SOME SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

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Abstract. In this paper we introduce a new concept of $\lambda$-strong convergence with respect to an Orlicz function and examine some properties of the resulting sequence spaces. It is also shown that if a sequence is $\lambda$-strongly convergent with respect to an Orlicz function then it is $\lambda$-statistically convergent.

1. Introduction

The concept of paranorm is closely related to linear metric spaces. It is a generalization of that of absolute value. Let $X$ be a linear space. A function $p : X \to \mathbb{R}$ is called paranorm, if

\begin{enumerate}
\item[(P.1)] $p(0) \geq 0$
\item[(P.2)] $p(x) \geq 0$ for all $x \in X$
\item[(P.3)] $p(-x) = p(x)$ for all $x \in X$
\item[(P.4)] $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$ (triangle inequality)
\item[(P.5)] if $(\lambda_n)$ is a sequence of scalars with $\lambda_n \to \lambda(n \to \infty)$ and $(x_n)$ is a sequence of vectors with $p(x_n - x) \to 0$ $(n \to \infty)$, then $p(\lambda_n x_n - \lambda x) \to 0$ $(n \to \infty)$ (continuity of multiplication by scalars).
\end{enumerate}

A paranorm $p$ for which $p(x) = 0$ implies $x = 0$ is called total. It is well known that the metric of any linear metric space is given by some total paranorm (cf. [14, Theorem 10.4.2, p.183]).

Let $\Lambda = (\lambda_n)$ be a non decreasing sequence of positive reals tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$.

The generalized de la Vallée-Poussin means is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be $(V, \lambda)$-summable to a number $\ell$ (see [2]) if $t_n(x) \to \ell$ as $n \to \infty$.

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We write
\[ [V, \lambda]_0 = \left\{ x = x_k : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| = 0 \right\} \]
\[ [V, \lambda] = \left\{ x = x_k : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - \ell e| = 0, \text{ for some } \ell \in C \right\} \]
and
\[ [V, \lambda]_\infty = \left\{ x = x_k : \sup_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| < \infty \right\}. \]

For the sets of sequences that are strongly summable to zero, strongly summable and strongly bounded by the de la Vallée-Poussin method. In the special case where \( \lambda_n = n \) for \( n = 1, 2, 3, \ldots \), the sets \([V, \lambda]_0\), \([V, \lambda]\) and \([V, \lambda]_\infty\) reduce to the sets \( \omega_0\), \( \omega \) and \( \omega_\infty\) introduced and studied by Maddox [5].

Following Lindenstrauss and Tzafriri [4], we recall that an Orlicz function \( M \) is a continuous, convex, non-decreasing function defined for \( x > 0 \) such that \( M(0) = 0 \) and \( M(x) \geq 0 \) for \( x > 0 \).

If convexity of Orlicz function \( M \) is replaced by \( M(x + y) \leq M(x) + M(y) \) then this function is called a modulus function, defined and discussed by Nakano [8], Ruckle [10], Maddox [6] and others.

Lindenstrauss and Tzafriri used the idea of Orlicz function to construct the sequence space
\[ l_M = \left\{ x = (x_k) : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty \text{ for some } \rho > 0 \right\}. \]
The space \( l_M \) with the norm
\[ \|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\} \]
becomes a Banach space which is called an Orlicz sequence space. For \( M(x) = x^p \), \( 1 \leq p < \infty \), the space \( l_M \) coincide with the classical sequence space \( l_p \).

Recently Parashar and Choudhary [9] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function \( M \), which generalized the well-known Orlicz sequence space \( l_M \) and strongly summable sequence spaces \([C, 1, p]_0\), \([C, 1, p]_1\) and \([C, 1, p]_\infty\). It may be noted that the spaces of strongly summable sequences were discussed by Maddox [5].

Quite recently E. Savas [11] has also used an Orlicz function to construct some sequence spaces.

In the present paper we introduce a new concept of \( \lambda \)-strong convergence with respect to an Orlicz function and examine some properties of the resulting sequence spaces. Furthermore it is shown that if a sequence is \( \lambda \)-strongly convergent with respect to an Orlicz function then it is \( \lambda \)-statistically convergent.

We now introduce the generalizations of the spaces of \( \lambda \)-strongly.
We define the following sequence spaces:

\[
[V, M, p] = \left\{ x = (x_k) : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} M\left( \frac{|x_k|}{\rho} \right)^{p_k} = 0 \text{ for some } l \text{ and } \rho > 0 \right\}
\]

\[
[V, M, p]_0 = \left\{ x = (x_k) : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} M\left( \frac{|x_k|}{\rho} \right)^{p_k} = 0 \text{ for some } \rho > 0 \right\}
\]

\[
[V, M, p]_\infty = \left\{ x = (x_k) : \sup_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} M\left( \frac{|x_k|}{\rho} \right)^{p_k} < \infty \text{ for some } \rho > 0 \right\}.
\]

We denote \([V, M, p], [V, M, p]_0\) and \([V, M, p]_\infty\) as \([V, M], [V, M]_0\) and \([V, M]_\infty\) when \(p_k = 1\) for all \(k\). If \(x \in [V, M]\) we say that \(x\) is of \(\lambda\)-strongly convergent with respect to the Orlicz function \(M\). If \(M(x) = x, p_k = 1\) for all \(k\), then \([V, M, p] = [V, \lambda], [V, M, p]_0 = [V, \lambda]_0\) and \([V, M, p]_\infty = [V, \lambda]_\infty\). If \(\lambda_n = n\) then, \([V, M, p], [V, M, p]_0\) and \([V, M, p]_\infty\) reduce the \([C, M, p]_0\) and \([C, M, p]_\infty\) which were studied Parashar and Choudhary [9].

2. Main Results

In this section we examine some topological properties of \([V, M, p], [V, M, p]_0\) and \([V, M, p]_\infty\) spaces.

**Theorem 1.** For any Orlicz function \(M\) and any sequence \(p = (p_k)\) of strictly positive real numbers, \([V, M, p], [V, M, p]_0\) and \([V, M, p]_\infty\) are linear spaces over the set of complex numbers.

**Proof.** We shall prove only for \([V, M, p]_0\). The others can be treated similarly. Let \(x, y \in [V, M, p]_0\) and \(\alpha, \beta \in C\). In order to prove the result we need to find some \(\rho_3 > 0\) such that

\[
\lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} M\left( \frac{\alpha x_k + \beta y_k}{\rho_3} \right)^{p_k} = 0.
\]

Since \(x, y \in [V, M, p]_0\), there exist a positive some \(\rho_1\) and \(\rho_2\) such that

\[
\lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} M\left( \frac{|x_k|}{\rho_1} \right)^{p_k} = 0 \quad \text{and} \quad \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} M\left( \frac{|y_k|}{\rho_2} \right)^{p_k} = 0.
\]
Define $\rho_3 = \max (2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $M$ is non-decreasing and convex,

\[
\frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{\alpha x_k + \beta y_k}{\rho_3} \right) \right]^{p_k} \leq \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|\alpha x_k|}{\rho_3} + \frac{|\beta y_k|}{\rho_3} \right) \right]^{p_k}
\]

\[
\leq \frac{1}{\lambda_n} \sum_{k \in I_n} \frac{1}{2^{p_k}} \left[ M \left( \frac{|x_k|}{\rho_1} \right) + M \left( \frac{|y_k|}{\rho_2} \right) \right]^{p_k}
\]

\[
\leq \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k|}{\rho_1} \right) + M \left( \frac{|y_k|}{\rho_2} \right) \right]^{p_k}
\]

\[
\leq K \cdot \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k|}{\rho_1} \right) \right]^{p_k} + K \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|y_k|}{\rho_2} \right) \right]^{p_k} \to 0
\]

as $n \to \infty$, where $K = \max (1, 2^{H-1})$, $H = \sup p_k$, so that $\alpha x + \beta y \in [V, M, p]_0$.

This completes the proof.

**Theorem 2.** For any Orlicz function $M$ and a bounded sequence $p = (p_k)$ of strictly positive real numbers, $[V, M, p]_0$ is a total paranormed spaces with

\[
g(x) = \inf \left\{ \rho^{p_n/H} : \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k|}{\rho} \right) \right]^{p_k} \right)^{1/H} \leq 1, \ n = 1, 2, 3, \ldots \right\}. \]

where $H = \max(1, \sup p_k)$.

**Proof.** Clearly $g(x) = g(-x)$. By using Theorem 1, for a $\alpha = \beta = 1$, we get $g(x + y) \leq g(x) + g(y)$. Since $M(0) = 0$, we get $\inf \{ \rho^{p_n/H} \} = 0$ for $x = 0$.

Conversely, suppose $g(x) = 0$, then

\[
\inf \left\{ \rho^{p_n/H} : \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k|}{\rho} \right) \right]^{p_k} \right)^{1/H} \leq 1 \right\} = 0.
\]

This implies that for a given $\varepsilon > 0$, there exists some $\rho_{\varepsilon} (0 < \rho_{\varepsilon} < \varepsilon)$ such that

\[
\left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k|}{\rho_{\varepsilon}} \right) \right]^{p_k} \right)^{1/H} \leq 1.
\]

Thus,

\[
\left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k|}{\varepsilon} \right) \right]^{p_k} \right)^{1/H} \leq \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k|}{\rho_{\varepsilon}} \right) \right]^{p_k} \right)^{1/H} \leq 1,
\]

for each $n$.

Suppose that $x_{nm} \neq 0$ for some $m \in I_n$. Let $\varepsilon \to 0$, then $\left( \frac{|x_{nm}|}{\varepsilon} \right) \to \infty$. It follows that

\[
\left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_{nm}|}{\varepsilon} \right) \right]^{p_k} \right)^{1/H} \to \infty
\]
which is a contradiction. Therefore \( x_{n_m} = 0 \) for each \( m \). Finally, we prove that scalar multiplication is continuous. Let \( \mu \) be any complex number. By definition
\[
g(\mu x) = \inf \left\{ \rho^{p_n/H} : \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|\mu x_k|}{\rho} \right) \right]^{p_k} \right)^{1/H} \leq 1, \quad n = 1, 2, 3, \ldots \right\}.
\]
Then
\[
g(\mu x) = \inf \left\{ (|\mu| s)^{p_n/H} : \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k|}{s} \right) \right]^{p_k} \right)^{1/H} \leq 1, \quad n = 1, 2, 3, \ldots \right\}
\]
where \( s = \rho/|\mu| \). Since \( |\mu|^{p_n} \leq \max(1, |\mu|^{\sup p_n}) \), we have
\[
g(\mu x) \leq (\max(1, |\mu|^{\sup p_n}))^{1/H} \times \inf \left\{ s^{p_n/H} : \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k|}{s} \right) \right]^{p_k} \right)^{1/H} \leq 1, \quad n = 1, 2, 3, \ldots \right\}
\]
which converges to zero as \( x \) converges to zero in \([V, M, p]_0\).

Now suppose \( \mu_m \to 0 \) and \( x \) is fixed in \([V, M, p]_0\). For arbitrary \( \varepsilon > 0 \), let \( N \) be a positive integer such that
\[
\frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k|}{\rho} \right) \right]^{p_k} < (\varepsilon/2)^H \quad \text{for some} \quad \rho > 0 \quad \text{and all} \quad n > N.
\]
This implies that
\[
\frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|x_k|}{\rho} \right) \right]^{p_k} < \varepsilon/2 \quad \text{for some} \quad \rho > 0 \quad \text{and all} \quad n > N.
\]
Let \( 0 < |\mu| < 1 \), using convexity of \( M \), for \( n > N \), we get
\[
\frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|\mu x_k|}{\rho} \right) \right]^{p_k} < \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ |\mu| M \left( \frac{|x_k|}{\rho} \right) \right]^{p_k} < (\varepsilon/2)^H.
\]
Since \( M \) is continuous everywhere in \([0, \infty)\), then for \( n \leq N \)
\[
f(t) = \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|tx_k|}{\rho} \right) \right]^{p_k}
\]
is continuous at 0. So there is \( 1 > \delta > 0 \) such that \( |f(t)| < (\varepsilon/2)^H \) for \( 0 < t < \delta \).
Let \( K \) be such that \( |\mu_m| < \delta \) for \( m > K \) then for \( m > K \) and \( n \leq N \)
\[
\left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|\mu_m x_k|}{\rho} \right) \right]^{p_k} \right)^{1/H} < \varepsilon/2.
\]
Thus
\[
\left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ M \left( \frac{|\mu_m x_k|}{\rho} \right) \right]^{p_k} \right)^{1/H} < \varepsilon.
\]
for \( m > K \) and all \( n \), so that \( g(\mu x) \to 0 \) (\( \mu \to 0 \)).

**Definition 2** ([1]). An Orlicz function \( M \) is said to satisfy \( \Delta_2 \)-condition for all values of \( u \), if there exists a constant \( K > 0 \) such that \( M(2u) \leq KM(u) \), \( u \geq 0 \).

It is easy to see that always \( K > 2 \). The \( \Delta_2 \)-condition is equivalent to the satisfaction of inequality \( M(lu) \leq K(l)M(u) \), for all values of \( u \) and for \( l > 1 \).

**Theorem 3.** For any Orlicz function \( M \) which satisfies \( \Delta_2 \)-condition, we have \([V, \lambda] \subseteq [V, M]\).

**Proof.** Let \( x \in [V, \lambda] \) so that

\[
T_n = \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - \ell| \to 0 \quad \text{as} \quad n \to \infty \quad \text{for some} \quad \ell.
\]

Let \( \varepsilon > 0 \) and choose \( \delta > 0 \) such that \( M(t) < \varepsilon \) for \( 0 \leq t \leq \delta \). Write \( y_k = |x_k - \ell| \) and consider

\[
\frac{1}{\lambda_n} \sum_{k \in I_n} M(|y_k|) = \sum_1 + \sum_2
\]

where the first summation is over \( y_k \leq \delta \) and the second summation over \( y_k > \delta \). Since, \( M \) is continuous

\[
\sum_1 < \lambda_n \varepsilon
\]

and for \( y_k > \delta \) we use the fact that \( y_k < y_k/\delta < 1 + y_k/\delta \). Since \( M \) is non-decreasing and convex, it follows that

\[
M(y_k) < M\left(1 + \delta^{-1}y_k\right) < \frac{1}{2}M(2) + \frac{1}{2}M\left(2\delta^{-1}y_k\right)
\]

Since \( M \) satisfies \( \Delta_2 \)-condition there is a constant \( K > 2 \) such that \( M\left(2\delta^{-1}y_k\right) \leq \frac{1}{2}K\delta^{-1}y_kM(2) \), therefore

\[
M(y_k) < \frac{1}{2}K\delta^{-1}y_kM(2) + \frac{1}{2}K\delta^{-1}y_kM(2)
\]

\[
= K\delta^{-1}y_kM(2).
\]

Hence

\[
\sum_2 M(y_k) \leq K\delta^{-1}M(2)\lambda_n T_n
\]

which together with \( \sum_1 \leq \varepsilon \lambda_n \) yields \([V, \lambda] \subseteq [V, M]\). This completes proof.

The method of the proof of Theorem 3 shows that for any Orlicz function \( M \) which satisfies \( \Delta_2 \)-condition; we have \([V, \lambda]_0 \subseteq [V, M]_0 \) and \([V, \lambda]_\infty \subseteq [V, M]_\infty \).

**Theorem 4.** Let \( 0 \leq p_k \leq q_k \) and \( (q_k/p_k) \) be bounded. Then \([V, M, q] \subseteq [V, M, p]\).

The proof of Theorem 4 used the ideas similar to those used in proving Theorem 7 of Parashar and Choudhary [9].

We now introduce a natural relationship between strong convergence with respect to an Orlicz function and \( \lambda \)-statistical convergence. Recently, Mursaleen [7] introduced the concept of statistical convergence as follows:
Definition 3. A sequence \( x = (x_k) \) is said to be \( \lambda \)-statistically convergent or \( s_\lambda \)-statistically convergent to \( L \) if for every \( \varepsilon > 0 \)
\[
\lim_{n} \frac{1}{\lambda_n} |\{ k \in I_n : |x_k - L| \geq \varepsilon \}| = 0,
\]
where the vertical bars indicate the number of elements in the enclosed set.

In this case we write \( s_\lambda \)-\( \lim x = L \) or \( x_k \to L(\lambda) \) and \( s_\lambda = \{ x : \exists L \in R : s_\lambda - \lim x = L \} \).

Later on, \( \lambda \)-statistical convergence was generalized by Savas [12].

We now establish an inclusion relation between \([V, M]\) and \( s_\lambda \).

Theorem 5. For any Orlicz function \( M \), \([V, M] \subset s_\lambda \).

Proof. Let \( x \in [V, M] \) and \( \varepsilon > 0 \). Then
\[
\frac{1}{\lambda_n} \sum_{k \in I_n} M \left( \frac{|x_k - \ell|}{\rho} \right) \geq \frac{1}{\lambda_n} \sum_{k \in I_n, |x_k - \ell| \geq \varepsilon} M \left( \frac{|x_k - \ell|}{\rho} \right) \geq \frac{1}{\lambda_n} M \left( \frac{\varepsilon}{\rho} \right) \cdot |\{ k \in I_n : |x_k - \ell| \geq \varepsilon \}|
\]
from which it follows that \( x \in s_\lambda \).

To show that \( s_\lambda \) strictly contains \([V, M]\), we proceed as in [7]. We define \( x = (x_k) \) by \( x_k = k \) if \( n - \sqrt{\lambda_n} + 1 \leq k \leq n \) and \( x_k = 0 \) otherwise. Then \( x \notin \ell_\infty \) and for every \( \varepsilon \) \((0 < \varepsilon \leq 1)\)
\[
\frac{1}{\lambda_n} |\{ k \in I_n : |x_k - 0| \geq \varepsilon \}| = \frac{\sqrt{\lambda_n}}{\lambda_n} \to 0 \quad \text{as} \quad n \to \infty
\]
i.e. \( x_k \to 0(s_\lambda) \), where \( [\ ] \) denotes the greatest integer function. On the other hand,
\[
\frac{1}{\lambda_n} \sum_{k \in I_n} M \left( \frac{|x_k - 0|}{\rho} \right) \to \infty \quad (n \to \infty)
\]
i.e. \( x_k \not\sim 0[V, M] \). This completes the proof. \( \square \)

References


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