(R, I)$. We say that $R$ satisfies unit $I$-stable range provided that for any $(I)$-unimodular row $(a_{11}, a_{12})$, there exist $u, v \in \text{GL}_1(R, I)$ such that $a_{11}u + a_{12}v = 1$. The condition above is very useful in the study of algebraic $K$-theory and it is more stronger than (ideal)-stable range condition. It is well known that $K_1(R, I) \cong \text{GL}_1(R, I)/V(R, I)$ provided that $R$ satisfies unit $I$-stable range, where $V(R, I) = \{(1 + ab)(1 + ba)^{-1} \mid 1 + ab \in U(R), (1 + ab)(1 + ba)^{-1} \equiv 1 \pmod{I}\}$ (see [2, Theorem 1.2]). In [3], $K_2$ group was studied for commutative rings satisfying unit ideal-stable range and it was shown that $K_2(R, I)$ is generated by $(a, b, c)$, provided that $R$ is a commutative ring satisfying unit $I$-stable range. We refer the reader to [4-10], the papers related to stable range conditions.

In this paper, we investigate representations of general linear groups for ideals of a ring and show that unit ideal-stable range condition is right and left symmetric.

Throughout, all rings are associative with identity. $M_n(R)$ denotes the ring of $n \times n$ matrices over $R$ and $\text{GL}_n(R, I)$ denotes the set $\{ A \in \text{GL}_n(R) \mid A \equiv I_n(\text{mod } M_n(I)) \}$, where $\text{GL}_n(R)$ is the $n$ dimensional general linear group of $R$ and $I_n = \text{diag}(1, \ldots, 1)_{n \times n}$. Write $B_{12}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $B_{21}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$. We always use $[u, v]$ to denote the matrix diag $(u, v)$.

Theorem 1. Let $I$ be an ideal of a ring $R$. Then the following properties are equivalent:

(1) $R$ satisfies unit $I$-stable range.
(2) For any $A \in \text{GL}_2(R, I)$, there exist $u, v, w \in \text{GL}_1(R, I)$ such that $A = [u, v]B_{21}(*)B_{12}(*)B_{21}(-w)$.

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Proof. (1) ⇒ (2) Pick $A = (a_{ij})_{2\times 2} \in GL_2(R, I)$. Then we have $u_1, v_1 \in GL_1(R, I)$ such that $a_{11}u_1 + a_{12}v_1 = 1$. So $a_{11} + a_{12}v_1^{-1} = u_1^{-1}$, hence,

$$\begin{align*}
\begin{pmatrix}
    u_1^{-1} & a_{12} \\
    a_{12} & a_{22}
\end{pmatrix}
\end{align*} = \begin{pmatrix}
    u_1^{-1} & a_{12} \\
    a_{12} & a_{22}
\end{pmatrix} \cdot \begin{pmatrix}
    a_{12} & 0 \\
    0 & a_{22}
\end{pmatrix} \cdot \begin{pmatrix}
    a_{12} & 0 \\
    0 & a_{22}
\end{pmatrix}
$$

Let $v = a_{22} - (a_{21} + a_{22}v_1^{-1})u_1a_{12}$. Then $AB_{21}(v_1u_1^{-1}) = B_{21}((a_{21} + a_{22}v_1^{-1})u_1) \begin{pmatrix}
    u_1^{-1} & a_{12} \\
    0 & v
\end{pmatrix}$. It follows from $A, B_{21}(v_1u_1^{-1}), B_{21}((a_{21} + a_{22}v_1^{-1})u_1) \in GL_2(R)$ that $\begin{pmatrix}
    u_1^{-1} & a_{12} \\
    0 & v
\end{pmatrix} \in GL_2(R)$. In addition, $\begin{pmatrix}
    u_1^{-1} & a_{12} \\
    0 & v
\end{pmatrix} = \begin{pmatrix}
    u_1^{-1} & 0 \\
    0 & v
\end{pmatrix} \begin{pmatrix}
    1 & u_1a_{12} \\
    0 & 1
\end{pmatrix}$ and $\begin{pmatrix}
    u_1^{-1} & a_{12} \\
    0 & v
\end{pmatrix} \in GL_2(R)$. This infers that $[u_1^{-1}, v] \in GL_2(R)$, and so $v \in U(R)$. Set $u = u_1^{-1}$, and $w = v_1u_1^{-1}$. Then $A = [u, v]B_{21}(*)B_{12}(*)B_{21}(-w)$. Clearly, $u, w \in GL_1(R, I)$. From $a_{22} \in 1 + I$ and $a_{12} \in I$, we have $v \in GL_1(R, I)$, as required.

(2) ⇒ (1) For any $(I)$-unimodular row $(a_{11}, a_{12})$, we get $A = (a_{ij})_{2\times 2} \in GL_2(R, I)$. So there exist $u, v, w \in GL_1(R, I)$ such that $A = [u, v]B_{21}(*)B_{12}(*)B_{21}(-w)$. Hence $AB_{21}(u) = [u, v]B_{21}(*)B_{12}(*)$, and then $a_{11} + a_{12}w = u$. That is, $a_{11}u + a_{12}wu^{-1} = 1$. As $u^{-1}, uw \in GL_1(R, I)$, we are done. □

Let $Z$ be the integer domain, $4Z$ the principal ideal of $Z$. Then $1 \in GL_1(Z, 4Z)$, while $-1 \notin GL_1(Z, 4Z)$. But we observe the following fact.

Corollary 2. Let $I$ be an ideal of a ring $R$. Then the following are equivalent:

1. $R$ satisfies unit $I$-stable range.
2. For any $A \in GL_2(R, I)$, there exist $u, v, w \in GL_1(R, I)$ such that $A = [u, v]B_{21}(w)B_{12}(*)B_{21}(*)$.

Proof. (1) ⇒ (2) Given any $A = (a_{ij})_{2\times 2} \in GL_2(R, I)$, then $A^{-1} \in GL_2(R, I)$. By Theorem 1, we have $u, v, w \in GL_1(R, I)$ such that $A^{-1} = [u, v]B_{21}(*)B_{12}(*)B_{21}(-w)$. Thus $A = B_{21}(w)B_{12}(*)B_{21}(*)[u^{-1}, v^{-1}] = [u^{-1}, v^{-1}]B_{21}(vwu^{-1})B_{12}(*)B_{21}(*)$. Clearly, $u^{-1}, v^{-1}, vwu^{-1} \in GL_1(R, I)$, as required.

(2) ⇒ (1) Given any $A = (a_{ij})_{2\times 2} \in GL_2(R, I)$, we have $u, v, w \in GL_1(R, I)$ such that $A^{-1} = [u, v]B_{21}(w)B_{12}(*)B_{21}(*)$, and so $A = B_{21}(*)B_{12}(*)B_{21}(-w)B_{21}(*)B_{12}(*)B_{21}(-vwu^{-1})$. It follows by Theorem 1 that $R$ satisfies unit $I$-stable range. □

Theorem 3. Let $I$ be an ideal of a ring $R$. Then the following are equivalent:

1. $R$ satisfies unit $I$-stable range.
2. For any $A \in GL_2(R, I)$, there exist $u, v, w \in GL_1(R, I)$ such that $A = [u, v]B_{12}(*)B_{21}(*)B_{12}(w)$.
3. For any $A \in GL_2(R, I)$, there exist $u, v, w \in GL_1(R, I)$ such that $A = [u, v]B_{12}(w)B_{21}(*)B_{12}(*)$.

Proof. (1) ⇒ (2) Observe that if $A \in GL_2(R, I)$, then the matrix $P^{-1}AP$ belongs to $GL_2(R, I)$, where $P = \begin{pmatrix}
    0 & 1 \\
    1 & 0
\end{pmatrix}$. Thus the formula in Theorem 1 can be replaced
Thus we have by use of (1)

\[
A = (P[u, v]P^{-1})(PB_{21}(*)P^{-1})(PB_{12}(*)P^{-1})(PB_{21}(-w)P^{-1}).
\]

That is, \( A = [v, u]B_{12}(*)B_{21}(-w) \), as required.

(2) \( \Rightarrow \) (1) For any \((I)\)-unimodular \((a_{11}, a_{12})\) row, \( \begin{pmatrix} a_{12} & * \\ a_{11} & * \end{pmatrix} \in GL_2(R, I) \). So we have \( u, v, w \in GL_4(R, I) \) such that

\[
\begin{pmatrix} a_{12} & * \\ a_{11} & * \end{pmatrix} = [u, v]B_{12}(*)B_{21}(*)B_{12}(-w).
\]

Thus \( a_{11} + a_{12}w = v \), hence, \( a_{11}v^{-1} + a_{12}vw^{-1} = 1 \). Obviously, \( v^{-1}, vw^{-1} \in GL_4(R, I) \), as required.

(2) \( \Leftrightarrow \) (3) is obtained by applying (1) \( \Leftrightarrow \) (2) to the inverse matrix of an invertible matrix \( A \).

Let \( I \) be an ideal of a ring \( R \). We use \( R^{op} \) to denote the opposite ring of \( R \) and use \( I^{op} \) to denote the corresponding ideal of \( I \) in \( R^{op} \).

**Corollary 4.** Let \( I \) be an ideal of a ring \( R \). Then the following are equivalent:

1. \( R \) satisfies unit \( I \)-stable range.
2. \( R^{op} \) satisfies unit \( I^{op} \)-stable range.

**Proof.** (2) \( \Rightarrow \) (1) Construct a map \( \varphi : M_2(R^{op}) \rightarrow M_2(R)^{op} \) by \( \varphi((a_{ij})_{2 \times 2}) = \left( \begin{smallmatrix} a_{ij} \end{smallmatrix} \right)_{2 \times 2}^{T^{op}} \). It is easy to check that \( \varphi \) is a ring isomorphism.

Given any \( A \in GL_2(R, I) \), \( \varphi^{-1}(P^{op}(A^{-1})^{op}(P^{-1})^{op}) \in GL_2(R^{op}, I^{op}) \), where \( P = [1, -1] \). By Theorem 1, there exist \( u^{op}, v^{op}, w^{op} \in GL_1(R^{op}, I^{op}) \) such that \( \varphi^{-1}(P^{op}(A^{-1})^{op}(P^{-1})^{op}) = [u^{op}, v^{op}]B_{21}(*)B_{12}(*)B_{12}(-w) \), whence

\[
P^{-1}A^{-1}P = B_{12}(-w)B_{21}(*)B_{12}([u, v]).
\]

This means that

\[
P^{-1}A = [u^{-1}, v^{-1}]B_{12}(*)B_{21}(*)B_{12}(w).
\]

So \( A = (P[u^{-1}, v^{-1}]P^{-1})(PB_{12}(*)P^{-1})(PB_{21}(*)P^{-1})(PB_{12}(w)P^{-1}) \). Hence \( A = [u^{-1}, v^{-1}]B_{12}(*)B_{21}(*)B_{12}(-w) \). Clearly, \( u^{-1}, v^{-1}, uwv^{-1} \in GL_1(R, I) \). According to Theorem 3, \( R \) satisfies unit \( I \)-stable range.

(1) \( \Rightarrow \) (2) is symmetric.

**Theorem 5.** Let \( I \) be an ideal of a ring \( R \). Then the following are equivalent:

1. \( R \) satisfies unit \( I \)-stable range.
2. For any \((I)\)-unimodular \((a_{11}, a_{12})\) row, there exist \( u, v \in GL_1(R, I) \) such that \( a_{11}u - a_{12}v = 1 \).
3. For any \( A \in GL_2(R, I) \), there exist \( u, v, w \in GL_1(R, I) \) such that

\[
\]

**Proof.** (1) \( \Leftrightarrow \) (2) Observe that \( \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL_2(R, I) \) if and only if

\[
\begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix} \in GL_2(R, I).
\]

Thus \( (a_{11}, -a_{12}) \) is an \((I)\)-unimodular row if and only if so is \((a_{11}, a_{12})\), as required.

(2) \( \Leftrightarrow \) (3) is similar to Theorem 1.

Let \( I \) be an ideal of a ring \( R \). As a consequence of Theorem 5, we prove that \( R \) satisfies unit \( I \)-stable range if and only if for any \( A \in GL_2(R, I) \), there exist
$u, v, w \in GL_1(R, I)$ such that $A = [u, v]B_{12(*)}B_{21(*)}B_{12(w)}$. We say that 
$
\begin{pmatrix}
a_{11} & \ast \\
a_{21} & \ast
\end{pmatrix}
$

is an $(I)$-unimodular column in case there exists $A = (a_{ij})_{2 \times 2} \in GL_2(R, I)$. By the symmetry, we can derive the following.

**Corollary 6.** Let $I$ be an ideal of a ring $R$. Then the following are equivalent:

1. $R$ satisfies unit $I$-stable range.
2. For any $(I)$-unimodular column $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$, there exist $u, v \in GL_1(R, I)$ such that $ua_{11} + va_{21} = 1$.
3. For any $(I)$-unimodular column $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$, there exist $u, v \in GL_1(R, I)$ such that $ua_{11} - va_{21} = 1$.

Suppose that $R$ satisfies unit $I$-stable range. We claim that every element in $I$ is an difference of two elements in $GL_1(R, I)$. For any $a \in I$, we have 
\[
\begin{pmatrix}
1 & a \\
a & 1 + a^2
\end{pmatrix} = B_{21(a)}B_{12(a)} \in GL_2(R, I).
\]
This means that $(1, a)$ is an $(I)$-unimodular. So we have some $u, v \in GL_1(R, I)$ such that $u + av = 1$. Hence $a = v^{-1} - uv^{-1}$, as asserted.

Let $I$ be an ideal of a ring $R$. Define $QM_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + c = b + d, a, b, c, d \in R \right\}$ and $QM_2(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + c = b + d, a, b, c, d \in I \right\}$. Define 
\[
QM_T^2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + b = c + d, a, b, c, d \in R \right\} \quad \text{and} \quad QM_T^2(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + b = c + d, a, b, c, d \in I \right\}.
\]
As an application of the symmetry of unit ideal-stable range condition, we derive the following.

**Theorem 7.** Let $I$ be an ideal of a ring $R$. Then the following are equivalent:

1. $R$ satisfies unit $I$-stable range.
2. $QM_2(R)$ satisfies unit $QM_2(I)$-stable range.
3. $QM_T^2(R)$ satisfies unit $QM_T^2(I)$-stable range.

**Proof.** (1) $\Rightarrow$ (2) Let $TM_2(R)$ denote the ring of all $2 \times 2$ lower triangular matrices over $R$, and let $TM_2(I)$ denote the ideal of all $2 \times 2$ lower triangular matrices over $I$.

If $(A_{11}, A_{12})$, where $A_{11} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ and $A_{12} = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}$, is a unimodular row, then $(a_{11}, b_{11})$ and $(a_{22}, b_{22})$ are unimodular rows, and so $a_{11}u_1 + b_{11}v_1 = 1$ and $a_{22}u_2 + b_{22}v_2 = 1$ for some $u_1, u_2, v_1, v_2 \in GL_1(R, I)$. Then there are matrices 
$U = \begin{pmatrix} u_1 \\ ** \\ u_2 \end{pmatrix}, V = \begin{pmatrix} v_1 \\ ** \\ v_2 \end{pmatrix}$ such that $A_{11}U + A_{12}V = I$. Now we construct a map $\psi : QM_2(R) \to TM_2(R)$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a + c \\ c & d - c \end{pmatrix}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in QM_2(R)$. For any $\begin{pmatrix} x \\ z \end{pmatrix} \in TM_2(R)$, we have 
$\psi\left( \begin{pmatrix} x & z \\ z & y + z \end{pmatrix} \right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
Thus we have \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R, I) \) such that \( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \). Therefore \( a + bu \in GL_1(R, I) \) and \( u \in GL_1(R, I) \), as desired.

(1) \( \Rightarrow \) (3) Clearly, we have an anti-isomorphism \( \psi : QT^2M_2(R) \to QM_2(R^{op}) \) given by \( \psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^{op} & c^{op} \\ b^{op} & d^{op} \end{pmatrix} \) for any \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in QT^2M_2(R) \). Hence \( QT^2M_2(R) \cong (QM_2(R^{op}))^{op} \). Likewise, we have \( QT^2M_2(I) \cong (QM_2(I^{op}))^{op} \). Thus we complete the proof by Corollary 4.

It follows by Theorem 7 that \( R \) satisfies unit 1-stable range if and only if so does \( QM_2(R) \) if and only if so does \( QM_2^2(R) \).

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References


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