ON THE STABILITY OF THE SOLUTIONS OF CERTAIN FIFTH ORDER NON-AUTONOMOUS DIFFERENTIAL EQUATIONS

A. I. Sadek

Abstract. Our aim in this paper is to present sufficient conditions under which all solutions of (1.1) tend to zero as $t \to \infty$.

1. Introduction

The equation studied here is of the form

$$(1.1) \quad x^{(5)} + f(t, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})x^{(4)} + \phi(t, \dddot{x}, \dddot{x}) + \psi(t, \dddot{x}) + g(t, \dot{x}) + e(t)h(x) = 0,$$

where $f, \phi, \psi, g, e$ and $h$ are continuous functions which depend only on the displayed arguments, $\phi(0, 0) = \psi(0, 0) = g(0, 0) = h(0) = 0$. The dots indicate differentiation with respect to $t$ and all solutions considered are assumed real.

Chukwu [3] discussed the stability of the solutions of the differential equation

$$x^{(5)} + ax^{(4)} + f_2(\dot{x}') + c\dot{x} + f_4(\dot{x}) + f_5(x) = 0.$$

In [1], sufficient conditions for the uniform global asymptotic stability of the zero solution of the differential equation

$$x^{(5)} + f_1(\dddot{x}')x^{(4)} + f_2(\dddot{x}') + f_3(\dddot{x}) + f_4(\dot{x}) + f_5(x) = 0$$

were investigated.


$$x^{(5)} + \phi(x, \dot{x}, \dddot{x}, x^{(4)})x^{(4)} + b\dddot{x} + h(x, \dot{x}) + g(x, \dddot{x}) + f(x) = 0,$$

$$x^{(5)} + \phi(x, \dot{x}, \dddot{x}, x^{(4)})x^{(4)} + \psi(\dddot{x}, \dddot{x}) + h(\dddot{x}) + g(\dddot{x}) + f(x) = 0.$$

We shall present here sufficient conditions, which ensure that all solutions of (1.1) tend to zero as $t \to \infty$. Many results have been obtained on asymptotic properties of non-autonomous equations of third order in Swich [5], Hara [4] and Yamamoto [8].

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2. Assumptions and theorems

We shall state the assumptions on the functions $f, \phi, \psi, g, e$ and $h$ appeared in the equation (1.1).

Assumptions:

(1) $h(x)$ is a continuously differentiable function in $\mathbb{R}^3$, and $e(t)$ is a continuously differentiable function in $\mathbb{R}^+ = [0, \infty)$.

(2) The function $g(t, y)$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^1$, and for the function $g(t, y)$ there exist non-negative functions $d(t), g_0(y)$ and $g_1(y)$ which satisfy the inequalities

$$d(t)g_0(y) \leq g(t, y) \leq d(t)g_1(y)$$

for all $(t, y) \in \mathbb{R}^+ \times \mathbb{R}^1$. The function $d(t)$ is continuously differentiable in $\mathbb{R}^+$. Let

$$\tilde{g}(y) \equiv \frac{1}{2}\{g_0(y) + g_1(y)\},$$

$\tilde{g}(y)$ and $\tilde{g}'(y)$ are continuous in $\mathbb{R}^1$.

(3) The function $\psi(t, z)$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^1$. For the function $\psi(t, z)$ there exist non-negative functions $c(t), \psi_0(z)$ and $\psi_1(z)$ which satisfy the inequalities

$$c(t)\psi_0(z) \leq \psi(t, z) \leq c(t)\psi_1(z)$$

for all $(t, z) \in \mathbb{R}^+ \times \mathbb{R}^1$. The function $c(t)$ is continuously differentiable in $\mathbb{R}^+$. Let

$$\tilde{\psi}(z) \equiv \frac{1}{2}\{\psi_0(z) + \psi_1(z)\},$$

$\tilde{\psi}(z)$ is continuous in $\mathbb{R}^1$.

(4) The function $\phi(t, z, w)$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^2$. For the function $\phi(t, z, w)$ there exist non-negative functions $b(t), \phi_0(z, w)$ and $\phi_1(z, w)$ which satisfy the inequalities

$$b(t)\phi_0(z, w) \leq \phi(t, z, w) \leq b(t)\phi_1(z, w)$$

for all $(t, z, w) \in \mathbb{R}^+ \times \mathbb{R}^2$. The function $b(t)$ is continuously differentiable in $\mathbb{R}^+$. Let

$$\tilde{\phi}(z, w) \equiv \frac{1}{2}\{\phi_0(z, w) + \phi_1(z, w)\},$$

$\tilde{\phi}(z, w)$ and $\partial \tilde{\phi}(z, w)/\partial z$ are continuous in $\mathbb{R}^2$.

(5) The function $f(t, y, z, w)$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^3$, and for the function $f(t, y, z, w)$ there exist functions $a(t), f_0(y, z, w)$ and $f_1(y, z, w)$ which satisfy the inequality

$$a(t)f_0(y, z, w) \leq f(t, y, z, w) \leq a(t)f_1(y, z, w)$$

for all $(t, y, z, w) \in \mathbb{R}^+ \times \mathbb{R}^3$. Further the function $a(t)$ is continuously differentiable in $\mathbb{R}^+$, and let

$$\tilde{f}(y, z, w) \equiv \frac{1}{2}\{f_0(y, z, w) + f_1(y, z, w)\},$$

$\tilde{f}(y, z, w)$ is continuous in $\mathbb{R}^3$. 
Theorem 1. Further to the basic assumptions (1)-(5), suppose the following \((\epsilon, \epsilon_1, \ldots, \epsilon_5\) are small positive constants):

(i) \(A \geq a(t) \geq a_0 \geq 1, B \geq b(t) \geq b_0 \geq 1, C \geq c(t) \geq c_0 \geq 1,\)
\[D \geq d(t) \geq d_0 \geq 1, E \geq e(t) \geq e_0 \geq 1, \text{ for } t \in \mathbb{R}^+;\]

(ii) \(\alpha_1, \ldots, \alpha_5\) are some constants satisfying
\[\alpha_1 > 0, \; \alpha_1 \alpha_2 - \alpha_3 > 0, \; (\alpha_1 \alpha_2 - \alpha_3) \alpha_3 - (\alpha_1 \alpha_4 - \alpha_5) \alpha_1 > 0,\]
\[\delta_0 := (\alpha_1 \alpha_3 - \alpha_2 \alpha_5)(\alpha_1 \alpha_2 - \alpha_3) - (\alpha_1 \alpha_4 - \alpha_5)^2 > 0, \; \alpha_5 > 0;\]
\[\Delta_1 := \frac{(\alpha_4 \alpha_3 - \alpha_2 \alpha_5)(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} - \{\alpha_1 d(t)^{\Delta_2}(y) - \alpha_5\} > 2\epsilon \alpha_2,\]

for all \(y\) and all \(t \in \mathbb{R}^+;\)

\[\Delta_2 := \frac{\alpha_4 \alpha_3 - \alpha_2 \alpha_5}{\alpha_1 \alpha_4 - \alpha_5} - \frac{(\alpha_1 \alpha_4 - \alpha_5) \gamma d(t)}{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)} - \frac{\epsilon}{\alpha_1} > 0,\]

for all \(y\) and all \(t \in \mathbb{R}^+\), where

\[\gamma := \begin{cases} \frac{\tilde{g}(y)}{y}, & y \neq 0 \\ \tilde{g}(0), & y = 0. \end{cases}\]

(iii) \(\epsilon_0 \leq \tilde{f}(y, z, w) - \alpha_1 \leq \epsilon_1\) for all \(z\) and \(w\).

(iv) \(\tilde{\phi}(0, 0) = 0, 0 \leq \tilde{\phi}(z, w)/w - \alpha_2 \leq \epsilon_2\) \((w \neq 0), \; \tilde{\phi}(z, w) \leq 0.\)

(v) \(\tilde{\psi}(0) = 0, 0 \leq \tilde{\psi}(z)/z - \alpha_3 \leq \epsilon_3\) \((z \neq 0).\)

(vi) \(\tilde{g}(0) = 0, \; \tilde{g}(y)/y \geq \frac{\delta_0}{4\epsilon}, \; (y \neq 0), \alpha_4 - \tilde{g}'(y) \leq \epsilon_4\) for all \(y\) and
\[\tilde{g}'(y) - \tilde{g}(y)/y \leq \alpha_5 \delta_0 / 4 \alpha_2^2 (\alpha_1 \alpha_2 - \alpha_3) \; (y \neq 0).\]

(vii) \(h(0) = 0, \; h(x) \; \text{sgn} \; x > 0 (x \neq 0), \; H(x) \equiv \int_0^x h(\xi)d\xi \to \infty \text{ as } |x| \to \infty\)

and
\[0 \leq \alpha_5 - h'(x) \leq \epsilon_5 \; \text{ for all } \; x.\]

(viii) \(\int_0^\infty \beta_0(t)dt < \infty, \; e'(t) \to 0 \text{ as } t \to \infty, \text{ where}\)
\[\begin{aligned}
\beta_0(t) &:= b'_+ (t) + c'_+ (t) + |d'(t)| + |e'(t)|, \\
b'_+ (t) &:= \max\{b'(t), 0\} \; \text{ and } \; c'_+ (t) := \max\{e'(t), 0\}.
\end{aligned}\]

(ix) \(|A(f_1 - f_0) + B(\phi_1 - \phi_0) + C(\psi_1 - \psi_0) + D(g_1 - g_0)| \leq \Delta(y^2 + z^2 + w^2 + u^2)^{1/2},\]

where \(\Delta\) is a non-negative constant.
Then every solution of (1.1) satisfies
\[ x(t) \to 0, \dot{x}(t) \to 0, \quad \ddot{x}(t) \to 0, \quad x^{(4)}(t) \to 0 \quad \text{as} \quad t \to \infty. \]

Next, considering the equation
\[ (2.5) \quad x^{(5)} + a(t)f(\dot{x}, \ddot{x})x^{(4)} + b(t)\phi(\dot{x}, \ddot{x}) + c(t)\psi(\ddot{x}) + d(t)g(\dot{x}) + e(t)h(x) = 0, \]
we can take the function \( g(y) \) in place of \( g_0(y) \) and \( g_1(y) \); the function \( \phi(y, z) \) in place of \( \phi_0(y, z) \) and \( \phi_1(y, z) \); the function \( \psi(z) \) in place of \( \psi_0(z) \) and \( \psi_1(z) \), and the function \( f(y, z, w) \) in place of \( f_0(y, z, w) \) and \( f_1(y, z, w) \) in the Assumptions (2) – (5). Thus in this case the functions \( \tilde{g}(y), \tilde{\phi}(y, z), \tilde{\psi}(z), \tilde{f}(y, z, w) \) coincide with \( g(x, y), \phi(y, z), \psi(z), f(y, z, w), g'(y), h'(x), \frac{\partial}{\partial z}\phi(y, z) \) and that these functions satisfy the following conditions:

(i) \( A \geq a(t) \geq a_0 \geq 1, \quad B \geq b(t) \geq b_0 \geq 1, \quad C \geq c(t) \geq c_0 \geq 1, \quad D \geq d(t) \geq d_0 \geq 1, \quad E \geq e(t) \geq e_0 \geq 1 \) for \( t \in \mathbb{R}^+ \).

(ii) \( \alpha_1, \ldots, \alpha_5 \) are some constants satisfying
\[ \alpha_1 > 0, \quad \alpha_1 \alpha_2 - \alpha_3 > 0, \quad (\alpha_1 \alpha_2 - \alpha_3)\alpha_3 - (\alpha_1 \alpha_4 - \alpha_5)\alpha_1 > 0, \]
\[ \delta_0 := (\alpha_4 \alpha_3 - \alpha_2 \alpha_5)(\alpha_1 \alpha_2 - \alpha_3) - (\alpha_1 \alpha_4 - \alpha_5)^2 > 0, \quad \alpha_5 > 0; \]
\[ \Delta_1 := \frac{(\alpha_4 \alpha_3 - \alpha_2 \alpha_5)(\alpha_1 \alpha_2 - \alpha_3) - (\alpha_1 d(t) g'(y) - \alpha_5)}{\alpha_1 \alpha_4 - \alpha_5} = \frac{\alpha_1 d(t)}{\alpha_1} > 0, \]
for all \( y \) and all \( t \in \mathbb{R}^+; \)
\[ \Delta_2 := \frac{\alpha_4 \alpha_3 - \alpha_2 \alpha_5}{\alpha_1 \alpha_4 - \alpha_5} - \frac{(\alpha_1 \alpha_4 - \alpha_5)\gamma d(t)}{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)} = \frac{\epsilon}{\alpha_1} > 0, \]
for all \( y \) and all \( t \in \mathbb{R}^+ \), where
\[ \gamma := \begin{cases} g(y)/y, & y \neq 0 \\ g'(0), & y = 0. \end{cases} \]

(iii) \( \epsilon_0 \leq f(y, z, w) - \alpha_1 \leq \epsilon_1, \) for all \( z \) and \( w. \)

(iv) \( \phi(0, 0) = 0, \quad 0 \leq \phi(z, w)/w - \alpha_2 \leq \epsilon_2 (w \neq 0), \quad \frac{\partial}{\partial z}\phi(z, w) \leq 0. \)

(v) \( \psi(0) = 0, \quad 0 \leq \psi(z)/z - \alpha_3 \leq \epsilon_3 \) (\( z \neq 0 \)).

(vi) \( g(0) = 0, \quad g(y)/y \geq \frac{E_{\alpha_4} \epsilon_4}{\alpha_4} (y \neq 0), \quad |\alpha_4 - g'(y)| \leq \epsilon_4 \) for all \( y \) and
\[ g'(y) - g(y)/y \leq \alpha_5 \delta_0/D\alpha_3^2(\alpha_1 \alpha_2 - \alpha_3) (y \neq 0). \]
(vi) \( h(0) = 0, \ h(x) \ \text{sgn } x > 0 \ (x \neq 0), \ H(x) \equiv \int_0^x h(\xi) \, d\xi \to \infty \) as \( |x| \to \infty \)

and

\[ 0 \leq \alpha_5 - h'(x) \leq \epsilon_5 \quad \text{for all } x. \]

(vii) \( \int_0^\infty \beta_0(t) \, dt < \infty, \ e'(t) \to 0 \) as \( t \to \infty \), where

\[
\begin{align*}
\beta_0(t) :&= b'_+(t) + c'_+(t) + |d''(t)| + |e'(t)|, \\
b'_+(t) :&= \max\{b'(t), 0\} \quad \text{and} \quad c'_+(t) := \max\{c'(t), 0\}.
\end{align*}
\]

Then every solution of (2.5) satisfies

\[ x(t), \dot{x}(t), \ddot{x}(t), \dddot{x}(t), x^{(4)}(t) \to 0 \quad \text{as } t \to \infty. \]

3. The Lyapunov function \( V_0(t, x, y, z, w, u) \)

We consider, in place of (1.1), the equivalent system

\[
\begin{align*}
\dot{x} &= y, \quad \dot{y} = z, \quad \dot{z} = w, \quad \dot{w} = u, \\
\dot{u} &= -f(t, y, z, w)u - \phi(t, z, w) - \psi(t, z) - g(t, y) - e(t)h(x).
\end{align*}
\]

The proof of the theorem is based on some fundamental properties of a continuously differentiable function \( V_0 = V_0(t, x, y, z, w, u) \) defined by

\[
2V_0 = u^2 + 2\alpha_1 uw + 2\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} u z + 2\delta y u + 2b(t) \int_0^u \tilde{\phi}(z, \omega) \, d\omega \\
+ \left\{ \alpha_1^2 - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \right\} w^2 + 2\alpha_3 + 2\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta \right\} w z \\
+ 2\alpha_1 \delta w y + 2d(t)w\tilde{g}(y) + 2e(t)wh(x) + 2\alpha_1c(t) \int_0^z \tilde{\psi}(\zeta) \, d\zeta \\
+ \left\{ \frac{\alpha_2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_4 - \alpha_1 \delta \right\} z^2 + 2\delta \alpha_2 y z + 2\alpha_1 d(t)z\tilde{g}(y) - 2\alpha_5 y z \\
+ 2\alpha_1c(t)zh(x) + 2\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}d(t) \int_0^z \tilde{g}(\eta) \, d\eta + (\delta \alpha_3 - \alpha_1 \alpha_3) y^2 \\
+ 2\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} e(t) y h(x) + 2\delta e(t) \int_0^x h(\xi) \, d\xi,
\end{align*}
\]

where

\[
\delta := \frac{\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{(\alpha_1\alpha_4 - \alpha_5)} + \epsilon.
\]

The properties of the function \( V_0 = V_0(t, x, y, z, w, u) \) are summarized in Lemma 1 and Lemma 2.
Lemma 1. Subject to the hypotheses (i)-(vii) of the theorem, there are positive constants \( D_7 \) and \( D_8 \) such that

\[
D_7 \{ H(x) + y^2 + z^2 + w^2 + u^2 \} \leq V_0 \leq D_8 \{ H(x) + y^2 + z^2 + w^2 + u^2 \}.
\]

Proof. We observe that \( 2V_0 \) in (3.2) can be rearranged as

\[
2V_0 = \left\{ u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y \right\}^2 + \frac{\alpha_4(\alpha_1\alpha_4 - \alpha_5)}{(\alpha_1\alpha_2 - \alpha_3)\gamma d(t)} \left( \frac{\alpha_1\alpha_2 - \alpha_3}{\alpha_1\alpha_4 - \alpha_5} \right) e(t) h(x) \nonumber \\
+ \frac{\alpha_1\alpha_2 - \alpha_3}{\alpha_1\alpha_4 - \alpha_5} \gamma d(t) y + \frac{\alpha_1}{\alpha_4} \gamma d(t) z + \frac{1}{\alpha_4} \gamma d(t) w \right\}^2 + \frac{\alpha_4\delta_0}{(\alpha_1\alpha_4 - \alpha_5)^2} \left( z + \frac{\alpha_5}{\alpha_4} y \right)^2 + \Delta_2 (w + \alpha_1 z)^2 \\
+ 2e \left( \frac{\alpha_4\alpha_3 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} \right) y z + \sum_{i=1}^{4} S_i,
\]

where

\[
S_1 := 2\delta e(t) \int_0^x h(\xi) d\xi - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{(\alpha_1\alpha_4 - \alpha_5)\gamma d(t)} e^2(t) h^2(x), \\
S_2 := \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)\gamma d(t)}{\alpha_1\alpha_4 - \alpha_5} \left\{ 2 \int_0^y \tilde{g}(\eta) d\eta - \tilde{g}(y) \right\} + \left( \frac{\delta\alpha_3 - \alpha_1\alpha_5}{\alpha_4(\alpha_1\alpha_4 - \alpha_5)^2} - \delta^2 \right) y^2, \\
S_3 := \frac{e}{\alpha_1} w^2 + 2b(t) \int_0^z \tilde{\phi}(z, \omega) d\omega - \alpha_2 w^2, \\
S_4 := 2\alpha_1 c(t) \int_0^z \tilde{\psi}(\zeta) d\zeta - \alpha_1\alpha_3 z^2.
\]

It can be seen from the estimates arising in the course of the proof of [2; Lemma 1] that

\[
2\alpha_5 \int_0^x h(\xi) d\xi - \tilde{h}^2(x) \geq 0, \\
S_1 \geq 2\epsilon e_0 \int_0^x h(\xi) d\xi.
\]

Since

\[
y\tilde{g}(y) = \int_0^y \tilde{g}(\eta) d\eta + \int_0^y \eta \tilde{g}'(\eta) d\eta,
\]
we have

\[ S_2 = \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)d(t)}{\alpha_1 \alpha_4 - \alpha_5} \left\{ 2 \int_0^y \tilde{g}(\eta) \, d\eta - y\tilde{g}(y) \right\} + \left[ \frac{\alpha_5 \delta_0}{\alpha_4(\alpha_1 \alpha_4 - \alpha_5)} - \epsilon \left\{ \epsilon + \frac{2 \alpha_5(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} - \alpha_3 \right\} \right] y^2 \]

\[ = \int_0^y \left[ \frac{2 \alpha_5 \delta_0}{\alpha_4(\alpha_1 \alpha_4 - \alpha_5)} - \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)d(t)}{\alpha_1 \alpha_4 - \alpha_5} \left\{ \tilde{g}'(\eta) - \tilde{g}(\eta) \right\} - 2 \epsilon \left\{ \epsilon + \frac{2 \alpha_5(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} - \alpha_3 \right\} \right] \eta \, d\eta \]

\[ \geq \int_0^y \left[ \frac{\alpha_5 \delta_0}{\alpha_4(\alpha_1 \alpha_4 - \alpha_5)} - 2 \epsilon \left\{ \epsilon + \frac{2 \alpha_5(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} - \alpha_3 \right\} \right] \eta \, d\eta , \]

by (vi) and (i)

\[ \geq \frac{\alpha_5 \delta_0}{4 \alpha_4(\alpha_1 \alpha_4 - \alpha_5)} y^2 , \]

provided that

\[ (3.7) \quad \frac{\alpha_5 \delta_0}{4 \alpha_4(\alpha_1 \alpha_4 - \alpha_5)} \geq \epsilon \left\{ \epsilon + \frac{2 \alpha_5(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} - \alpha_3 \right\} , \]

which we now assume. From (i), (iv) and (v) we find

\[ S_3 = \frac{\epsilon}{\alpha_1} w^2 + 2 b(t) \int_0^w \tilde{\phi}(z, \omega) d\omega = \epsilon \alpha_2 w^2 \]

\[ \geq \frac{\epsilon}{\alpha_1} w^2 + 2 \int_0^w \left\{ \frac{\tilde{\phi}(z, \omega)}{\omega} - \alpha_2 \right\} \omega \, d\omega \geq \frac{\epsilon}{\alpha_1} w^2 , \]

\[ S_4 = 2 \alpha_1 c(t) \int_0^z \tilde{\psi}(\zeta) d\zeta - \alpha_3 z^2 \geq 2 \alpha_1 \int_0^z \left\{ \frac{\tilde{\psi}(\zeta)}{\zeta} - \alpha_3 \right\} \zeta \, d\zeta \geq 0 . \]

On gathering all of these estimates into (3.5) we deduce

\[ 2 V_0 \geq \left\{ w + \alpha_1 w + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} z + \delta y \right\}^2 + \frac{\alpha_4 \delta_0}{(\alpha_1 \alpha_4 - \alpha_5)^2} (z + \alpha_5 y)^2 \]

\[ + \Delta_2(w + \alpha_1 z)^2 + 2 \epsilon \alpha_0 \int_0^x h(\xi) \, d\xi + \frac{\alpha_5 \delta_0}{4 \alpha_4(\alpha_1 \alpha_4 - \alpha_5)} y^2 + \frac{\epsilon}{\alpha_1} \]

\[ + 2 \epsilon \left( \frac{\alpha_4 \alpha_3 - \alpha_2 \alpha_5}{\alpha_1 \alpha_4 - \alpha_5} \right) y z , \]

by (ii) and (vi). It is clear that there exist sufficiently small positive constants \( D_1, \ldots, D_5 \) such that

\[ 2 V_0 \geq D_1 H(x) + 2 D_2 y^2 + 2 D_3 z^2 + D_4 w^2 + D_5 u^2 + 2 \epsilon \left( \frac{\alpha_4 \alpha_3 - \alpha_2 \alpha_5}{\alpha_1 \alpha_4 - \alpha_5} \right) y z . \]
Let
\[ S_5 := D_2 y^2 + 2\epsilon \left( \frac{\alpha_4 \alpha_3 - \alpha_2 \alpha_5}{\alpha_1 \alpha_4 - \alpha_5} \right) y z + D_3 z^2. \]

By using the inequality \(|yz| \leq \frac{1}{2}(y^2 + z^2)|\), we obtain
\[ S_5 \geq D_2 y^2 + D_3 z^2 - \epsilon \left( \frac{\alpha_4 \alpha_3 - \alpha_2 \alpha_5}{\alpha_1 \alpha_4 - \alpha_5} \right) (y^2 + z^2) \geq D_6 (y^2 + z^2), \]
for some \(D_6 > 0, D_6 = \frac{1}{2} \min\{D_2, D_3\}\), if
\begin{equation}
(3.8) \quad \epsilon \leq (\alpha_1 \alpha_4 - \alpha_5)/(2(\alpha_4 \alpha_3 - \alpha_2 \alpha_5)) \min\{D_2, D_3\},
\end{equation}
which we also assume. Then
\[ 2V_0 \geq D_1 H(x) + (D_2 + D_6) y^2 + (D_3 + D_6) z^2 + D_4 w^2 + D_5 u^2. \]

Consequently there exists a positive constant \(D_7\) such that
\[ V_0 \geq D_7 \{H(x) + y^2 + z^2 + w^2 + u^2\}, \]
provided \(\epsilon\) is so small that (3.7) and (3.8) hold. From (i), (iv), (v), (vi) and (3.6) we can verify that there exists a positive constant \(D_8\) satisfying
\[ V_0 \leq D_8 \{H(x) + y^2 + z^2 + w^2 + u^2\}. \]
Thus (3.4) follows. \(\square\)

**Lemma 2.** Assume that all conditions of the theorem hold. Then there exist positive constants \(D_i\) \((i = 11, 12)\) such that
\begin{equation}
(3.9) \quad \dot{V}_0 \leq -D_{12}(y^2 + z^2 + w^2 + u^2) + D_{11}\beta_0 V_0.
\end{equation}
Proof. From (3.2) and (3.1) it follows that (for \( y, z, w \neq 0 \))

\[
\frac{d}{dt} V_0 \leq -u^2 \{ a(t) \tilde{f}(y, z, w) - \alpha_1 \} \\
- w^2 \left[ \frac{b(t) \tilde{\phi}(z, w)}{w} \right] - \left\{ \alpha_3 + \frac{\alpha_1 \alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} - \delta \right\} \\
- z^2 \left[ \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3) c(t) \tilde{\psi}(z)}{\alpha_1 \alpha_4 - \alpha_5} - \left\{ \delta \alpha_2 + a_1 d(t) \tilde{g}'(y) - \alpha_5 \right\} \right] \\
- y^2 \left\{ \delta d(t) \tilde{\psi}(y) \right\} - \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} e(t) h'(x) \\
+ w b(t) \int_0^w \frac{\partial}{\partial z} \tilde{\phi}(z, \omega) \, d\omega - a_1 w \omega a(t) \{ \tilde{f}(y, z, w) - \alpha_1 \} - u z c(t) \left\{ \tilde{\psi}(z) \right\} \frac{1}{z} - \alpha_3 \\
- \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} u z a(t) \{ \tilde{f}(y, z, w) - \alpha_1 \} \\
- \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} w z b(t) \left\{ \tilde{\phi}(z, w) \right\} \frac{1}{w} - \alpha_2 \} \\
- \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} w z d(t) \{ \tilde{\phi}(z, \omega) \} - \delta y w a(t) \{ \tilde{f}(y, z, w) - \alpha_1 \} - y w e(t) \{ \alpha_5 - h'(x) \} \\
- \delta y w b(t) \left\{ \tilde{\phi}(z, w) \right\} \frac{1}{w} - \alpha_2 \} - \alpha_1 y w e(t) \{ \alpha_5 - h'(x) \} - \delta y w e(t) \left\{ \tilde{\psi}(z) \right\} \frac{1}{z} - \alpha_3 \} \\
+ \left\{ \alpha_2 y w + \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} u z + \alpha_1 y w \right\} \{ 1 - a(t) \} \\
+ \left\{ \frac{\alpha_2 y w (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} w z + \delta \alpha_2 y w \right\} \{ 1 - b(t) \} + (\alpha_3 w z + \delta \alpha_3 y z) \{ 1 - c(t) \} \\
- \frac{\alpha_4 w z \{ 1 - d(t) \} - (\alpha_5 y w + \alpha_1 \alpha_5 y z) \{ 1 - e(t) \}}{1} \\
+ \frac{1}{2} \{ a(t) (f_1 - f_0) + b(t) (\phi_1 - \phi_0) + c(t) (\psi_1 - \psi_0) + d(t) (g_1 - g_0) \} \\
\} \\
(3.10) \\
\{ u + a_1 w + \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} z + \delta y \} + \frac{\partial V_0}{\partial t}.
\]

By (i) and (iii), \( a(t) \tilde{f}(y, z, w) - \alpha_1 \geq \epsilon_0 \). From (i), (iv) and (3.3) we have (for \( w \neq 0 \))

\[
\alpha_1 \left[ \frac{b(t) \tilde{\phi}(z, w)}{w} \right] \frac{1}{w} - \left\{ \alpha_3 + \frac{\alpha_1 \alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} - \delta \right\} \\
\geq \alpha_1 \left\{ \frac{\tilde{\phi}(z, w)}{w} - \alpha_2 \right\} + \left\{ \alpha_1 \alpha_2 - \alpha_3 + \delta - \frac{\alpha_1 \alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \right\} \geq \epsilon.
\]

By using (i), (v), (3.3) and (2.2) we obtain (for \( z \neq 0 \))

\[
\frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3) c(t) \tilde{\psi}(z)}{\alpha_1 \alpha_4 - \alpha_5} \frac{1}{z} - \left\{ \delta \alpha_2 + a_1 d(t) \tilde{g}'(y) - \alpha_5 \right\} \\
\geq \left\{ \frac{\alpha_4 \alpha_3 - \alpha_2 \alpha_5 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \right\} \left\{ a_1 d(t) \tilde{g}'(y) - \alpha_5 \right\} - \epsilon \alpha_2 \geq \epsilon \alpha_2.
\]
From (i), (vi) and (vii) we find (for \( y \neq 0 \))

\[
\delta d(t) \frac{\bar{y}(y)}{y} = \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} e(t) h'(x) \\
\geq \alpha_4 E + \frac{\alpha_4 E(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \left\{ \alpha_5 - h'(x) \right\} \geq \alpha_4 E.
\]

Therefore, the first four terms involving \( u^2, w^2, z^2 \) and \( y^2 \) in (3.10) are majorizable by

\[-(\epsilon_0 u^2 + \epsilon w^2 + \epsilon \alpha_2 z^2 + \epsilon \alpha_4 E y^2).\]

Let \( R(t, x, y, z, w, u) \) denote the sum of the remaining terms in (3.10). By using hypotheses (i), (iii)–(vii) and the inequalities

\[
|uw| \leq \frac{1}{2}(u^2 + w^2), \quad |uz| \leq \frac{1}{2}(u^2 + z^2), \quad |uy| \leq \frac{1}{2}(u^2 + y^2), \\
|wz| \leq \frac{1}{2}(w^2 + z^2), \quad |wy| \leq \frac{1}{2}(w^2 + y^2), \quad |yz| \leq \frac{1}{2}(y^2 + z^2);
\]

it follows that

\[
|R(t, x, y, z, w, u)| \leq D_9(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5)(y^2 + z^2 + w^2 + u^2) \\
+ \frac{1}{2}\left\{ a(t)(f_1 - f_0) + b(t)(\phi_1 - \phi_0) + c(t)(\psi_1 - \psi_0) + d(t)(g_1 - g_0) \right\} \\
\left\{ u + \alpha_1 w + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} z + \delta y \right\} + \frac{\partial V_0}{\partial t},
\]

for some \( D_9 > 0 \). Thus, after substituting in (3.10), one obtains

\[
\dot{V}_0 \leq -(\epsilon_0 u^2 + \epsilon w^2 + \epsilon \alpha_2 z^2 + \epsilon \alpha_4 E y^2) + D_9(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5)(y^2 + z^2 + w^2 + u^2) \\
+ \frac{1}{2}\left\{ a(t)(f_1 - f_0) + b(t)(\phi_1 - \phi_0) + c(t)(\psi_1 - \psi_0) + d(t)(g_1 - g_0) \right\} \\
\left\{ u + \alpha_1 w + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} z + \delta y \right\} + \frac{\partial V_0}{\partial t} \\
\leq -\frac{1}{2}\min\{\epsilon_0, \epsilon, \epsilon \alpha_2, \epsilon \alpha_4 E\}(y^2 + z^2 + w^2 + u^2) \\
+ \frac{1}{2}\left\{ a(t)(f_1 - f_0) + b(t)(\phi_1 - \phi_0) + c(t)(\psi_1 - \psi_0) + d(t)(g_1 - g_0) \right\} \\
\left\{ u + \alpha_1 w + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} z + \delta y \right\} + \frac{\partial V_0}{\partial t},
\]

provided that

\[
D_9(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5) \leq \frac{1}{2}\min\{\epsilon_0, \epsilon, \epsilon \alpha_2, \epsilon \alpha_4 E\}.
\]
Now we assume that \( D_9 \) and \( \epsilon_1, \ldots, \epsilon_5 \) are so small that (3.12) holds. The case \( y, z, w = 0 \) is trivially dealt with. From (3.2) we find

\[
\frac{\partial V_0}{\partial t} = b'(t) \int_0^w \tilde{\omega}(z, \omega) \, d\omega + \alpha_1 c'(t) \int_0^x \tilde{\omega}(\zeta) \, d\zeta \\
+ d'(t) \left\{ \omega \tilde{\varphi}(y) + \alpha_1 z \tilde{\varphi}(y) + \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \int_0^y \tilde{\varphi}(\eta) \, d\eta \right\} \\
+ e'(t) \left\{ \omega h(x) + \alpha_1 z h(x) + \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} y h(x) + 2\delta \int_0^x h(\xi) \, d\xi \right\}.
\]

From (iv), (v), (vi), (3.6) and (3.4) we can find a positive constant \( D_{10} \) which satisfies

\[
\frac{\partial V_0}{\partial t} \leq D_{10} \{ b'_+ (t) + c'_+ (t) + |d'(t)| + |e'(t)| \} \{ H(x) + y^2 + z^2 + w^2 \}
\]

(3.13)

where \( D_{12} = \frac{1}{4} \min \{ \epsilon_0, \epsilon, c \alpha_2, c \alpha_4 E \} \), and \( D_{13} = \max \left\{ 1, \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5}, \delta \right\} \)
then from (3.11), (3.13) and (ix) we obtain the estimate

\[
\dot{V}_0 \leq -2D_{12} (y^2 + z^2 + w^2 + u^2) + 2D_{13} \Delta (y^2 + z^2 + w^2 + u^2) + D_{11} \beta_0 V_0.
\]

Let \( \Delta \) be fixed, in what follows, to satisfy \( \Delta = \frac{D_{14}}{2D_{13}} \). With this limitation on \( \Delta \) we find

\[
\dot{V}_0 \leq -D_{12} (y^2 + z^2 + w^2 + u^2) + D_{11} \beta_0 V_0.
\]

Now (3.9) is verified and the lemma is proved.

4. COMPLETION OF THE PROOF OF THEOREM 1

Define the function \( V(t, x, y, z, w, u) \) as follows

\[
V(t, x, y, z, w, u) = e^{-\int_0^t \beta_0 (\tau) \, d\tau} V_0 (t, x, y, z, w, u).
\]

Then one can verify that there exist two functions \( U_1 \) and \( U_2 \) satisfying

\[
U_1 (\| \bar{x} \|) \leq V(t, x, y, z, w, u) \leq U_2 (\| \bar{x} \|),
\]

for all \( \bar{x} = (x, y, z, w, u) \in \mathbb{R}^5 \) and \( t \in \mathbb{R}^+ \); where \( U_1 \) is a continuous increasing positive definite function, \( U_1 (r) \to \infty \) as \( r \to \infty \) and \( U_2 \) is a continuous increasing function.

Along any solution \((x, y, z, w, u)\) of (3.1) we have

\[
\dot{V} = e^{-\int_0^t \beta_0 (\tau) \, d\tau} \left\{ V_0 - \beta(t) V_0 \right\} \\
\leq -D_{12} e^{-\int_0^t \beta_0 (\tau) \, d\tau} (y^2 + z^2 + w^2 + u^2).
\]

Thus we can find a positive constant \( D_{14} \) such that

\[
\dot{V} \leq -D_{14} (y^2 + z^2 + w^2 + u^2).
\]

From the inequalities (4.2) and (4.3), we obtain the uniform boundedness of all solutions \((x, y, z, w, u)\) of (3.1) [9; Theorem 10.2].
Consider a system of differential equations

\[ \dot{x} = F(t, \bar{x}), \]

where \( F(t, \bar{x}) \) is continuous on \( \mathbb{R}^+ \times \mathbb{R}^n \), \( F(t, 0) = 0 \).

The following lemma is well-known [9].

**Lemma 3.** Suppose that there exists a non-negative continuously differentiable scalar function \( V(t, \bar{x}) \) on \( \mathbb{R}^+ \times \mathbb{R}^n \) such that \( \dot{V}(4.4) \leq -U(\|\bar{x}\|) \), where \( U(\|\bar{x}\|) \) is positive definite with respect to a closed set \( \Omega \) of \( \mathbb{R}^n \). Moreover, suppose that \( F(t, \bar{x}) \) of system (4.4) is bounded for all \( t \) when \( \bar{x} \) belongs to an arbitrary compact set in \( \mathbb{R}^n \) and that \( F(t, \bar{x}) \) satisfies the following two conditions with respect to \( \Omega \):

1. \( F(t, \bar{x}) \) tends to a function \( H(\bar{x}) \) for \( \bar{x} \in \Omega \) as \( t \to \infty \), and on any compact set in \( \Omega \) this convergence is uniform.

2. Corresponding to each \( \epsilon > 0 \) and each \( \bar{y} \in \Omega \), there exist a \( \delta, \delta = \delta(\epsilon, \bar{y}) \) and \( T, T = T(\epsilon, \bar{y}) \) such that if \( t \geq T \) and \( \|\bar{x} - \bar{y}\| < \delta \), we have \( \|F(t, \bar{x}) - F(t, \bar{y})\| < \epsilon \).

Then every bounded solution of (4.4) approaches the largest semi-invariant set of the system \( \dot{x} = H(\bar{x}) \) contained in \( \Omega \) as \( t \to \infty \).

From the system (3.1) we set

\[
F(t, \bar{x}) = \begin{bmatrix}
y \\
z \\
w \\
u \\
- f(t, y, z, w)u - \phi(t, z, w) - \psi(t, z) - g(t, y) - e(t)h(x) \end{bmatrix}.
\]

It is clear that \( F \) satisfies the conditions of Lemma 3. Let \( U(\|\bar{x}\|) = D_1 D_2(y^2 + z^2 + w^2 + u^2) \), then

\[ \dot{V}(t, x, y, z, w, u) \leq -U(\|\bar{x}\|) \]

and \( U(\|\bar{x}\|) \) is positive definite with respect to the closed set \( \Omega := \{ (x, y, z, w, u) \mid x \in \mathbb{R}, y = 0, z = 0, w = 0, u = 0 \} \). It follows that in \( \Omega \)

\[ F(t, \bar{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -e(t)h(x) \end{bmatrix} \]
According to condition (viii) of the theorem and the boundedness of \( e \), we have
\[ e(t) \to e_\infty \text{ as } t \to \infty, \]
where \( 1 \leq e_0 \leq e_\infty \leq E \). If we set
\[ (4.7) \quad H(\dot{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -e_\infty h(x) \end{bmatrix}, \]
then the conditions on \( H(\dot{x}) \) of Lemma 3 are satisfied. Since all solutions of (3.1)
are bounded, it follows from Lemma 3 that every solution of (3.1) approaches the
largest semi-invariant set of the system \( \dot{x} = H(\dot{x}) \) contained in \( \Omega \) as \( t \to \infty \). From
\( (4.7); \dot{x} = H(\dot{x}) \) is the system
\[ \dot{x} = 0, \quad y = 0, \quad \dot{z} = 0, \quad \dot{w} = 0 \quad \text{and} \quad \dot{u} = -e_\infty h(x), \]
which has the solutions
\[ x = k_1, \quad y = k_2, \quad z = k_3, \quad w = k_4, \quad \text{and} \quad u = k_5 - e_\infty h(k_1)(t - t_0). \]
In order to remain in \( \Omega \), the above solutions must satisfy
\[ k_2 = 0, \quad k_3 = 0, \quad k_4 = 0 \quad \text{and} \quad k_5 - e_\infty h(k_1)(t - t_0) = 0 \quad \text{for all } t \geq t_0, \]
which implies \( k_5 = 0, \quad h(k_1) = 0 \), and thus \( k_1 = k_5 = 0 \).
Therefore the only solution of \( \dot{x} = H(\dot{x}) \) remaining in \( \Omega \) is \( \bar{x} = 0 \), that is, the
largest semi-invariant set of \( \dot{x} = H(\dot{x}) \) contained in \( \Omega \) is the point \((0, 0, 0, 0, 0)\). Consequently we obtain
\[ x(t), \dot{x}(t), \ddot{x}(t), \dot{x}^2(t), x^{(4)}(t) \to 0 \quad \text{as} \quad t \to \infty. \]

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