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# THE TANAKA–WEBSTER CONNECTION FOR ALMOST $\mathcal{S}$-MANIFOLDS AND CARTAN GEOMETRY

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Abstract. We prove that a CR-integrable almost $\mathcal{S}$-manifold admits a canonical linear connection, which is a natural generalization of the Tanaka–Webster connection of a pseudo-hermitian structure on a strongly pseudoconvex CR manifold of hypersurface type. Hence a CR-integrable almost $\mathcal{S}$-structure on a manifold is canonically interpreted as a reductive Cartan geometry, which is torsion free if and only if the almost $\mathcal{S}$-structure is normal. Contrary to the CR-codimension one case, we exhibit examples of non normal almost $\mathcal{S}$-manifolds with higher CR-codimension, whose Tanaka–Webster curvature vanishes.

## 1. Introduction

In [3] D. E. Blair initiated the study of the differential geometry of manifolds carrying an $U(k) \times O(s)$-structure. These are exactly the manifolds $M$ which admit an $f$-structure, i.e. a tensor field $\varphi$ of type $(1,1)$ with constant rank $2k$, and such that $\varphi^3 + \varphi = 0$. This kind of structure was investigated first by K. Yano in [15]. An $f$–structure provides a splitting of the tangent bundle

$$TM = \mathrm{Ker}(\varphi) \oplus \mathrm{Im}(\varphi)$$

and the restriction $J$ of $\varphi$ to $\mathcal{D} = \mathrm{Im}(\varphi)$ is a partial complex structure, that is $J^2 = -\mathrm{Id}$. Hence $M$ is an almost CR manifold having CR-dimension $k$ and CR-codimension $s = n - 2k$, where $n = \dim_{\mathbf{R}} M$. Actually, an $f$-structure is equivalent to an almost CR structure $(\mathcal{D}, J)$ together with the choice of a complementary subbundle to $\mathcal{D}$ in $TM$. Here we restrain our attention to the case where the subbundle $\mathrm{Ker}(\varphi)$ is trivial, i.e. the structure group can be further reduced to $U(k) \times I_s$. In this case $M$ is called an $f$-manifold with parallelizable kernel ($f{\cdot}$pk manifold). From the CR point of view, this is equivalent to the triviality of the annihilator $\mathcal{D}^0 M$ of the analytic tangent bundle $\mathcal{D}$, which is the subbundle of the cotangent bundle $T^*M$ whose fiber is $\mathcal{D}^0_x M = \{\eta \in T^*_x M \mid \eta(X) = 0 \ \forall X \in \mathcal{D}_x\}$.

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Notice that $\mathcal{D}^0 M$ is automatically trivial for any orientable almost CR manifold of hypersurface type ($s = 1$); in this case, a trivialization $\eta$ of $\mathcal{D}^0 M$ is usually called a pseudohermitian structure and $(M, \mathcal{D}, J, \eta)$ is called a pseudohermitian manifold.

A *metric $f$·pk manifold* is an $f$·pk manifold endowed with a Riemannian metric $g$ such that

$$g(X,Y) = g(\varphi X, \varphi Y) + \sum_{i=1}^{s} \eta^i(X)\eta^i(Y) \tag{1}$$

where $\{\eta^i\}_{i=1,\dots,s}$ is a fixed trivialization of $\mathcal{D}^0 M$. Notice that $\varphi$ is then skew--symmetric with respect to $g$.

In [4] an *almost $\mathcal{S}$-manifold* is defined as a metric $f$·pk manifold such that

$$d\eta^i = \Phi \qquad i = 1, \dots, s \tag{2}$$

where $\Phi$ is the fundamental 2-form of the $f$·pk structure, defined as usual by $\Phi(X,Y) = g(X, \varphi Y)$.

This notion is a natural generalization of the concept of contact metric structure, which corresponds to the case $s = 1$ (cf. [2]).

It is known that an orientable almost CR manifold $(M, \mathcal{D}, J)$ of hypersurface type is an almost $\mathcal{S}$-manifold with underlying almost CR structure $(\mathcal{D}, J)$ if and only if *i*) $J$ is partially integrable, i.e. $[X,Y] - [JX, JY] \in \mathcal{D}$ for all sections $X, Y$ of $\mathcal{D}$, and *ii*) a pseudohermitian structure $\eta$ can be chosen with positive definite Levi form $\mathcal{L}_\eta$. Recall that $\mathcal{L}_\eta$ is defined by $\mathcal{L}_\eta(X,Y) = d\eta(JX, Y)$ for all $X, Y \in \mathcal{D}$. When these two conditions are satisfied, a pseudohermitian structure $\eta$ as in *ii*) uniquely determines an $f$-structure $\varphi$ extending $J$ and a compatible metric $g$ satisfying the above conditions (1) and (2) with $\eta^1 = \eta$. If moreover *i*) is replaced by CR-integrability, $(M, \mathcal{D}, J)$ is called a strongly pseudoconvex CR manifold (see e.g. [11]).

The strongly pseudoconvex CR manifolds have been investigated by several authors, and one of their fundamental properties is the existence of a unique linear connection $\tilde{\nabla}$ such that the tensors $\varphi, \eta, g$ are all $\tilde{\nabla}$-parallel and whose torsion satisfies

$$\tilde{T}(X,Y) = 2\Phi(X,Y)\xi \quad \text{for all} \quad X, Y \in \mathcal{D}, \tag{3}$$

$$\tilde{T}(\xi, \varphi X) = -\varphi \tilde{T}(\xi, X) \quad \text{for all} \quad X \in \mathcal{X}(M)\,. \tag{4}$$

Here $\xi$ is the dual vector field of $\eta$ with respect to the metric $g$.

This connection was introduced first by N. Tanaka in [10], and independently by Webster in [14]. We remark that $\tilde{\nabla}$ actually depends not only on the CR structure but also on the choice of the pseudohermitian structure $\eta$.

In this paper we provide a geometrical characterization of condition (2), showing that a metric $f$·pk manifold admits a connection $\tilde{\nabla}$ having the same formal properties as (3)-(4) (cf. (6)–(7) in sec. 2), with the additional requirement that $\tilde{T}$ vanishes on $\mathrm{Ker}(\varphi)$, if and only if (2) holds and the almost CR structure $(\mathcal{D}, J)$ is integrable. This connection is uniquely determined and hence we call it the Tanaka–Webster connection of a CR-integrable almost $\mathcal{S}$-manifold.

This result is also interpreted from the point of view of Cartan's method of equivalence, showing that the datum of a CR-integrable almost $\mathcal{S}$-structure on a manifold admits a canonical interpretation as a reductive Cartan geometry (cf. [9]).

We also obtain that, as a Cartan geometry, a CR-integrable almost $\mathcal{S}$-structure is *torsion free* if and only if the tensor field

$$N = [\varphi, \varphi] + 2d\eta^i \otimes \xi_i$$

vanishes, where $[\varphi, \varphi]$ is the Nijenhuis torsion of $\varphi$, while $\{\xi_i\}$ is the $g$-orthonormal frame of $\mathrm{Ker}(\varphi)$ dual of $\{\eta^i\}$. This is the *normality condition* considered by Blair in [3], where an almost $\mathcal{S}$-manifold satisfying $N = 0$ is called an $\mathcal{S}$-manifold.

Finally, we exhibit examples of $\tilde{\nabla}$-flat *non* normal almost $\mathcal{S}$-manifolds with CR codimension $s > 1$. This is interesting since it is easily seen that a strongly pseudoconvex CR manifold of hypersurface type with vanishing Tanaka–Webster curvature is necessarily normal.

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## 2. The Tanaka–Webster connection of a CR-integrable almost $\mathcal{S}$-manifold

Let $M^{2k+s}$ be a metric $f{\cdot}\mathrm{pk}$ manifold with structure $(\varphi, \xi_i, \eta^i, g)$.

Let $\nabla$ be the Levi–Civita connection of $g$. Denote by $Q$ the tensor field of type $(1, 2)$ on $M$ defined by

$$\begin{aligned} Q(X,Y) := (\nabla_X\varphi)Y + \Phi(X,\varphi Y)\bar{\xi} - g(h_j X, Y)\xi_j \\ - \bar{\eta}(Y)\varphi^2 X + \eta^j(Y) h_j X\,. \end{aligned} \tag{5}$$

Here and in the following the sum symbol for repeated indices is omitted. In this formula $\bar{\xi} := \sum_{i=1}^{s} \xi_i$, $\bar{\eta} := \sum_{i=1}^{s} \eta_i$, while $h_i$ is the operator $h_i = \frac{1}{2}\mathcal{L}_{\xi_i}\varphi$. $\Phi$ denotes the fundamental 2-form defined by $\Phi(X,Y) = g(X, \varphi Y)$.

For basic properties of almost $\mathcal{S}$-manifolds, we refer the reader to [4]. In particular, we have the following:

**Proposition 2.1** ([4])**.** *Assume that $M$ is an almost $\mathcal{S}$-manifold. Then:*

1) *Each $h_i$ is a self-adjoint operator anti-commuting with $\varphi$.*
2) *Each $h_i$ vanishes on* $\mathrm{Ker}(\varphi)$ *and takes values in $\mathcal{D}$.*
3) *For each $i, j = 1, \dots, s$ we have*

$$\begin{aligned} &\nabla_{\xi_i}\varphi = 0,\ \nabla_{\xi_i}\xi_j = 0\,, \\ &\nabla_X \xi_i = -\varphi(X) - \varphi h_i(X)\,. \end{aligned}$$

4) *$M$ is CR-integrable, that is the partial complex structure $J$ induced by $\varphi$ on $\mathcal{D} = \mathrm{Im}(\varphi)$ is formally integrable, if and only if $Q \equiv 0$.*

In this section we prove the following geometric characterization of the CR--integrable almost $\mathcal{S}$-manifolds:

**Theorem 2.2.** *Let $M$ be a metric $f\cdot pk$-manifold with structure $(\varphi,\xi_i,\eta^i,g)$. Then $M$ is a CR–integrable almost $\mathcal{S}$-manifold if and only if it admits a linear connection $\tilde\nabla$ with the following properties:*

1) $\tilde\nabla\varphi=0,\ \tilde\nabla g=0$ *and* $\tilde\nabla\eta^i=0$ *for each* $i\in\{1,\dots,s\}$;
2) *The torsion $\tilde T$ of $\tilde\nabla$ satisfies:*

$$\tilde T(X,Y)=2\Phi(X,Y)\bar\xi \quad \text{for all} \quad X,Y\in\mathcal{D}\,, \tag{6}$$

$$\tilde T(\xi_i,\varphi X)=-\varphi\tilde T(\xi_i,X) \quad \text{for all} \quad X\in\mathcal{X}(M)\,,\quad i\in\{1,\dots,s\}\,, \tag{7}$$

$$\tilde T(\xi_i,\xi_j)=0\,,\quad i,j\in\{1,\dots,s\}\,. \tag{8}$$

*Such a linear connection $\tilde\nabla$ is uniquely determined.*

Notice that in the case $s=1$, condition (8) is vacuous, and a CR-integrable almost $\mathcal{S}$-manifold is a strictly pseudoconvex CR manifold of hypersurface type (cf. e.g. [11], [13]); hence $\tilde\nabla$ coincides with the Tanaka–Webster connection (cf. [10], [13], [11]). For this reason, we shall adopt the name Tanaka–Webster connection to refer to $\tilde\nabla$ also in the higher CR-codimension case.

We remark that the factor 2 in (6) appears since we follow the convention of [5] for the exterior derivative (the same convention is adopted in Blair's book [2]).

To prove Theorem 2.2 we start by defining a tensor field $H$ of type $(1,2)$, $H:\mathcal{X}(M)\times\mathcal{X}(M)\to\mathcal{X}(M)$, such that

$$\begin{aligned} H(X,Y)=\Phi(X,Y)\bar\xi+\bar\eta(Y)\varphi(X)+\bar\eta(X)\varphi(Y)\\ +\Phi(h_jX,Y)\xi_j+\eta^j(Y)\varphi h_j(X)\,. \end{aligned}$$

**Lemma 2.3.** *For all $X,Y,Z\in\mathcal{X}(M)$ we have:*

$$g(H(X,Y),Z)+g(H(X,Z),Y)=0\,; \tag{9}$$

*moreover, if $M$ is an almost $\mathcal{S}$-manifold:*

$$H(X,Y)-H(Y,X)=2\Phi(X,Y)\bar\xi+\eta^j(Y)\varphi h_j(X)-\eta^j(X)\varphi h_j(Y) \tag{10}$$

**Proof.** Notice that for all $X,Y,Z\in\mathcal{X}(M)$ we have

$$\begin{aligned} g(H(X,Y),Z)=\Phi(X,Y)\bar\eta(Z)+\Phi(Z,X)\bar\eta(Y)+\Phi(Z,Y)\bar\eta(X)\\ +\Phi(h_jX,Y)\eta^j(Z)+\Phi(Z,h_jX)\eta^j(Y)\,; \end{aligned}$$

interchanging $Y$ and $Z$ in this formula we get

$$\begin{aligned} g(H(X,Z),Y)=\Phi(X,Z)\bar\eta(Y)+\Phi(Y,X)\bar\eta(Z)+\Phi(Y,Z)\bar\eta(X)\\ +\Phi(h_jX,Z)\eta^j(Y)+\Phi(Y,h_jX)\eta^j(Z)\,, \end{aligned}$$

and (9) follows. To prove (10), it suffices to observe that, assuming that $M$ is an almost $\mathcal{S}$ manifold, then the operators $h_j$ are self–adjoint and they anti-commute with $\varphi$; this yields

$$\Phi(h_jX,Y)=\Phi(h_jY,X)$$

for all $X,Y\in\mathcal{X}(M)$, and this implies (10). □

**Lemma 2.4.** *Assume that $M$ admits a linear connection $\tilde{\nabla}$ satisfying properties* 1), 2) *stated in Theorem 2.2. Then we have:*

i) $\tilde{\nabla}\xi_i = 0 \quad i \in \{1, \ldots, s\}$;

ii) $\tilde{\nabla}_Z X \in \mathcal{D}$ *for all* $X \in \mathcal{D}$ *and* $Z \in \mathcal{X}(M)$;

iii) $[\xi_i, \mathcal{D}] \subset \mathcal{D}$;

iv) *For all* $X \in \mathcal{X}(M)$ *and for each* $i \in \{1, \ldots, s\}$, *we have*

$$\tilde{T}(\xi_i, X) = -\varphi h_i(X) = -\frac{1}{2}N(X, \xi_i)\,. \tag{11}$$

**Proof.** i) Since $\tilde{\nabla}$ is metric, we get, for all $X, Y \in \mathcal{X}(M)$:

$$\begin{aligned} g(\tilde{\nabla}_X \xi_i, Y) &= X \cdot g(\xi_i, Y) - g(\xi_i, \tilde{\nabla}_X Y) \\ &= X \cdot \eta^i(Y) - \eta^i(\tilde{\nabla}_X Y) = (\tilde{\nabla}_X \eta^i)(Y) = 0\,. \end{aligned}$$

ii) This is clear since

$$\eta^i(\tilde{\nabla}_Z X) = Z \cdot \eta^i(X) - (\tilde{\nabla}_Z \eta^i)X = 0\,;$$

iii) Expanding formula (7), and using i), we have

$$\tilde{\nabla}_{\xi_i}\varphi X - [\xi_i, \varphi X] = -\varphi\tilde{\nabla}_{\xi_i} X + \varphi[\xi_i, X]\,;$$

using $\tilde{\nabla}\varphi = 0$, this equation can be rewritten as follows:

$$2\varphi(\tilde{\nabla}_{\xi_i} X) = [\xi_i, \varphi X] + \varphi[\xi_i, X]\,. \tag{12}$$

Notice that this formula implies that for all $X \in \mathcal{X}(M)$, we have $[\xi_i, \varphi X] \in \mathcal{D}$, thus proving iii). Now, assume that $X \in \mathcal{D}$; applying $\varphi$ to both sides of (12), we get

$$-2\tilde{\nabla}_{\xi_i} X = \varphi[\xi_i, \varphi X] - [\xi_i, X]$$

which implies

$$\tilde{T}(\xi_i, X) = -\frac{1}{2}\{\varphi[\xi_i, \varphi X] + [\xi_i, X]\}\,. \tag{13}$$

On the other hand, by definition

$$h_i(X) = \frac{1}{2}\{[\xi_i, \varphi X] - \varphi[\xi_i, X]\}$$

so that

$$\varphi h_i(X) = \frac{1}{2}\{\varphi[\xi_i, \varphi X] + [\xi_i, X]\}\,.$$

This proves the equality

$$\tilde{T}(\xi_i, X) = -\varphi h_i(X)$$

for $X \in \mathcal{D}$. Since by hypothesis $\tilde{T}(\xi_i, \xi_j) = 0$, in force of i) we also have $[\xi_i, \xi_j] = 0$, and this gives $h_i(\xi_j) = 0$. Hence we conclude that the above equality is actually valid for all $X \in \mathcal{X}(M)$. The lemma is proved. □

**Proof of Theorem 2.2.** Define a linear connection $\tilde{\nabla}$ on $M$ by

$$\tilde{\nabla} := \nabla + H \tag{14}$$

where $\nabla$ is the Levi–Civita connection relative to $g$. We have

$$\begin{aligned}(\tilde{\nabla}_X\varphi)Y &= (\nabla_X\varphi)Y + H(X,\varphi Y) - \varphi H(X,Y)\\ &= (\nabla_X\varphi)Y + \Phi(X,\varphi Y)\bar{\xi} + g(h_jX,\varphi^2 Y)\xi_j\\ &\quad - \bar{\eta}(Y)\varphi^2 X - \eta^j(Y)\varphi^2 h_j X\\ &= Q(X,Y) + \eta^k(Y)(\eta^k(h_jX) - \eta^j(h_kX))\xi_j\,.\end{aligned}$$

Notice that when $M$ is an almost $\mathcal{S}$-manifold, according to Prop. 2.1, since the operators $h_j$ take values in $\mathcal{D}$, the above formula simplifies to

$$\tilde{\nabla}\varphi = Q\,. \tag{15}$$

Now, assume that $M$ is a CR-integrable almost $\mathcal{S}$-manifold. Then $Q = 0$, and (15) yields $\tilde{\nabla}\varphi = 0$. Moreover, since $\nabla g = 0$, it is an immediate consequence of (9) that $\tilde{\nabla} g = 0$. Using the formula (Prop. 2.1)

$$\nabla_X\xi_i = -\varphi(X) - \varphi h_i(X)\,,$$

we also get

$$\begin{aligned}(\tilde{\nabla}_X\,\eta^i)Y &= Xg(Y,\xi_i) - \eta^i(\tilde{\nabla}_XY)\\ &= g(\nabla_XY,\xi_i) + g(Y,\nabla_X\xi_i) - \eta^i(\nabla_XY) - \eta^i(H(X,Y))\\ &= -g(Y,\varphi X) - g(Y,\varphi h_iX) - g(H(X,Y),\xi_i) = 0\,.\end{aligned}$$

Finally, notice that

$$\tilde{T}(X,Y) = H(X,Y) - H(Y,X)\,;$$

by virtue of (10), taking into account that each $h_i$ vanishes on $\mathrm{Ker}(\varphi)$, this implies that $\tilde{T}$ has properties (6)–(8). We have thus proved the existence of a linear connection having the properties stated in the theorem, under the assumption that $M$ is a CR-integrable almost $\mathcal{S}$-manifold. To show the converse, we first prove that the equations

$$d\eta^i(X,Y) = \Phi(X,Y) \quad i\in\{1,\dots,s\} \tag{16}$$

hold as a consequence of the existence of $\tilde{\nabla}$. Indeed, if $X,Y\in\mathcal{D}$, from (6) we have

$$\tilde{\nabla}_XY - \tilde{\nabla}_YX - [X,Y] = 2\Phi(X,Y)\bar{\xi}\,,$$

which gives

$$g(\tilde{\nabla}_XY,\xi_i) - g(\tilde{\nabla}_YX,\xi_i) - \eta^i([X,Y]) = 2\Phi(X,Y)\,.$$

Observe that, since $\tilde{\nabla}$ is metric and $\xi_i$ is parallel with respect to $\tilde{\nabla}$, we have $g(\tilde{\nabla}_XY,\xi_i) = g(\tilde{\nabla}_YX,\xi_i) = 0$. Hence $\eta^i([X,Y]) = -2\Phi(X,Y)$ and this shows that (16) holds for $X,Y\in\mathcal{D}$. Using iii) in the above lemma, we also get $d\eta^i(\xi_k,X) = 0 = \Phi(\xi_k,X)$ for $X\in\mathcal{D}$ and since $[\xi_k,\xi_j] = 0$ (see the proof of iv) in the same lemma), we also have $d\eta^i(\xi_k,\xi_j) = 0 = \Phi(\xi_k,\xi_j)$. These facts imply (16), that is $M$ is an almost $\mathcal{S}$-manifold. To conclude the proof of the theorem, we make the following

**Claim**: *Let $\tilde{\nabla}$ be a linear connection satisfying conditions* 1) *and* 2) *in Theorem 2.2; then $\tilde{\nabla}$ is given by formula* (14).

Clearly, this implies the uniqueness assertion about $\tilde{\nabla}$. Moreover, since $M$ is an almost $\mathcal{S}$-manifold, using (15) again, we get $Q = 0$, that is $M$ is CR–integrable. To prove the claim, set $\nabla' := \tilde{\nabla} - H$; then $\nabla'$ is a linear connection. We just have to verify that $\nabla'$ is metric and without torsion. Since $\tilde{\nabla}$ is metric, we obtain

$$Xg(Y,Z) = g(\nabla'_X Y, Z) + g(Y, \nabla'_X Z) + g(H(Z,X),Y) + g(Y, H(X,Z))$$

for all $X, Y, Z \in \mathcal{X}(M)$, and in force of (9) this implies that $\nabla'$ is metric. Clearly, the condition that $\nabla'$ be torsionless is equivalent to

$$\tilde{T}(X,Y) = H(X,Y) - H(Y,X)\,;$$

taking into account (10), the validity of this equation is an immediate consequence of the formulas

$$\tilde{T}(X,Y) = 2\Phi(X,Y)\bar{\xi}\,, \quad \tilde{T}(\xi_i, Z) = -\varphi h_i(Z)\,, \quad X, Y \in \mathcal{D},\ Z \in \mathcal{X}(M)$$

which hold by assumption on $\tilde{\nabla}$ and by virtue of Lemma 2.4. This completes the proof of Theorem 2.2. □

**Corollary 2.5.** *Let $M$ be a CR-integrable almost $\mathcal{S}$-manifold with Tanaka–Webster connection $\tilde{\nabla}$. Then $M$ is normal, i.e. the tensor $N = [\varphi, \varphi] + 2d\eta^i \otimes \xi_i$ vanishes, if and only if*

$$\tilde{T}(\xi_i, X) = 0, \quad \textit{for all} \quad X \in \mathcal{D},\ i \in \{1, \ldots, s\}\,.$$

We end this section with a remark on the relationship between Theorem 2.2 and a result of R. Mizner [8]. Let $M$ be an almost $\mathcal{S}$-manifold with structure $(\varphi, \xi_i, \eta^i, g)$. Denote by $TM^{\mathbf{C}}$ the complexified tangent bundle of $M$, and let $\mathcal{H}$ be the complex version of the almost CR structure $(\mathcal{D}, J)$, namely the distribution $\mathcal{H} \subset TM^{\mathbf{C}}$ defined by

$$\mathcal{H}_p = \{Z \in \mathcal{D}_p^{\mathbf{C}} \mid JZ = iZ\} = \{X - iJX \mid X \in \mathcal{D}_p\}\,.$$

It is easily verified that the almost CR structure under consideration is partially integrable, namely $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H} \oplus \bar{\mathcal{H}}$. Moreover the 1-forms $\{\eta^1, \ldots, \eta^s\}$ make up an *annhilating frame*, i.e. a globally defined frame for the annihilator $\mathcal{D}^0 M$ of $\mathcal{D}$. In the terminology of Mizner ([8], p. 1341), such a frame is *nondegenerate* of type $\{1, \ldots, s\}$. This means that at each point $p \in M$, and for each $j \in \{1, \ldots, s\}$, $\eta^j \circ \mathcal{L}_p$ is a nondegenerate hermitian form on $\mathcal{H}_p$, where

$$\mathcal{L}_p : \mathcal{H}_p \times \mathcal{H}_p \to T_p M^{\mathbf{C}} / \mathcal{H}_p^{\mathbf{C}}$$

is the Levi form (cf. e.g. [8], p. 1340). We recall that $\mathcal{L}_p$ is defined by

$$\mathcal{L}_p(Z_p, W_p) = i\pi[Z, \bar{W}]_p\,, \quad Z_p, W_p \in \mathcal{H}_p$$

where $Z$ and $W$ are arbitrary extensions of the tangent vectors $Z_p, W_p$ to sections of $\mathcal{H}$. In the present situation, if $Z \in \mathcal{H}_p$, $Z = X - iJX$, with $X \in \mathcal{D}_p$, we have

$$\begin{aligned}(\eta^j \circ \mathcal{L}_p)(Z_p) &= i\eta^j([Z, \bar{Z}]_p) = -2i\, d\eta^j(Z, \bar{Z}) \\ &= -2i\Phi(Z, \bar{Z}) = -2ig(Z, J\bar{Z}) \\ &= -2g(Z, \bar{Z}) = -4g(X, X)\end{aligned}$$

so that $\eta^j \circ \mathcal{L}_p$ is negative definite. The main result in [8] states that a globally defined nondegenerate annhilating frame for a partially integrable almost CR structure canonically determines an affine connection $\nabla'$. This connection is uniquely determined by the following requirements. Consider the decomposition of $TM^{\mathbf{C}}$

$$TM^{\mathbf{C}} = E_1 \oplus E_2 \oplus E_3 \oplus \cdots \oplus E_{s+2}$$

where $E_1 := \mathcal{H}$, $E_2 := \bar{\mathcal{H}}$, and for each $i \in \{1, \ldots, s\}$, $E_{i+2}$ is the complex line bundle spanned by $\xi_i$. Then $\mathcal{E} = \{E_1, \ldots, E_{s+2}\}$ is an almost product structure, whose *torsion* is the skew-symmetric bilinear map $\tau : TM^{\mathbf{C}} \times TM^{\mathbf{C}} \to TM^{\mathbf{C}}$ defined as follows:

$$\tau := \frac{1}{2} \sum_{i=1}^{s+2} \pi_i [\pi_i, \pi_i] \, ,$$

where $\pi_i : TM^{\mathbf{C}} \to E_i$ denotes the natural projection, and $[\pi_i, \pi_i]$ is the Nijenhuis torsion of $\pi_i$. It is known that for all $i, j \in \{1, \ldots, s+2\}$ and for all $Z_i \in \Gamma E_i$, $Z_j \in \Gamma E_j$:

$$\tau(Z_i, Z_j) = \sum_{k \neq i,j} [Z_i, Z_j]_k$$

where $[Z_i, Z_j]_k = \pi_k \, [Z_i, Z_j]$. Then Mizner's connection $\nabla'$ is the unique affine connection on $M$ whose $\mathbf{C}$-linear extension to $TM^{\mathbf{C}}$ satifies the following conditions:

1. $\nabla'$ is a parallelizing connection for $\mathcal{E}$;
2. $T'_{ij} = -\tau_{ij}$ for all distinct $i, j \in \{1, \ldots, s+2\}$;
3. $\nabla'_{\xi_i} \xi_i = 0$ for all $i \in \{1, \ldots, s\}$
4. $\nabla'_X \tau_{123} = 0$ for any $X \in \Gamma\mathcal{H}$.

Here $T'$ is the torsion of $\nabla'$, and we have adopted the following convention: for a map $F : TM^{\mathbf{C}} \times TM^{\mathbf{C}} \to TM^{\mathbf{C}}$, and for all $i, j, k \in \{1, \ldots, s+2\}$,

$$F_{ij} : E_i \times E_j \to TM^{\mathbf{C}} \, , \quad F_{ijk} : E_i \times E_j \to E_k$$

denote the maps obtained from $F$ in the obvious way.

**Theorem 2.6.** *Let $M$ be an almost $\mathcal{S}$-manifold with structure $(\varphi, \xi_i, \eta^i, g)$, and let $\nabla'$ be its Mizner's connection according to the above discussion. Then the following conditions are equivalent:*

(a) *$M$ is CR-integrable;*

(b) *$T'(Z, W) = 0$ for all $Z, W \in \Gamma\mathcal{H}$.*

*When* (a) *or* (b) *holds, $\nabla'$ coincides with the Tanaka–Webster connection $\tilde{\nabla}$ of $M$ according to Theorem 2.2.*

**Proof.** To prove (b)⇒(a), it suffices to use the following relation which holds as a consequence of the conditions defining $\nabla'$ (for a proof see [8], p. 1353):

$$T'_{iij} = -\tau_{iij} \quad \text{for all distinct} \quad i, j \in \{1, \ldots, s+2\}.$$

Assuming (b), applying this relation for $i = 1$ we get $\tau_{11j}(Z, W) = 0$, for all sections $Z, W$ of $\mathcal{H}$, which means $[Z, W]_j = 0$ for all $j \geq 2$. This proves that $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$, i.e. $M$ is CR-integrable.

In order to prove that (a)⇒(b), it suffices to show that if $M$ is CR-integrable, then the Tanaka–Webster connection $\tilde{\nabla}$ coincides with $\nabla'$. After this, $(b)$ follows from

$$\tilde{T}(X, Y) = 2\Phi(X, Y)\bar{\xi}, \quad X, Y \in \Gamma\mathcal{D}. \tag{17}$$

Indeed, if $X, Y \in \Gamma\mathcal{D}$, then

$$\begin{aligned}&\tilde{T}(X - iJX, Y - iJY)\\ &\quad = 2\{\Phi(X, Y) - i\Phi(X, \varphi Y) - i\Phi(\varphi X, Y) - \Phi(\varphi X, \varphi Y)\}\bar{\xi} = 0\end{aligned}$$

and this yields $\tilde{T}(Z, W) = 0$ for all $Z, W \in \Gamma\mathcal{H}$. Hence we verify that $\tilde{\nabla} = \nabla'$ showing that $\tilde{\nabla}$ satisfies the above conditions 1. – 4. It is clear that, since $\varphi$ and the $\xi_i$ are all $\tilde{\nabla}$-parallel, then $\tilde{\nabla}$ parallelizes $\mathcal{E}$, and morever $\tilde{\nabla}$ satisfies condition 3. To prove 2, we consider first the case where $i = 1$ and $j = 2$. Let $Z = X - iJX$ and $\bar{W} = Y + iJY$ be arbitrary sections of $\mathcal{H}$ and $\bar{\mathcal{H}}$ respectively, where $X, Y \in \Gamma\mathcal{D}$. Then, using (13):

$$\begin{aligned}\tilde{T}_{12}(Z, \bar{W}) &= 2\{\Phi(X, Y) + i\Phi(X, \varphi Y) - i\Phi(\varphi X, Y) + \Phi(\varphi X, \varphi Y)\}\bar{\xi}\\ &= 4\{\Phi(X, Y) - i\Phi(\varphi X, Y)\}\bar{\xi}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\tau_{12}(Z, \bar{W}) = \sum_{k \neq 1,2} [Z, \bar{W}]_k &= \sum_{t=1}^{s} \eta^t([X, Y])\xi_t\\ &\quad + i\sum_{t=1}^{s} \eta^t([X, \varphi Y])\xi_t - i\sum_{t=1}^{s} \eta^t([\varphi X, Y])\xi_t + \sum_{t=1}^{s} \eta^t([\varphi X, \varphi Y])\xi_t\\ &= -2\Phi(X, Y)\bar{\xi} - 2i\Phi(X, \varphi Y)\bar{\xi} + 2i\Phi(\varphi X, Y)\bar{\xi} - 2\Phi(\varphi X, \varphi Y)\bar{\xi}\end{aligned}$$

and this implies $\tilde{T}_{12} = -\tau_{12}$. Next we treat the case where $i = 1$ and $j > 2$. Using (16), setting $t = j - 2$, we have

$$\begin{aligned}\tilde{T}_{1j}(Z, \xi_t) &= \tilde{T}_{1j}(X, \xi_t) - i\tilde{T}_{1j}(\varphi X, \xi_t)\\ &= \frac{1}{2}\{\varphi[\xi_t, \varphi X] + [\xi_t, X]\} - \frac{i}{2}\{-\varphi[\xi_t, X] + [\xi_t, \varphi X]\}\\ &= \frac{1}{2}\{[\xi_t, X] + i\varphi[\xi_t, X]\} - \frac{i}{2}\{[\xi_t, \varphi X] + i\varphi[\xi_t, \varphi X]\}\\ &= [\xi_t, X]_2 - i[\xi_t, \varphi X]_2 = [\xi_t, Z]_2.\end{aligned}$$

Now, since $[\xi_t, \mathcal{D}] \subset \mathcal{D}$, we have $[Z, \xi_t] \in \Gamma(\mathcal{H} \oplus \bar{\mathcal{H}})$, hence

$$\tau_{1j}(Z, \xi_t) = [Z, \xi_t]_2$$

so that $\tilde{T}_{1j} = -\tau_{1j}$. The verification of 2. when $i = 2$ and $j > 2$ is similar. For the case when $i, j \geq 3$, observe that both sides of 2. vanish. This completes the verification of 2. As to property 4, it is a consequence of $\tilde{\nabla} g = 0$ and $\tilde{\nabla}\xi_1 = 0$, since

$$\begin{aligned}\tau_{123}(Z, \bar{W}) &= [Z, \bar{W}]_3 = \eta^1([Z, \bar{W}])\xi_1 \\ &= -2\Phi(Z, \bar{W})\xi_1 = 2ig(Z, \bar{W})\xi_1 \,.\end{aligned}$$

We conclude that $\tilde{\nabla} = \nabla'$ and this completes the proof. □

We remark that our approach in the determination of the Tanaka–Webster connection of an almost $\mathcal{S}$-manifold provides an explicit formula for $\tilde{\nabla}$ involving the Levi-Civita connection of the metric $g$ (cf. (14)).

## 3. CR-integrable almost $\mathcal{S}$-structures as Cartan geometries

As an application of Theorem 2.2, in this section we give a canonical interpretation of the notion of CR–integrable almost $\mathcal{S}$-structure on a manifold as a Cartan geometry with an appropriate reductive model Klein geometry. About this notion, we shall follow the terminology and notations in R. Sharpe's book [9], Chap. 5.

Consider the real vector space

$$V := \mathbf{R}^{2k} \,\oplus\, \mathbf{R}^s = \mathsf{D} \oplus \mathsf{D}^\perp$$

where $k \geq 1$, $s \geq 1$. We denote by $\{x_1, \ldots, x_{2k}, e_1, \ldots, e_s\}$ the standard basis and by $g_o$ the standard inner product on $V$. Moreover, let $J : \mathsf{D} \to \mathsf{D}$ be the complex structure associated to the matrix

$$\begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}$$

with respect to the basis $\{x_1, \ldots, x_{2k}\}$ of $\mathsf{D}$. Let $f : V \to V$ be the endomorphism defined by

$$f(Z) = \begin{cases} JZ & \text{if} \quad Z \in \mathsf{D} \\ 0 & \text{if} \quad Z \in \mathsf{D}^\perp \end{cases} .$$

We also set $e := \sum_{i=1}^s e_i \in \mathsf{D}^\perp$ and we denote by $\Phi_o$ the 2-form on $V$ such that

$$\Phi_o(x, y) := g_o(x, fy)$$

for all $x, y \in V$.

Now let $M$ be a smooth manifold of dimension $n = 2k + s$; we denote by $L(M)$ the bundle of frames of $M$; we think of $L(M)$ as the $GL(V)$-principal fibre bundle over $M$ consisting of all linear isomorphisms $u : V \to T_xM$, $x \in M$. The following proposition is standard:

**Proposition 3.1.** *There is a natural bijective correspondence between metric $f{\cdot}pk$ structures $\zeta = (\varphi, \xi_i, \eta^i, g)$ of rank $2k$ and $U(k) \times I_s$-reductions $Q_\zeta$ of the bundle $L(M)$. A frame $u \in L_x(M)$ belongs to $Q_\zeta$ if and only if*

$$\varphi_x \circ u = u \circ f \,, \quad u^*(g_x) = g_o \,, \quad u(e_i) = \xi_i(x) \,.$$

*Moreover, a linear connection on $M$, with covariant differentiation $\nabla$, is reducible to $Q_\zeta$ if and only if*

$$\nabla\varphi = \nabla g = \nabla\xi_i = 0\,.$$

Next we introduce a Lie algebra structure on the vector space

$$\mathsf{g} = \mathsf{u}(\mathsf{k}) \oplus V \tag{18}$$

as follows. We set

$$[x,y] := -2\Phi_o(x,y)e,\ \ [A,x] := A\cdot x =: -[x,A],\ \ [A,B] := AB - BA$$

for all $x, y \in V$ and $A, B \in \mathsf{u}(\mathsf{k})$; here $A \cdot x$ denotes the natural action of $\mathsf{u}(\mathsf{k})$ on $V$. We remark that the validity of the Jacobi identity for $[\,,\,]$ is based on the fact that each $A \in \mathsf{u}(\mathsf{k})$ acts as a skew-symmetric endomorphism of $V$ with respect to $g_o$, commuting with $f$.

The adjoint representation of $U(k) \times I_s$ on its Lie algebra $\mathsf{u}(\mathsf{k})$ extends to a representation, still denoted by $Ad : U(k) \times I_s \to Aut(\mathsf{g})$ such that

$$Ad(h)(x) = h\cdot x\,,\quad Ad(h)(A) = hAh^{-1}\quad \text{for all}\quad x\in V,\, A\in \mathsf{u}(\mathsf{k})\,.$$

Hence the Klein pair $(\mathsf{g}, \mathsf{u}(\mathsf{k}))$ is a *model geometry with group $H = U(k) \times I_s \subset GL(V)$* according to R. Sharpe's definition in [9], page 174. Notice that the representation $Ad$ and the induced representation $Ad_V$ of $H$ on $V$ are faithful, so that the model is *effective* and of *first–order*. Moreover, the decomposition $\mathsf{g} = \mathsf{u}(\mathsf{k}) \oplus V$ is a *reductive* one, namely $V$ is an $Ad(H)$-submodule of $\mathsf{g}$. This property implies the following characterization of Cartan connections with model $(\mathsf{g}, \mathsf{u}(\mathsf{k}))$ and group $H$ (see e.g. [1] or [9], Appendix A):

**Proposition 3.2.** *Up to gauge equivalence, every Cartan geometry on $M$ modeled on $(\mathsf{g}, \mathsf{u}(\mathsf{k}))$ with group $H$ is given by $(Q, \omega)$ where $Q$ is an $H$-reduction of the bundle $L(M)$, and $\omega = \gamma + \theta$, where $\gamma : TQ \to \mathsf{u}(\mathsf{k})$ is a principal connection form on $Q$, while $\theta : TQ \to V$ is the canonical form given by*

$$\theta_u(Y) = u^{-1}(\pi_* Y),\quad \pi : Q \to M \quad \textit{natural projection}$$

*for each frame $u \in Q$ and $Y \in T_u Q$.*

We recall that two Cartan geometries $(P, \omega)$ and $(Q, \omega')$ on a manifold $M$, having the same Klein model, are called *gauge equivalent* it there is a bundle isomorphism $\Psi : P \to Q$ covering the identity $i_M$, such that $\Psi^*\omega' = \omega$.

In order to get a canonical interpretation of CR-integrable almost $\mathcal{S}$-structures as Cartan geometries, we need to restrain our attention to a special class of the latter, which we shall call *normal* Cartan geometries. Their characterization is done by means of the corresponding curvature function. We recall that the *curvature form* $\Omega$ of a Cartan geometry $(P, \omega)$ modeled on $(\mathsf{g}, \mathsf{u}(\mathsf{k}))$ is the $\mathsf{g}$-valued 2-form on $P$, such that

$$\Omega(X,Y) = d\omega(X,Y) + \frac{1}{2}[\omega(X), \omega(Y)]\,.$$

Denote by $C^2(V,\mathsf{g})$ the real vector space of alternating bilinear maps $\psi : V\times V\to\mathsf{g}$. This is an $H$–module under the left action

$$h\cdot\psi(\cdot,\cdot) := Ad(h)\psi\big(Ad_V(h^{-1})\cdot, Ad_V(h^{-1})\cdot\big)\,.$$

The *curvature function* of $(P,\omega)$ is the smooth map $K : P\to C^2(V,\mathsf{g})$ defined by

$$K(u)(X,Y) := \Omega_u(\omega^{-1}X,\omega^{-1}Y)\,.$$

A Cartan geometry is called *torsion free* if $K_V=0$, where $K_V(u) = \mathrm{pr}_V\circ K(u)$. Now consider the subspace $\mathcal{M}$ of $C^2(V,\mathsf{g})$ consisting of the bilinear maps $\psi : V\times V\to\mathsf{g}$ such that

$$\psi_V(x,y)=\psi_V(e_i,e_j)=0\,,\quad \psi_V(e_i,fx)=-f\psi_V(e_i,x)$$

for all $x,y\in\mathsf{D}$.

**Remark 3.3.** *$\mathcal{M}$ is an $H$–submodule of $C^2(V,\mathsf{g})$.*

This is an immediate consequence of the fact that the decomposition $V = \mathbf{C}^k\oplus\mathbf{R}^s$ is $H$-invariant and that $H$ acts by complex linear maps on $\mathbf{C}^k$.

According to this remark, we define an $\mathcal{M}$-*normal* Cartan geometry on $M$, modeled on $(\mathsf{g},\mathsf{u}(\mathsf{k}))$ with group $H$, to be one which is of curvature type $\mathcal{M}$, i.e. $K(P)\subset\mathcal{M}$. This is in accordance with the general prescription in [9], page 201. Notice that normality is preserved under gauge equivalence.

Now we can state the main result of this section.

**Theorem 3.4.** *Let $M$ be a real manifold of dimension $2k+s$. There is a natural bijection between the set of CR–integrable almost $\mathcal{S}$–structures of rank $2k$ on $M$ and the set of $\mathcal{M}$-normal Cartan geometries on $M$ modeled on $(\mathsf{g},\mathsf{u}(\mathsf{k}))$, with group $H=U(k)\times I_s$, modulo gauge equivalence. Moreover, the $\mathcal{S}$-structures correspond to the torsion free Cartan geometries.*

Before starting the proof, we make the following remark:

**Lemma 3.5.** *Maintaining the notation in Proposition 3.2, let $Q$ be an $H$-reduction of $L(M)$, and let $\omega=\gamma+\theta$ be a Cartan geometry on $M$ modeled on $(\mathsf{g},\mathsf{u}(\mathsf{k}))$ with group $H$. We denote by $\tilde\nabla$ the linear connection induced by the principal connection $\gamma$. Let $K$ denote the curvature function of $\omega$, and let $\tilde T$ denote the torsion tensor of $\tilde\nabla$. Then for each frame $u\in Q_x$, we have the following formula:*

$$2uK_V(u)(X,Y)=\tilde T(uX,uY)+u[X,Y]\,,\quad \textit{for all}\quad X,Y\in V\,. \tag{19}$$

**Proof.** This is a standard computation, cf. [9] or [5]. □

**Proof of Theorem 3.4.** Fix a CR-integrable almost $\mathcal{S}$-structure $\zeta=(\varphi,\xi_i,\eta^i,g)$; according to Proposition 3.1, $\zeta$ gives rises canonically to a reduction $Q_\zeta$ of $L(M)$ to the group $H$. Moreover on $M$ we have the Tanaka–Webster connection $\tilde\nabla$ according to Theorem 2.2. Since the tensor fields $\varphi,g,\xi_i$ are all parallel with respect to $\tilde\nabla$, this connection reduces to a principal connection $\gamma$ on $Q_\zeta$. Let $\theta$ be the canonical form of $Q_\zeta$ and set $\omega_\zeta=\gamma+\theta$. Then $(Q_\zeta,\omega_\zeta)$ is a Cartan geometry modeled on $(\mathsf{g},\mathsf{u}(\mathsf{k}))$ with group $H$ (Proposition 3.2). Using formula (19) we see that $(Q_\zeta,\omega_\zeta)$

is a normal geometry. Indeed, for all $X, Y \in \mathsf{D}$ we have $[X,Y] = -2\Phi_o(X,Y)e$; if $u \in Q_\zeta(x)$, $x \in M$, it follows

$$u[X,Y] = -2\Phi(x)(uX, uY)\bar{\xi}_x\,,$$

and on the other hand, taking into account property (6) of $\tilde{\nabla}$, since $uX, uY \in \mathcal{D}(x)$, we have

$$\tilde{T}(uX, uY) = 2\Phi(x)(uX, uY)\bar{\xi}_x\,.$$

It follows from (19) that $uK_V(u)(X,Y) = 0$, that is $K_V(u)(X,Y) = 0$. Since $[e_i, e_j] = 0$, in the same way we can verify that $K_V(\mathsf{D}^\perp, \mathsf{D}^\perp) = 0$. Finally, using property (7) of $\tilde{\nabla}$, we get

$$\begin{aligned} 2uK_V(u)(e_i, fX) &= \tilde{T}(\xi_i(x), \varphi(uX)) = -\varphi_x\tilde{T}(\xi_i(x), uX) \\ &= -2\varphi_x uK_V(u)(e_i, X) = -2ufK_V(u)(e_i, X) \end{aligned}$$

whence $K_V(u)(e_i, fX) = -fK_V(u)(e_i, X)$.

Hence to each CR-integrable almost $\mathcal{S}$-structure $\zeta$ we have associated a normal Cartan geometry $\mathcal{C}_\zeta = (Q_\zeta, \omega_\zeta)$ modeled on $(\mathsf{g}, \mathsf{u}(\mathsf{k}))$ with group $H$. Clearly, the map $\zeta \mapsto \mathcal{C}_\zeta$ is injective. Notice that, according to corollary 2.5, $\zeta$ is normal, i.e. it is an $\mathcal{S}$-structure, if and only if $\tilde{T}(\xi_i, Z) = 0$ for all $Z \in \mathcal{D}$. Using again (19), we easily see that this is equivalent to $K_V(u)(e_i, X) = 0$ for all $u \in Q_\zeta$ and $X \in \mathsf{D}$. By definition of $\mathcal{M}$, this is equivalent to $K_V = 0$, that is to $\mathcal{C}_\zeta$ being torsion free.

To conclude the proof of the theorem, it suffices to verify that, up to gauge equivalence, every normal Cartan geometry $(P, \omega)$ with model $(\mathsf{g}, \mathsf{u}(\mathsf{k}))$ and group $H$ is given by $\mathcal{C}_\zeta$ for some CR-integrable almost $\mathcal{S}$-structure on $M$. We know from Proposition 3.2 that $(P, \omega)$ is gauge equivalent to $\mathcal{C} = (Q, \omega')$ where $Q$ is a reduction of $L(M)$ to $H$, and $\omega' = \gamma + \theta$, where $\gamma$ is a principal connection form on $Q$. There exists a unique metric $f{\cdot}$pk structure $\zeta = (\varphi, \xi_i, \eta^i, g)$ on $M$ such that $Q = Q_\zeta$. To $\gamma$ there corresponds a linear connection $\tilde{\nabla}$; clearly, $\tilde{\nabla}\varphi = \tilde{\nabla}g = \tilde{\nabla}\xi_i = 0$. Moreover, using the $\mathcal{M}$-normality of $(Q, \omega')$, we see as above that the torsion $\tilde{T}$ of $\tilde{\nabla}$ satisfies the conditions (6)–(8) in Theorem 2.2. Hence $\zeta$ is actually a CR-integrable almost $\mathcal{S}$-structure and $\tilde{\nabla}$ is the corresponding Tanaka–Webster connection. In particular, it follows that $\mathcal{C}_\zeta = \mathcal{C}$ and this concludes the proof of the theorem. □

**Examples.** We end by discussing examples of homogeneous non normal almost $\mathcal{S}$-manifold whose Tanaka–Webster curvature vanishes. Notice that, for the case $s = 1$, a manifold with this properties does not exist. Namely, it can be easily verified by using the Bianchi identity that a contact metric manifold with vanishing Tanaka–Webster curvature is necessarily Sasakian.

Set

$$\mathsf{m} = \mathbf{R}^{2k} \oplus \mathbf{R}^s = V_1 \oplus V_2,\ s \geq 2$$

and denote by $\{X_1, \dots, X_k, JX_1, \dots, JX_k\}$ the standard basis of $\mathbf{R}^{2k}$ endowed with the complex structure $J$ associated with the matrix $\begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}$. Moreover let $\{\xi_1, \dots, \xi_s\}$ denote the natural basis of $V_2$ and let $g$ be the inner product on $\mathsf{m}$

obtained by declaring the basis $\{X_i, JX_i, \xi_j\}$ to be orthonormal. Let $\varphi : \mathsf{m} \to \mathsf{m}$ be the natural $f$-structure on $\mathsf{m}$, i.e. $\varphi$ is the endomorphism which coincides with $J$ on $V_1$ and vanishes on $V_2$.

We also denote by $U$ the endomorphism of $\mathsf{m}$ which is associated to the matrix

$$\begin{pmatrix} I_k & 0 & 0 \\ 0 & -I_k & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Notice that $U\varphi = -\varphi U$.

We denote by $\mathsf{h}$ the Lie subalgebra of $End(\mathsf{m})$ consisting of all endomorphisms which vanish on $V_2$ and annihilate the tensors $\varphi$, $g$ and $U$ when extended to the tensor algebra of $\mathsf{m}$ as derivations. We remark that

$$A \in \mathsf{so(k)} \mapsto \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

provides a Lie-algebra isomorphism $\mathsf{so(k)} \cong \mathsf{h}$. In particular, $\mathsf{h}$ is compact semisimple provided $k \geq 3$.

Now we define a Lie algebra structure on $\mathsf{g} := \mathsf{h} \oplus \mathsf{m}$ as follows:

$$[X,Y] := -2g(X,JY)e,\ \ [v,X] := a(v)UX = -[X,v]$$
$$[A,X] = A \cdot X = -[X,A],\ \ [A,v] := 0,\ [v,w] := 0, [A,B] := AB - BA$$

for each $X, Y \in V_1, v, w \in V_2$, $A \in \mathsf{h}$. Here $e := \sum_i \xi_i \in V_2$, and $a : V_2 \to \mathbf{R}$ is a fixed non null linear functional such that $a(e) = 0$.

Let $G$ be the connected and simply connected Lie group with Lie algebra $\mathsf{g}$ and let $H$ denote the analytic subgroup corresponding to the subalgebra $\mathsf{h}$. Assuming $k \geq 3$, we have that $H$ is compact, so that $M = G/H$ is a reductive homogeneous space. The tensors $\varphi$ and $g$ on the reductive summand $\mathsf{m}$ are $Ad(H)$-invariant, and $Ad(h)\xi_i = \xi_i$, for each $h \in H$. Then $(\varphi, \xi_i, \eta^i, g)$, where the $\eta^i$ are the dual forms of the $\xi_i$, canonically determine a $G$-invariant metric $f{\cdot}$pk structure on $M$. The canonical $G$-invariant linear connection $\tilde{\nabla}$ satisfies the conditions 1), 2) in Theorem 2.2. Indeed, since the structure is $G$-invariant, the tensor fields $\varphi$, $\eta^i$ and $g$ are all parallel with respect to $\tilde{\nabla}$. Moreover at the point $o = H$, under the natural identification $T_oM \cong \mathsf{m}$, we have the formula $\tilde{T}_o(Z,W) = -[Z,W]$ for the torsion of $\tilde{\nabla}$, which implies that $\tilde{\nabla}$ satisfies properties (6)–(8), according to the definition of the Lie bracket $\mathsf{m} \times \mathsf{m} \to \mathsf{m}$; in particular notice that (7) holds since $[v, JX] = a(v)UJX = -J[v,X]$, for each $v \in V_2$ and $X \in V_1$. Hence $M$ is a homogeneous almost $\mathcal{S}$-manifold, which is not normal according to Corollary 2.5. Finally, $\tilde{\nabla}$ has vanishing Tanaka–Webster curvature because $[\mathsf{m}, \mathsf{m}] \subset \mathsf{m}$.

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