THE CONTACT SYSTEM FOR $A$-JET MANIFOLDS

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Abstract. Jets of a manifold $M$ can be described as ideals of $\mathcal{C}^\infty(M)$. This way, all the usual processes on jets can be directly referred to that ring. By using this fact, we give a very simple construction of the contact system on jet spaces. The same way, we also define the contact system for the recently considered $A$-jet spaces, where $A$ is a Weil algebra. We will need to introduce the concept of derived algebra.

Although without formalization, jets are present in the work of S. Lie (see, for instance, [6]; § 130, pp. 541) who does not assume a fibered structure on the concerned manifold; on the contrary, this assumption is usually done nowadays in the more narrow approach given by the jets of sections.

It is an old idea to consider the points of a manifold other than the ordinary ones. This can be traced back to Plücker, Grassmann, Lie or Weil. Jets are ‘points’ of a manifold $M$ and can be described as ideals of its ring of differentiable functions $[9, 13]$. Indeed, the $k$-jets of $m$-dimensional submanifolds of $M$ are those ideals $\mathfrak{p} \subset \mathcal{C}^\infty(M)$ such that $\mathcal{C}^\infty(M)/\mathfrak{p}$ is isomorphic to $\mathbb{R}_m^k \overset{\text{def}}{=} \mathbb{R}[\epsilon_1, \ldots, \epsilon_m]/(\epsilon_1, \ldots, \epsilon_m)^{k+1}$ (where the $\epsilon$’ are undetermined variables).

This point of view was introduced in the Ph. D. thesis of J. Rodríguez, advised by the second author [13]. Subsequently, several applications were done showing the improvement given by this approach with respect to the usual one: formal integrability theory [10], Lie equations and Lie pseudogroups [7, 8], differential invariants [12] and transformations of partial differential equations [3]. Even the present paper may be placed into that series.

The main advantage of considering jets as ideals is the following. All the operations on the space of $(m, k)$-jets $J^k_m M$ are directly referred to $\mathcal{C}^\infty(M)$, making the usual processes much more transparent and natural. In particular, the tangent space $T_p J^k_m M$ is given by classes of derivations from $\mathcal{C}^\infty(M)$ to $\mathcal{C}^\infty(M)/\mathfrak{p}$ (where two of these derivations are considered as equivalent if they agree on

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p ⊂ C∞(M)). As a result, the very functions \( f \in p \) define canonically \( C^\infty(M)/p \)-linear maps \( \partial_p f : T_p J^k_m M \to C^\infty(M)/p \) whose real components span the cotangent space \( T^*_p J^k_m M \) (Corollary 1.5).

We will construct the contact system starting from the following remark. Let \( p \) be the unique point of \( M \) such that \( p \subset m_p \) (where \( m_p \) denotes the ideal of the functions vanishing on \( p \)). When \( f \) runs over \( p \) and \( D_p \) runs over the tangent spaces to jet prolongations of \( m \)-dimensional submanifolds \( X \subset M \), the set of the values of \( \partial_p f (D_p) \) equals \( m^p_p/p \).

As a consequence, it is natural to define the contact system by composing each \( \partial_p f \) with the projection \( C^\infty(M)/p \to C^\infty(M)/p + m^p_p \) (Definition 1.6). The resulting maps annihilate all the tangent vectors to jet prolongations of \( m \)-dimensional submanifolds. This way, the basic properties of the contact system are easily established.

On the other hand, for each Weil algebra \( A \) (finite local rational commutative \( \mathbb{R} \)-algebra), we can define an \( A \)-jet on \( M \) as an ideal \( p \subset C^\infty(M) \) such that \( C^\infty(M)/p \) is isomorphic to \( A \). The set of \( A \)-jets \( J^A M \) can be also endowed with an smooth structure [1]. The way we have defined the contact system for \( (m, k) \)-jets can be translated into \( A \)-jets. All we have to do is looking for a suitable substitute for \( C^\infty(M)/p + m^p_p \). Such a substitute turns to be the derived algebra associated with \( C^\infty(M)/p \) (Proposition 3.9). Once this is done, we can proceed as in the case of \( A = \mathbb{R}^k_m \).

**Notation.** Let \( \phi : A \to B \) be an \( \mathbb{R} \)-algebra morphism; by \( \text{Der}_{\mathbb{R}}(A, B)_{\phi} \) we will denote the set of \( \mathbb{R} \)-derivations from \( A \) to \( B \) where \( B \) is considered as an \( A \)-module via \( \phi \). When \( \phi \) is implicitly assumed, we will omit it. The characters \( \alpha \), \( \beta \) will be reserved to denoting multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_k), \beta = (\beta_1, \ldots, \beta_k) \in \mathbb{N}^k \) (typically, \( k \) will be \( n \) or \( m \)). Besides, we will denote by \( 1_j \) the multi-indices \( (1_j)_i = \delta_{ij} \).

1. **The contact system on Jet spaces**

In the whole of this paper, \( M \) will be a smooth manifold of dimension \( n \). Besides, ‘submanifold’ will mean ‘locally closed submanifold’. When \( X \) is a closed submanifold of \( M \), \( I_X \) will be the ideal of \( C^\infty(M) \) consisting of the functions vanishing on \( X \). When \( X \) is only locally closed, one would replace \( M \) by the open set \( U \) into which \( X \) is a closed submanifold but, for the sake of simplicity in the exposition, that will be implicitly understood.

Let us consider an \( m \)-dimensional submanifold \( X \subset M \), its associated ideal \( I_X \subset C^\infty(M) \), and a point \( p \in X \). The class of the submanifolds having at \( p \) a contact of order \( k \) with \( X \) is naturally identified with the ideal \( p \overset{\text{def}}{=} I_X + m^k_{p+1} \subset C^\infty(M) \). Moreover, an isomorphism \( C^\infty(M)/p \cong \mathbb{R}^k_m \) is deduced by taking local coordinates \( \{ x_i, y_j \} \) centered at \( p \) and such that \( I_X = (y_j) \).

**Definition 1.1.** A jet of dimension \( m \) and order \( k \) (or, simply, an \((m, k)\)-jet) of \( M \) is, by definition, an ideal \( p \subset C^\infty(M) \) such that \( C^\infty(M)/p \cong \mathbb{R}^k_m \). The set of \((m, k)\)-jets of \( M \) will be denoted by \( J^k_m M \).
Given \( p \in J^k_m M \), there is a unique point \( p \in M \) such that \( p \subset m_p \). This way, it is deduced a map \( J^k_m M \to M, \; p \mapsto p \).

The smooth structure on \( J^k_m M \) is obtained in the following way (see [13, 9]). Let \( (U; x_1, \ldots, x_n) \) be a local chart of \( M \). Now, let us choose \( m \) coordinates, for instance \( x_1, \ldots, x_m \), and let us consider the subset \( J^k U \) given by those jets \( p \in J^k_m U \) such that \( \mathbb{R}[x_1, \ldots, x_m]/p \cap \mathbb{R}[x_1, \ldots, x_m] \simeq \mathcal{C}^\infty(U)/p \). So, with each function \( f \in \mathcal{C}^\infty(U) \) we can associate a unique polynomial \( P_f(x) \) of degree \( \leq k \) such that \( f - P_f \in p \).

Let us denote by \( y_j \) the coordinate \( x_{m+j} \). Then we have

\[
P_{y_j}(x) = \sum_{|\alpha| \leq k} y_{j\alpha}(p) \frac{(x - x(p))^\alpha}{\alpha!},
\]

for suitable numbers \( y_{j\alpha}(p) \). Besides, \( p \) is spanned by the functions \( y_j - P_{y_j} \) together with \( m_p^{k+1} \). So the set of functions \( \{x_i, y_j, y_{j\alpha}\} \) provides one with a coordinate system on \( J^k_m U \).

By taking in the above process all the possible choices of \( m \) elements of \( \{x_1, \ldots, x_n\} \) in all the local charts of \( M \) we get an atlas on \( J^k_m M \).

The following basic statement was proved in [13] (see also [1, 9]).

**Theorem 1.2.** For each \( p \in J^k_m M \) the following isomorphism holds,

\[
T_p J^k_m M \simeq D_p/D'_p
\]

where \( D_p = \text{Der}_\mathbb{R}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)/p) \) and \( D'_p = \{D \in D_p \mid Df = 0, \forall f \in p\} \).

The correspondence in the above theorem is locally given by

\[
\left( \frac{\partial}{\partial x_i} \right)_p = \left[ \frac{\partial}{\partial x_i} \right]_p, \quad \left( \frac{\partial}{\partial y_{j\alpha}} \right)_p = \left[ \frac{(x - x(p))^\alpha}{\alpha!} \frac{\partial}{\partial y_j} \right]_p
\]

where \([D]_p \) denotes the class of a derivation \( D \in D_p \) modulo \( D'_p \) (see [9], pp. 744-45, for this calculation).

**Remark 1.3.** Since Theorem 1.2 it is deduced that the tangent space at a jet \( p \in J^k_m M \) is naturally provided with the structure of \( \mathcal{C}^\infty(M)/p \)-module.

**Corollary 1.4.** Each function \( f \in p \) defines an \( \mathcal{C}^\infty(M)/p \)-linear map

\[
\vartheta_p f : T_p J^k_m M \to \mathcal{C}^\infty(M)/p; \quad D_p = [D]_p \mapsto [Df]_p
\]

where \([Df]_p \) denotes the class of the function \( Df \) modulo \( p \).

The local expression of \( \vartheta_p f \) is given by

\[
\vartheta_p f = \sum_i \left[ \frac{\partial f}{\partial x_i} \right]_p d_p x_i + \sum_{j, \alpha} \left[ \frac{(x - x(p))^\alpha}{\alpha!} \frac{\partial f}{\partial y_j} \right]_p d_p y_{j\alpha}.
\]

**Corollary 1.5.** For each jet \( p \), the cotangent space \( T_p^* J^k_m M \) is spanned by the real components of the \( \vartheta_p f, \; f \in p \):\n
\[
T_p^* J^k_m M = \text{Real components of } \{ \vartheta_p f \mid f \in p \}.
\]
Proof. Given $D_p \in T_p J^k_m M$, there exist at least a function $f \in \mathfrak{p}$ such that $\delta_p f(D_p) \neq 0$ (elsewhere, $D_p = 0$); so, also a real component of $\delta_p f$ is not vanishing on $D_p$.

Let us denote by $\delta'_p f$ the following composition

$$T_p J^k_m M \xrightarrow{\delta_p f} \mathcal{C}^\infty(M)/\mathfrak{p} \xrightarrow{\pi'} \mathcal{C}^\infty(M)/\mathfrak{p'},$$

where $\mathfrak{p'} \overset{\text{def}}{=} \mathfrak{p} + \mathfrak{m}^k_p$.

**Definition 1.6.** The distribution of tangent vectors $\mathcal{C}$ given by

$$\mathcal{C}_p \overset{\text{def}}{=} \bigcap_{f \in \mathfrak{p}} \ker(\delta'_p f) \subset T_p J^k_m M$$

will be called the contact distribution on $J^k_m M$. The Pfaffian system associated with $\mathcal{C}$ will be called the contact system on $J^k_m M$ and we will denote it by $\Omega$.

In order to get the local expression of $\Omega$ let us consider the functions $f_j = y_j - P_{y_j} \in \mathfrak{p}$ (thus, $\mathfrak{p} = (f_j) + \mathfrak{m}^{k+1}_p$). From relations (1.1)-(1.3) we get

$$\delta'_p f_j = \sum_{|\alpha| \leq k-1} \left[ \frac{(x - x(p))^\alpha}{\alpha!} \right]_{\mathfrak{p}'} \partial_p y_{j\alpha} - \sum_{i, |\alpha| \leq k} y_{j\alpha} (p) \left[ \frac{(x - x(p))^{\alpha-1}_i}{(\alpha - 1)_i!} \right]_p dx_i,$$

$$= \sum_{|\alpha| \leq k-1} \left[ \frac{(x - x(p))^\alpha}{\alpha!} \right]_{\mathfrak{p}'} (\partial_p y_{j\alpha} - \sum_{i} y_{j\alpha+1, i}(p) \partial_p x_i).$$

Because $\delta'_p \mathfrak{m}^{k+1}_p = 0$, we deduce that the contact system $\Omega$ is generated by the 1-forms

$$(1.4) \quad \omega_{j\alpha} \overset{\text{def}}{=} dy_{j\alpha} - \sum_{i} y_{j\alpha+1, i} dx_i,$$

which are the real components of the $\delta'_p f_j$.

Since (1.4) it is obvious that $\Omega$ is the usual contact system. Nevertheless, in the rest of this section we will explain why $\Omega$ is well behaved.

Let $X$ be an $m$-dimensional submanifold of $M$; each $(m, k)$-jet $q \in J^k_m X$ is necessarily of the form $q \in \mathfrak{m}^k_p$, where $\mathfrak{m}_p \subset \mathcal{C}^\infty(X)$ denotes the maximal ideal of a point $p \in X$. Accordingly, an identification $J^k_m X \approx X$ arises. Moreover, if $I_X \subset \mathcal{C}^\infty(M)$ denotes the ideal associated with $X$, we can consider the inclusion $X \approx J^k_m X \hookrightarrow J^k_m M$ by $p \mapsto I_X + \mathfrak{m}^{k+1}_p$. That defines an immersion which will be called the $k$-jet prolongation of $X$. The ideal of $J^k_m X$ into $J^k_m M$ is the prolongation of $I_X$ to $\mathcal{C}^\infty(J^k_m M)$ (see [9]). As a result, and taking into account the characterization of the tangent spaces given in Theorem 1.2, we obtain item (2) of the following statement (item (1) is easy).

**Theorem 1.7.** Let $X$ be an $m$-dimensional submanifold of $M$ and consider its jet prolongation $J^k_m X$ immersed into $J^k_m M$.

(1) A jet $p \in J^k_m M$ belongs to $J^k_m X$ if and only if $p \supset I_X$. 


(2) A vector \( D_p \in T_p J^k_m M \) is tangent to \( J^k_m X \) if and only if
\[
\mathcal{D}_p f(D_p) = 0, \quad \forall f \in I_X.
\]

Let us suppose \( p = (y_j) + m^k_{p+1} \) where \( \{x_i, y_j\} \) are local coordinates around \( p \in M \). All the submanifolds \( X \) such that \( I_X \subset p \) are locally given by equations \( y_j = P_j(x); j = 1, \ldots, n - m \), where \( P_j(x) \in m_{p}^{k+1} \). As a consequence,

**Lemma 1.8.** Given a jet \( p \in J^k_m M \), the set of the values \( \mathcal{D}_p f(D_p) \) when \( f \) runs over \( p \) and \( D_p \) runs over the tangent spaces to jet prolongations of \( m \)-dimensional submanifolds \( X \subset M \), equals \( m^k_p / p \).

According to the above lemma, if \( f \in p \) then \( \mathcal{D}_p f \) annihilates each vector which is tangent to a jet prolongation \( J^k_m X \). This is why Definition 1.6 gives the usual contact system.

From this point the basic properties of \( \Omega \) could be deduced. However, we have preferred to do it in the more general context of \( A \)-jets where a similar construction of the contact system will be carried on (see below).

2. \( A \)-jets

It is well known that a manifold \( M \) can be recovered as the set of \( \mathbb{R} \)-algebra morphisms \( \mathcal{C}^\infty(M) \rightarrow \mathbb{R} \); also the tangent bundle \( TM \) is obtained by taking the morphisms with values in \( \mathbb{R}[\varepsilon]/\varepsilon^2 \). In general we can consider the morphisms taking values in an algebra \( A \). This concept comes back to Weil [14], who called them ‘points \( A \)-proches’ of \( M \).

**Definition 2.1.** A commutative \( \mathbb{R} \)-algebra \( A \) is called a Weil algebra if it is finite dimensional, local and rational. Let us denote by \( m_A \) the maximal ideal of \( A \). The integer \( k \) such that \( m_A^{k+1} = 0 \), \( m_A^k \neq 0 \), will be called the order of \( A \) and denoted by \( o(A) \). The dimension of \( m_A/m_A^2 \) will be called the width of \( A \) and denoted by \( w(A) \).

The main examples of Weil algebras are the rings of truncated polynomials \( \mathbb{R}_m^k \) (here, \( o(\mathbb{R}_m^k) = k \) and \( w(\mathbb{R}_m^k) = m \)). On the other hand, if \( m_p \) denotes the maximal ideal associated to a point \( p \) in a manifold \( M \), the quotient \( \mathcal{C}^\infty(M)/m_p^{k+1} \) is also a Weil algebra isomorphic to \( \mathbb{R}_m^k \) where \( n \) is the dimension of \( M \) (an isomorphism is induced by taking local coordinates).

**Definition 2.2.** Let \( M \) be a manifold and \( A \) a Weil algebra. An \( \mathbb{R} \)-algebra morphism
\[
p^A: \mathcal{C}^\infty(M) \rightarrow A
\]
is called an \( A \)-point (or \( A \)-velocity) of \( M \). The set of \( A \)-points of \( M \) will be called the Weil bundle of \( A \)-points of \( M \) and denoted by \( M^A \). We will say that an \( A \)-point \( p^A \) is regular if it is surjective. The set of regular \( A \)-points of \( M \) will be denoted by \( M^A \).

To simplify notation, when \( A = \mathbb{R}_m^k \), we will write \( M^A_m \) instead of \( M^{k^m} \). For instance, \( M^0_m = M \) (for any \( m \)) and \( M^1 = TM \).
If we compose an \( A \)-point \( p^A \) with the canonical projection \( A \to A/\mathfrak{m}_A = \mathbb{R} \) we obtain an \( \mathbb{R} \)-point of \( M \), that is, an ordinary point \( p \in M \); this defines a projection \( M^A \to M \).

On the other hand, each function \( f \in C^\infty(M) \) defines a map
\[
f^A : M^A \to A,
\]
by the tautological rule \( f^A(p^A) \overset{\text{def}}{=} p^A(f) \).

For the proof of the following statement see [5] or [9].

**Theorem 2.3.** There exists a differentiable structure on \( M^A \) determined by the condition that the maps \( f^A \) are smooth (\( M^A \) is a dense open set of \( M^A \)). Furthermore, \( M^A \to M \) is a fiber bundle with typical fiber \( \text{Hom}(\mathbb{R}^k_n; A) \), where \( n = \dim M \) and \( k = o(A) \).

**Remark 2.4.** If \( \{y_j\} \) is a local chart on \( M \) and \( \{a_\alpha\} \) is a basis of \( A \), then the collection of functions \( y_j^\alpha \) determined by the rule \( y_j^\alpha(p^A) = \sum \alpha y_j^\alpha(p^A) a_\alpha \), define a local chart on \( M^A \).

The next proposition is straightforward (see, for instance, [2]).

**Proposition 2.5.**
1. Let \( \psi : A \to B \) be a morphism of Weil algebras. For each smooth manifold \( M \), \( \psi \) induces a differentiable map
\[
\psi_M : M^A \to M^B ; \quad p^A \mapsto \psi_M(p^A) \overset{\text{def}}{=} \psi \circ p^A.
\]
2. Let \( \phi : M \to N \) be an smooth map between the smooth manifolds \( M \) and \( N \). For each Weil algebra \( A \), \( \phi \) induces a differentiable map
\[
\phi^A : M^A \to N^A ; \quad p^A \mapsto \phi^A(p^A) \overset{\text{def}}{=} \phi \circ p^A
\]
where \( \phi^* \) stands for the map induced between the rings of functions of \( M \) and \( N \).

The following theorem was given in [9].

**Theorem 2.6.** There is a natural identification
\[
T_{p^A}M^A \simeq \text{Der}_\mathbb{R}(C^\infty(M), A)_{p^A}
\]
where each \( X \in T_{p^A}M^A \) is related to the derivation \( X' \in \text{Der}_\mathbb{R}(C^\infty(M), A)_{p^A} \) determined by \( X'(f) = X(f^A) \in A, f \in C^\infty(M) \) (where \( X \) derives componentwise the vector-valued function \( f^A \)).

**Remark 2.7.** According with this theorem, the tangent maps corresponding with Proposition 2.5 are given respectively by
\[
(\psi_M)_* : D_{p^A} \to T_{\psi_M(p^A)}M^B, \quad (\phi^A)_* : D_{p^A} \circ \phi^* \in T_{\phi^*(p^A)}N^A.
\]

Next, we will generalize the notion of jet for any Weil algebra \( A \).

**Definition 2.8.** An \( A \)-jet on \( M \) is, by definition, an ideal \( p \subset C^\infty(M) \) such that \( C^\infty(M)/p \simeq A \). The space of \( A \)-jets of \( M \) will be denoted by \( J^A M \).
We have a surjective map \( \operatorname{Ker}: \tilde{M}^A \to J^A M \) which associates with each \( A \)-point its kernel. The group \( \operatorname{Aut}(A) \) acts on \( \tilde{M}^A \) by composition and there is an obvious equivalence between the set of orbits of this action and \( J^A M \).

The proof of the following two theorems was given in [1].

**Theorem 2.9.** On \( J^A M \) there exists an smooth structure such that

\[
\operatorname{Ker}: \tilde{M}^A \to J^A M
\]

is a principal fiber bundle with structure group \( \operatorname{Aut}(A) \).

**Remark 2.10.** In particular, \( J^k M \) is the quotient manifold of \( \tilde{M}^k_m \) under the action of \( \operatorname{Aut}(\mathbb{R}^k_m) \).

**Theorem 2.11.** For each \( p \in J^A M \), the following isomorphism holds,

\[
T_p J^A M \simeq D_p / D'_p
\]

where \( D_p = \operatorname{Der}_{\mathbb{R}}(C^\infty(M), C^\infty(M)) / p \) and \( D'_p = \{ D \in D_p \mid Df = 0, \forall f \in p \} \).

As a result and similarly to the case of \((m,k)\)-jets, each function \( f \in p \) defines a \( C^\infty(M) / p \)-linear map

\[
\tilde{\alpha}_p f: T_p J^A M \to C^\infty(M) / p
\]

and Corollaries 1.4–1.5 also hold for \( A \)-jets with the same proof.

On the other hand, each smooth map \( \phi: M \to N \) induces a new map between the corresponding \( A \)-Weil bundles, \( \phi^A: \tilde{M}^A \to \tilde{N}^A \) (Definition 2.5). However, the condition of regularity of an \( A \)-point is not, in general, preserved, that is, \( \phi^A(M^A) \not\subseteq \tilde{N}^A \). This is why we give the following definition (see [2]).

**Definition 2.12.** Let \( \phi: M \to N \) be a differentiable map. An \( A \)-point \( p^A \in M^A \) will be called \( \phi \)-regular if \( \phi^A(p^A) = p^A \circ \phi^* \in \tilde{N}^A \). The set of \( \phi \)-regular \( A \)-points of \( M^A \) will be denoted by \( \tilde{M}^A_\phi \).

The proof of the following propositions is not difficult (see [2]).

**Proposition 2.13.** The set of \( \phi \)-regular \( A \)-points, \( \tilde{M}^A_\phi \), is an open subset of \( M^A \) (eventually the empty set).

The set of jets of \( \phi \)-regular \( A \)-points will be denoted by \( J^A_\phi M \). In particular, we have a principal fiber bundle \( \operatorname{Ker}: \tilde{M}^A_\phi \to J^A_\phi M \).

**Proposition 2.14.** The map \( \phi: M \to N \) induces maps

\[
\tilde{M}^A_\phi \xrightarrow{\phi^A} \tilde{N}^A, \quad p^A \mapsto \phi^A(p^A) \overset{\text{def}}{=} p^A \circ \phi^*
\]

\[
J^A_\phi M \xrightarrow{j^A_\phi} J^A N, \quad p \mapsto j^A_\phi(p) \overset{\text{def}}{=} (\phi^*)^{-1} p
\]

in such a way that \( \operatorname{Ker} \circ \phi^A = j^A_\phi \circ \operatorname{Ker} \).
Example 2.15. 1) If \( \gamma: X \to M \) is an immersion, then \( \gamma^*: C^\infty(M) \to C^\infty(X) \) is surjective on germs; so \( \tilde{X}^A = \tilde{X}^A \) and \( J^A X = J^A X \). In particular, \( J^A \) defines a functor in the category of differentiable manifolds with immersions (see [4]). When \( \gamma \) is the inclusion of an \( m \)-dimensional submanifold \( X \subset M \) and \( A = \mathbb{R}^m \), \( j^A \gamma \) gives the jet prolongation of \( X \).

2) If \( \pi: M \to X \) is a fiber bundle and \( s \) is a section of \( \pi \), then we have induced maps \( \pi^A, s^A, j^A \pi \) and \( j^A s \) such that \( \pi^A \circ s^A = id_{\tilde{X}^A} \) and \( j^A \pi \circ J^A s = id_{J^A X} \). When \( A = \mathbb{R}^m \) and \( m = \dim X \), \( J^A M \) equals the well known bundle of \( k \)-jets of sections of \( \pi \).

Proposition 2.16. Let \( \phi: M \to N \), \( A \) be as above. The tangent map corresponding to \( j^A \phi \) at a point \( p \in J^A X \) sends each \( D_p \in T_p J^A X \) to

\[
(j^A \phi)_* D_p = \left[ [\phi^*]^{-1} \circ D \circ \phi^* \right]_{j^A \phi(p)} j^A N,
\]

where \( [\phi^*] \) denotes the isomorphism \( C^\infty(M)/p \cong C^\infty(N)/j^A \phi(p) \) induced by \( \phi^* \).

Proof. It follows from Remark 2.7 and Theorem 2.11 (see [2] for details).

Definition 2.17. Let \( i: X \hookrightarrow M \) be an \( m \)-dimensional submanifold of \( M \), where \( m = w(A) \); then \( J^A i: J^A X \hookrightarrow J^A M \) will be called the \( A \)-jet prolongation of \( X \).

Theorem 1.7 remain valid for \( A \)-jet prolongations. It can be shown by means of an attentive inspection of the definitions. There is just a difference: as a rule, \( J^A X \) can not be identified with \( X \).

Definition 2.18. Let \( p \in J^A M \), and \( p \in M \) be its projection (that is, \( p \subset m_p \subset C^\infty(M) \)) and let us denote by \( m \) the maximal ideal of \( C^\infty(M)/p \) (i.e., \( m = m_p/p \)).

A local chart \( \{x_1, \ldots, x_m, y_1, \ldots, y_{m-k} \} \) (where \( m = w(A) \)) in a neighborhood of \( p \) will be called adapted to the jet \( p \) if it holds

1. The classes of \( \{x_i\} \) modulo \( m^2 \) generate \( m/m^2 \).
2. The functions \( y_j \) belong to \( p \) and they are linearly independent modulo \( m^2_p \).

It is easily deduced the existence of local charts adapted to a given jet.

Lemma 2.19. Let \( \{x_i, y_j\} \) be a local chart adapted to a jet \( p \in J^A M \); then, there exists polynomials \( Q_s(x) \), \( \deg(Q_s) \leq o(A) = k \) such that

\[
p = (y_j) + (Q_s(x)) + m_p^{k+1}.
\]

Proof. By hypothesis we have an epimorphism

\[
\mathbb{R}[x_1, \ldots, x_m]/(x_1, \ldots, x_m)^{k+1} \hookrightarrow C^\infty(M)/m_p^{k+1} \longrightarrow C^\infty(M)/p,
\]

whose kernel is generated by a finite number polynomials \( Q_s(x) \). This way we get an isomorphism

\[
\mathbb{R}[x_1, \ldots, x_m]/(Q_s) + (x_1, \ldots, x_m)^{k+1} \cong C^\infty(M)/p
\]

from which we deduce the statement.

Remark 2.20. We have \( Q_s \in m^2_p \), elsewhere \( w(A) \) could not be \( m \), but lower.

The proof of Corollaries 2.21 and 2.22 below is straightforward.
Corollary 2.21. Let $X$ be an $m$-dimensional submanifold of $M$, and $p \in J^A M$ be an $A$-jet containing $I_X$. There exists local coordinates $\{x_i, y_j\}$ such that the local equations of $X$ into $M$ are

$$y_j = P_j(x),$$

for suitable functions $P_j(x) \in p$.

Corollary 2.22. Let $X \overset{i}{\hookrightarrow} M$ be as above, and $p = j^A i(q)$ where $q \in J^A X$ and $j^A i: J^A X \rightarrow J^A M$ is the jet prolongation of $i$. Besides, let $\{x_i, y_j\}$ be a local chart adapted to $p$. Then the tangent map is given by

$$T_q J^A X \overset{(j^A i)_*}{\longrightarrow} T_p J^A M; \quad \left[ \frac{\partial}{\partial x_i} \right]_q \mapsto \left[ \frac{\partial}{\partial x_i} + \sum_j \frac{\partial P_j(x)}{\partial x_i} \frac{\partial}{\partial y_j} \right]_p.$$

3. Derived algebra of a Weil algebra

Each Weil algebra $A$ has several canonically defined ideals; examples of which are the powers of its maximal ideal. We show here two more of them which are a key point in order to obtain a contact system for $A$-jet spaces.

Definition 3.1. Let $\mathcal{W}$ be the category whose objects are the Weil algebras and whose morphisms are the Weil algebra isomorphisms.

A functor $\mathcal{W} \overset{F}{\longrightarrow} \mathcal{W}$ will be called an equivariant projection of Weil algebras if for each $A \in \mathcal{W}$ there is an epimorphism $A \overset{\pi_F}{\longrightarrow} F(A)$ such that for any isomorphism $A \overset{\psi}{\rightarrow} B$ of Weil algebras we have

$$\pi_F \circ \psi = F(\psi) \circ \pi_F.$$ 

Example 3.2. For each positive integer $j$ we define the functor $F_j: \mathcal{W} \rightarrow \mathcal{W}$ which maps a Weil algebra $A$ to $F_j(A) \overset{\text{def}}{=} A_j = A/\mathfrak{m}_A^{j+1}$, where $\mathfrak{m}_A$ is the maximal ideal of $A$: because any isomorphism $A \overset{\psi}{\rightarrow} B$ holds $\psi(\mathfrak{m}_A) = \mathfrak{m}_B$, we deduce that $F_j$ is an equivariant projection ($A_j$ is the $j$-th underlying algebra of $A$, see [4]).

The proof of the following lemma is straightforward.

Lemma 3.3. Each equivariant projection $F$ defines a group morphism

$$\text{Aut}(A) \overset{F}{\longrightarrow} \text{Aut}(F(A)); \quad g \mapsto F(g).$$

The projections $\pi_F: A \rightarrow F(A)$ induce maps of Weil bundles $\pi_F: M^A \rightarrow M^{F(A)}$ for each smooth manifold $M$ (Proposition 2.5). The equivariance property of $\pi_F$ ensures that we have an induced map at the level of jet spaces. This way,

Theorem 3.4. Given an smooth manifold $M$ and a Weil algebra $A$, each equivariant projection $F$ defines a differentiable map

$$\pi_F: J^A M \rightarrow J^{F(A)} M.$$

Remark 3.5. By the very definition, $F(\mathcal{C}^\infty(M)/p) = \mathcal{C}^\infty(M)/\pi_F(p)$.  

Corollary 3.6. Under the identification in Theorem 2.11, the tangent map corresponding to \( \pi_F \) is given by

\[
T_p J^A M \left( \pi_F \right) = T_{\pi_F(p)} f^M M; \quad D_p = \left[ D \right]_p \mapsto (\pi_F \circ D)_p = [\pi_F \circ D]_{\pi_F(p)},
\]

where \( \pi_F \circ D \in \text{Der}\_R(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)/\pi_F(p)) \).

Let \( w(A) = m \) and \( o(A) = k \).

Definition 3.7. For each given epimorphism \( H : \mathbb{R}^{k+1}_m \to A \) we define

\[
I'_H \overset{\text{def}}{=} \{ D_H P \mid P \in \ker H, D_H \in \text{Der}_R(\mathbb{R}^{k+1}_m, A) \}. \]

Lemma 3.8. Let \( \psi : A \cong B \) be an isomorphism of Weil algebras and let \( H : \mathbb{R}^{k+1}_m \to A, \overline{H} : \mathbb{R}^{k+1}_m \to B \) be algebra epimorphisms. Then \( \psi(I'_H) = I'_B \).

Proof. By Lemma 1 in the Appendix there exists an automorphism \( g \in \text{Aut}(\mathbb{R}^{k+1}_m) \) such that \( \overline{H} \circ g = \psi \circ H \). Moreover, \( g \) establishes an isomorphism

\[
\psi_g : \text{Der}_R(\mathbb{R}^{k+1}_m, A)_H \cong \text{Der}_R(\mathbb{R}^{k+1}_m, B)_\overline{H},
\]

defined by \( \psi_g(D_{\overline{H}}) \overset{\text{def}}{=} \psi \circ D_{\overline{H}} \circ g^{-1} \).

If \( D_H P \in I'_H \), then \( \psi(D_H P) = \psi_g(D_{\overline{H}})(gP) \in I'_B \). So that \( \psi(I'_H) \) is included into \( I'_B \). By symmetry the proof is finished.

From this lemma it follows that \( I'_H \) is not depending on \( H \); let us denote it by \( I'_A \). By using again the lemma above we also deduce

Proposition 3.9. If \( \psi : A \cong B \) is a Weil algebra isomorphism, then \( \psi(I'_A) = I'_B \).

This way,

\[
F'(A) \overset{\text{def}}{=} A/I'_A
\]

defines an equivariant projection (Definition 3.1) where \( \pi_F' \overset{\text{def}}{=} \pi' \) is the natural epimorphism \( A \to A' \). We will call \( A' \) the derived algebra of \( A \).

Remark 3.10. The ideal \( I'_A \) is just the first Fitting ideal of the module of differentials \( \Omega_{A/R} \).

Computation of \( A' \). Let \( A = \mathbb{R}[\epsilon_1, \ldots, \epsilon_m]/I \) where \( I = (Q_s(\epsilon)) + (\epsilon_1, \ldots, \epsilon_m)^{k+1} \) (the \( Q_s \) are suitable polynomials of degree lower than \( k + 1 \)). Let us consider the projection

\[
\mathbb{R}^{k+1}_m = \mathbb{R}[\epsilon_1, \ldots, \epsilon_m]/(\epsilon_1, \ldots, \epsilon_m)^{k+2} \overset{H}{\to} \mathbb{R}[\epsilon_1, \ldots, \epsilon_m]/I,
\]
in such a way that \( \ker H = (Q_s(\epsilon)) + (\epsilon_1, \ldots, \epsilon_m)^{k+1} \) mod \( (\epsilon_1, \ldots, \epsilon_m)^{k+2} \). On the other hand, \( \text{Der}_R(\mathbb{R}^{k+1}_m, A)_H \) is spanned by the partial derivatives \( \partial/\partial \epsilon_i \). So we see that \( I' = (\partial Q_s/\partial \epsilon_i) + (\epsilon_1, \ldots, \epsilon_m)^k \) mod \( I \) and then

\[
A' = \mathbb{R}[\epsilon_1, \ldots, \epsilon_m]/((Q_s) + (\partial Q_s/\partial \epsilon_i) + (\epsilon_1, \ldots, \epsilon_m)^k).
\]

In particular, \( (\mathbb{R}^k_m)' = \mathbb{R}^{k-1}_m \) and \( (\mathbb{R}^k_m \otimes \mathbb{R}^l_n)' = \mathbb{R}^{k-1}_m \otimes \mathbb{R}^{l-1}_n \).
Remark 3.11. Because \((\mathbb{R}_n^k)' = \mathbb{R}^{k-1}_m\), the notation \(\pi'\) used here, is compatible with that of Section 1. Indeed, we think that \(A'\), better than \(A/m_A^k\), is the natural generalization of \(\mathbb{R}^{k-1}_m\).

By applying Proposition 3.5 we have an induced map
\[
\pi': J^A M \longrightarrow J^{A'} M
\]
which takes each \(p \in J^A M\) to the kernel of the composition
\[
C^\infty(M) \longrightarrow C^\infty(M)/p \xrightarrow{\pi'} (C^\infty(M)/p)' .
\]

Corollary 3.12. If \(\{x_i, y_j\}\) is a local chart adapted to a jet \(p \in J^A M\) such that \(p = (y_j) + (Q_s(x)) + m_p^{k+1}\) for suitable polynomials \(Q_s\) (Lemma 2.19), then
\[
\pi'(p) = (y_j) + (Q_s(x)) + (\partial Q_s/\partial x_i) + m_p^k.
\]

There is a second ideal canonically associated to any Weil algebra \(A\). Let us take an epimorphism \(H: \mathbb{R}^{k+1}_m \rightarrow A\) as above and define the following set
\[
\hat{I}_H \overset{\text{def}}{=} \{ H(P) \in I_A' \mid D_H(P) \in I_A, \forall D_H \in \text{Der}_\mathbb{R}(\mathbb{R}^{k+1}_m, A)_H \}.
\]

It is straightforward to check that \(\hat{I}_H\) is an ideal of \(A\). A similar reasoning like that used for \(I_A'\), shows that \(\hat{I}_H\) is not depending on \(H\). Let us denote this ideal by \(\hat{I}_A\). Then we also have

Proposition 3.13. If \(\psi: A \overset{\sim}{\longrightarrow} B\) is an isomorphism of Weil algebras, then \(\psi(I_A) = \hat{I}_B\). In particular,
\[
\hat{F}(A) \overset{\text{def}}{=} \hat{A} = A/\hat{I}_A ;
\]
defines an equivariant projection.

Example 3.14. The algebras \(A = \mathbb{R}_n^k = \mathbb{R}[\epsilon_1, \ldots, \epsilon_n]/(\epsilon_1, \ldots, \epsilon_n)^{k+1}\) hold \(\hat{I}_A = 0\). Indeed, let \(\mathbb{R}^{k+1}_m \rightarrow \mathbb{R}_n^k\) be the natural projection and denote by \(m\) the ideal \((\epsilon_1, \ldots, \epsilon_n)\), then \(I'_m = m^\mathbb{R}\). On the other hand, if a polynomial \(P \in \mathbb{R}^{k+1}_m\) verifies \(\partial P/\partial x_i \in I'_m = m^\mathbb{R}\), \(i = 1, \ldots, m\), then necessarily \(P\) belongs to \(m^{k+1}\) and so \(H(P) = 0\). However, \(\epsilon_1\epsilon_2\) defines a non trivial element of \(\hat{I}_A\) when \(A = \mathbb{R}[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2)\).

4. The contact system on \(A\)-jets

In this section we will construct the contact system on \(A\)-jet spaces. The way we have defined the contact system for \((m, k)\)-jets (Section 1) can be mostly translated to the new context. However, there is a number of necessary modifications we will focuses ourselves on.

Let \(p\) be an \(A\)-jet on \(M\) and \(\{x_i, y_j\}\) a local chart adapted to \(p\) such that \(p = (y_j) + (Q_s(x)) + m_p^{k+1}\). Taking into account Corollaries 2.21 and 2.22, the set of values \(p_f(D_p) \in C^\infty(M)/p\), where \(f\) runs over \(p\) and \(D_p\) runs over the tangent spaces to \(m\)-dimensional submanifolds of \(M\), equals to
\[
((\partial Q_s/\partial x_i) + m_p^k)/p
\]
(compare with Lemma 1.8).

Let us consider the epimorphism
\[ C^\infty(M)/p \rightarrow C^\infty(M)/(p + (\partial Q_s/\partial x_i) + m^k) \]
and observe that \( p + (\partial Q_s/\partial x_i) + m^k \) equals \( \pi'(p) \) (see computation (3.1)).

As in the case of \((m,k)\)-jets, if \( f \in p \) we can define
\[ \mathcal{A}_p f \overset{\text{def}}{=} \pi' \circ \mathcal{A}_p f : T_p J^A M \rightarrow C^\infty(M)/\pi'(p) \]
where \( \pi' \) denotes the canonical projection of \( C^\infty(M)/p \) onto \( C^\infty(M)/\pi'(p) \). From the above discussion it follows that \( \mathcal{A}_p f \) vanishes on the tangent subspaces \( T_p J^A X \subset T_p J^A M \).

**Remark 4.1.** For each tangent vector \( D_p \in T_p J^A M \), we have
\[ \mathcal{A}_p f(D_p) = \mathcal{A}_{p'} f(\pi'(D_p)) \]
where \( p' \) denotes \( \pi'(p) \in J^A M \).

**Remark 4.2.** By the very definition and using the above notation we have \( \mathcal{A}_p(Q_s) = 0 \) and \( \mathcal{A}_p m^k = 0 \) (i.e., \( \mathcal{A}_p f = 0 \) if \( f \in (Q_s) + m^k \)).

**Definition 4.3.** The distribution of tangent vectors \( C \) given by
\[ C_p \overset{\text{def}}{=} \bigcap_{f \in p} \ker(\mathcal{A}_p f) \subset T_p J^A M \]
will be called the contact distribution on \( J^A M \). The Pfaffian system associated with \( C \) will be called the contact system on \( J^A M \) and we will denote it by \( \Omega \).

**Remark 4.4.** Let \( \phi : N \rightarrow M \) be a differentiable map. It is deduced from the definition of the contact system that the jet prolongation \( j^A \phi : J^A N \rightarrow J^A M \) is a contact transformation, i.e., \( (j^A \phi)_* C_q \subseteq C_{(j^A \phi)_* q} \) for each \( q \in J^A N \).

Since the construction of \( C \) and the discussion before Remark 4.1 we have,

**Proposition 4.5.** Let \( X \) be a submanifold of \( M \) with \( \dim X = w(A) = m \). The prolongation \( J^A X \subset J^A M \) is a solution of the contact distribution.

**Lemma 4.6.** Let \( p_0 \in J^A M \), \( f \in p_0 \) and \( \{x_i, y_j\} \) a local chart adapted to \( p_0 \). For each jet \( p \) in a neighborhood of \( p_0 \) there exists a polynomial \( P_{f,p} = P_{f,p}(x) \) of degree \( \leq o(A) \), such that
\[ f - P_{f,p} \in p \text{.} \]
Moreover, the coefficients of \( P_{f,p} \) can be choosen in such a way that they depend smoothly on \( p \).

**Proof.** Let \( p^A_0 \) be a regular \( A \)-velocity with \( \ker p^A_0 = p_0 \) and let \( A \) be a set of multi-indices such that \( \{p^A_0(x)^\alpha\}_{\alpha \in A} \) is a basis of \( A \).

Now, let us consider \( a_i = p^A_0(x_i) \) and \( b_i = p^A(x_i - x(p)) \) in Lemma 2 of the Appendix. We deduce the existence of differentiable functions \( \Phi_\beta \) in a neighborhood of \( p^A_0 \) such that
\[ p^A(f) = \sum_{\alpha \in A} \Phi_\alpha(p^A) p^A(x - x(p))^\alpha \]
provided that $p^A$ is near enough of $p_0^A$. So, $f - \sum \Phi_\alpha(p^A)(x - x(p))^\alpha \in \ker p^A$.

Finally, by taking a local section $s$ of $\ker M^A \rightarrow J^A M$ defined around $p_0$ and such that $s(p_0) = p_0^A$, we can choose the polynomials in the statement to be

$$P_{f,p} \doteq \sum_{\alpha \in \Lambda} \Phi_\alpha(s(p))(x - x(p))^\alpha.$$  \[ \square \]

**Theorem 4.7.** The contact distribution is smooth.

**Proof.** Let $p_0 \in J^A M$. The incident subspace to $C_{p_0}$ is generated by the real components of the $\delta_{p_0}f$ when $f$ runs over $p_0$.

This way, the theorem follows if each $\delta_{p_0}f, f \in p_0$, can be extended in the following sense: for all jet $p$ in a neighborhood of $p_0$, there is a suitable $C^\infty(M)/p$-linear map $\omega_p: T_p J^A M \rightarrow C^\infty(M)/p'$ fulfilling

1. $\omega_p$ annihilates each vector $D_p \in C_p$.
2. $\omega_p$ depends smoothly on $p$.
3. $\omega_{p_0} = \delta_{p_0}f$.

Since the lemma above, items (1) and (2) hold if we take $\omega_p \doteq \delta_p(f - P_{f,p})$. Item (3) also holds because $\delta_{p_0}P_{f,p_0} = 0$ (see Remark 4.2).  \[ \square \]

**Proposition 4.8.** The vector subspace $C_p$ equals the linear span of the tangent spaces at $p$ of the $m$-dimensional submanifolds $X$ such that $I_X \subset p$:

$$C_p = \sum_{j_X \subset p} T_p J^A X.$$  

**Proof.** Let a fix local coordinates $\{x_i, y_j\}$ adapted to $p$. A tangent vector $D_p = \left[ \sum_i a_i \frac{\partial}{\partial x_i} + \sum_j b_j \frac{\partial}{\partial y_j} \right]_p$ (where we can assume that $a_i, b_j$ are polynomials in the $x_i$) belongs to $C_p$ if and only if $p'(b_j) = 0$. So, $b_j \in p' \doteq p'(p)$ and therefore $b_j = \sum_{\alpha} b_{\alpha}^j \frac{\partial}{\partial x_{\alpha}}$, for suitable polynomials $b_{\alpha}^j(x)$ (see Corollary 3.12). If we denote by $H_i^j$ the sum $\sum_s b_{\alpha}^j Q_s$ we will have $D_p = \left[ \sum_i a_i \frac{\partial}{\partial x_i} + \sum_{\alpha} \frac{\partial}{\partial x_{\alpha}} + \sum_{\alpha} \frac{\partial H_i^j}{\partial x_{\alpha}} \frac{\partial}{\partial y_{\alpha}} \right]_p$.

Next, let us consider the following submanifolds: $X_0 = \{ y_j = 0 \} \xrightarrow{i_0} M$, $X_h = \{ y_j = H_i^j(x) \} \xrightarrow{i_h} M$, $h = 1, \ldots, m$. Then, a calculation gives

$$D_p = (j^A i_0)_* \left[ \sum_i a_i \frac{\partial}{\partial x_i} \right]_p + \sum_h (j^A i_h)_* \left[ \frac{\partial}{\partial x_h} \right]_p - (j^A i_0)_* \left[ \frac{\partial}{\partial x_h} \right]_p$$

which belongs to $T_p J^A X_0 + \sum_h T_p J^A X_h$.  \[ \square \]

**Remark 4.9.** An easy consequence follows. Let $\pi': J^A M \rightarrow J^{A'} M$ be the natural projection and $p' = \pi'(p)$, $p \in J^A M$. If $w(A') = w(A) = m$, then $\pi'_* C_p \subset C_{p'}$.

**Lemma 4.10.** Let $U \subset J^A M$ be a solution of the contact system and $p$ a jet in $U$. Then $\dim \pi'_* T_p U \leq \dim J^{A'} \mathbb{R}^m$ where $m = w(A)$. Moreover, if $p \supset I_X$, where $I_X$ is the ideal of a given $m$-dimensional submanifold $X$, then $\pi'_* T_p U \subset T_{p'} J^{A'} X$. 

Proof. It is sufficient to show the second part in the claim. If \( p \supset I_X \), also we have \( p' \supset I_X \) (that is, \( p' \in J^A X \)). Then, by using Remark 4.1, for each given tangent vector \( D_p \in T_p U \subset C_p \) we have
\[
\partial_p f(\pi'_p D_p) = \partial_p f(D_p) = 0, \quad \forall f \in I_X.
\]

From the version of Theorem 1.7 in the case of \( A \)-jets, it follows that \( \pi'_p D_p \in T_p J^A X \).

**Lemma 4.11.** Let \( U \subset J^A M \) be a solution of the contact system which contains \( J^A X \), where \( X \) is an \( m \)-dimensional submanifold of \( M \). If \( p \in J^A X \), there exist a neighborhood of \( \pi'(p) = p' \) where
\[
\pi'(U) = J^A X.
\]

**Proof.** By applying the lemma above to the inclusion \( J^A X \subseteq U \) we have
\[
T_{p'} J^A X \subseteq \pi'_p T_p U \subseteq T_{p'} J^A X.
\]

So, the equality holds and the dimension of \( \pi'_p T_p U \) is the highest possible. Therefore, the rank of \( \pi|_U \) is constant in a neighborhood of \( p \). We deduce that, in a neighborhood of \( p' \), \( \pi'(U) \) is a submanifold. Moreover, also locally, \( \pi'(U) \) contains \( J^A X \) and \( \text{dim} \pi'(U) = \text{dim} J^A X \). As a consequence, near of \( p' \), \( \pi'(U) = J^A X \).

Finally, the proof of the maximality of the solutions \( J^A X \) requires an additional hypothesis on the algebra \( A \).

**Theorem 4.12.** Let us suppose that \( \widehat I_A = 0 \). The prolongations \( J^A X \subseteq J^A M \) (with \( \text{dim} X = m = w(A) \)) are maximal solutions of the contact system. In other words, if \( J^A X \subseteq U \subseteq J^A M \) where \( U \) is a solution of the contact system, then \( \text{dim} J^A X = \text{dim} U \).

**Proof.** Let \( p \in J^A X \subseteq U \) with \( p' = \pi'(p) \) and let us suppose that \( \overline{p} \in U \) is another jet such that \( \pi'((\overline{p})) = \pi'(p) = p' \) and \( \overline{p} \notin J^A X \).

In a suitable local chart \( \{x_i, y_j\} \) we have \( I_X = (y_j) \) and
\[
\overline{p} = (y_j - P_j(x)) + (\overline{Q}_j(x)) + m^{k+1},
\]
for certain polynomials \( P_j(x), \overline{Q}_j(x) \), where at least one among the \( P_j \), say \( P_{j_0}(x) \), is not in \( \overline{p} \) (elsewhere, \( \overline{p} \supset I_X \), and then \( \overline{p} \in J^A X \), in contradiction with the above assumption).

For each given index \( i \), let us pick a tangent vector \( D_{\overline{p}} = [D]_{\overline{p}} \in T_{\overline{p}} U \) such that \( \pi'_p D_{\overline{p}} = \frac{\partial}{\partial x_i} |_{\overline{p}'} \in T_{\overline{p}} J^A X \), which is always possible according to Lemma 4.11. From \( U \) being a solution of the contact system, we get
\[
0 = \partial_{\overline{p}} (y_{j_0} - P_{j_0})(D_{\overline{p}}) = \partial_{\overline{p}} (y_{j_0} - P_{j_0}) (\pi'_p D_{\overline{p}}) = - \frac{\partial P_{j_0}}{\partial x_i} |_{\overline{p}'}.
\]

It is deduced that \( \frac{\partial P_{j_0}}{\partial x_i} \in \pi'(\overline{p}) = p' \). Moreover, \( P_{j_0} \in \pi'(\overline{p}) \) because \( y_{j_0} - P_{j_0} \in \overline{p} \in \pi'(\overline{p}) \) and \( y_{j_0} \in p \subset p' = \pi'(\overline{p}) \). This way, we have a polynomial \( P_{j_0} \not\in \overline{p} \) but
\[ P_{j_0}, \frac{\partial P_0}{\partial x_i} \in \pi'(\mathcal{P}), \ i = 1, \ldots, m. \] As a consequence, \( P_{j_0} \) belongs to the ideal \( \tilde{I} \) of \( \mathcal{C}^\infty(M)/\mathcal{P} \cong A \) and then \( \tilde{I}_A \neq 0. \)

**Corollary 4.13.** On the spaces \( J^k_m \) \( M \) the prolongations of \( m \)-dimensional submanifolds of \( M \) are maximal solutions of the contact system.

**Proof.** It is sufficient to taking into account Example 3.14.

---

**Appendix**

**Lemma 1.** Let \( H, \overline{H} : \mathbb{R}^n_k \to A \) be \( \mathbb{R} \)-algebra epimorphisms; then there exists an automorphism \( g \in \text{Aut}(\mathbb{R}^n_k) \) such that \( H = \overline{H} \circ g. \)

**Proof.** If the classes of \( a_1, \ldots, a_m \) generate \( m_A/\mathbb{m}_A^2 \), one easily deduces that each element in \( A \) can be obtained as a polynomial on \( a_1, \ldots, a_m \). It is not difficult to see that elements \( x_1, \ldots, x_n \) can be chosen in \( \mathbb{R}^n_k \) such that they generate the maximal ideal and we have \( H(x_i) = a_i \) if \( i \leq m \) and \( H(x_{m+j}) = 0. \) Analogously, we can choose a elements \( \overline{x}_1, \ldots, \overline{x}_n \) which hold the same property with respect to \( \overline{H} \). Finally, we define \( g \) by the condition of mapping the first basis to the second one.

**Lemma 2.** Let \( \{a_i\} \) be a basis of \( m_A \) modulo \( \mathbb{m}_A^2 \) and let us choose a collection of multi-indices \( \Lambda \) such that the set \( \{a^\alpha\}_{\alpha \in \Lambda} \) is a basis of \( m_A \). Then, there exist rational functions \( \Psi_{\alpha\beta}, \alpha, \beta \in \Lambda \) such that for any other basis \( \{b_i\} \) of \( m_A \) modulo \( \mathbb{m}_A^2 \), near enough of \( \{a_i\} \) we have

\[ a^\alpha = \sum_{\beta \in \Lambda} \Psi_{\alpha\beta}(\lambda_{i\sigma}) b^\beta, \quad \alpha \in \Lambda, \]

where \( b_i = \sum_{i \sigma \in \Lambda} \lambda_{i\sigma} a^\sigma. \)

**Proof.** Let us suppose the multiplication law on \( A \) being \( a^\alpha a^\sigma = \sum_{\gamma \in \Lambda} c_{\alpha\sigma}^{\gamma} a^\gamma \), \( c_{\alpha\sigma}^{\gamma} \in \mathbb{R} \) (structure constants).

Because each \( b_i \) is near enough of \( a_i \), \( i = 1, \ldots, m \) we deduce that the set of powers \( \{b^\beta\}_{\beta \in \Lambda} \) is also a basis of \( m_A \).

From \( b_i = \sum_{i \sigma \in \Lambda} \lambda_{i\sigma} a^\sigma \) we can write each \( b^\beta \) as a linear combination of the \( a^\alpha \), \( \alpha \in \Lambda \) whose coefficients are polynomials in the \( \lambda_{i\sigma} \) (multiplication law of \( A \)). These linear relations can be inverted and we get the required expressions for \( a^\alpha. \)

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**References**


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