

**APPROXIMATION OF SOME REGULAR DISTRIBUTION
IN $S'(\mathcal{R})$ BY FINITE, CONVEX, LINEAR
COMBINATIONS OF BLASCHKE DISTRIBUTIONS**

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0. INTRODUCTION

0.1. Some background on Blaschke products and Marshall's theorem for functions on H^∞ .

Let U be the open, unit disc in the plane, $T = \partial U$. $H^\infty(U)$ is the space of all bounded analytic functions $f(z)$ on U , for which the norm is defined by

$$\|f\|_{H^\infty} = \sup_{z \in U} |f(z)|.$$

If $f \in H^\infty(U)$, then the radial boundary function

$$f^*(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta})$$

is defined almost everywhere on T with respect to the Lebesgue measure on T and $\log |f^*(e^{i\theta})| \in L^1(T)$.

Let $\{z_n\}$ be a sequence of points in U such that

$$(0.1.1) \quad \sum_{n=1}^{\infty} (1 - |z_n|) < \infty.$$

Let m be the number of z_n equal to 0. Then the infinite product

$$(0.1.2) \quad B(z) = z^m \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}$$

converges on U . The function $B(z)$ of the form (0.1.2) is called Blaschke product. $B(z)$ is in $H^\infty(U)$, and the zeros of $B(z)$ are precisely the points z_n , each zero

2000 *Mathematics Subject Classification*: 46Fxx, 46F20, 62E17.

Key words and phrases: Blaschke product, upper Blaschke distribution, tempered distribution, approximation.

Received September 24, 2001.

having multiplicity equal the number of times it occurs in the sequence $\{z_n\}$. Moreover $|B(z)| \leq 1$ and $|B^*(e^{i\theta})| = 1$ a.e.

For the needs of our subsequent work we will define the Blaschke product in the upper half plane Π^+ . In the upper half plane Π^+ , condition (0.1.1) is replaced by

$$(0.1.3) \quad \sum_{n=1}^{\infty} \frac{y_n}{1 + |z_n|^2} < \infty, \quad z_n = x_n + iy_n \in \Pi^+$$

and the Blaschke product with zeros z_n is

$$(0.1.4) \quad B(z) = \left(\frac{z-i}{z+i} \right)^m \prod_{n=1}^{\infty} \frac{|z_n^2+1|}{z_n^2+1} \frac{z-z_n}{z-\bar{z}_n}.$$

Note. If the number of zeros z_n in (0.1.2) or (0.1.4) is finite, then we call $B(z)$ finite Blaschke product.

For the needs of our subsequent work we will state the Marshall's theorem for approximation of functions of $H^\infty(U)$ by finite, convex, linear combinations of Blaschke products. The theorem is given in [6].

Marshall's theorem. *Let $f \in H^\infty(U)$ and $\|f\|_{H^\infty} \leq 1$. Then for every $\varepsilon > 0$, there are Blaschke products $B_1(z), B_2(z), \dots, B_n(z)$ and positive numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, $\sum_{k=1}^n \lambda_k = 1$ such that*

$$\|f(z) - \sum_{k=1}^n \lambda_k B_k(z)\|_{H^\infty} < \varepsilon.$$

0.2. Some notions of distributions and Blaschke distribution.

For a function f , $f: \Omega \rightarrow C^n$, $\Omega \subseteq \mathcal{R}^n$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_j \in \mathcal{N} \cup \{0\}$, $x \in \Omega$, $D_x^\alpha f$ denotes the differential operator

$$D_x^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n.$$

$C^\infty(\mathcal{R}^n)$ denotes the space of all complex valued infinitely differentiable functions on \mathcal{R}^n and $C_0^\infty(\mathcal{R}^n)$ denotes the subspace of $C^\infty(\mathcal{R}^n)$ that consists of those functions of $C^\infty(\mathcal{R}^n)$ which have compact support. Support of a function f , denoted by $\text{supp}(f)$, is the closure of $\{x \mid f(x) \neq 0\}$ in \mathcal{R}^n .

$D = D(\mathcal{R}^n)$ denotes the space of $C_0^\infty(\mathcal{R}^n)$ functions in which convergence is defined in the following way: a sequence $\{\varphi_\lambda\}$ of functions $\varphi_\lambda \in D$ converges to $\varphi \in D$ in D as $\lambda \rightarrow \lambda_0$ if and only if there is a compact set $K \subset \mathcal{R}^n$ such that $\text{supp}(\varphi_\lambda) \subseteq K$ for each λ , $\text{supp}(\varphi) \subseteq K$ and for every n -tuple α of nonnegative integers the sequence $\{D_t^\alpha \varphi_\lambda(t)\}$ converges to $D_t^\alpha \varphi(t)$ uniformly on K as $\lambda \rightarrow \lambda_0$.

$D' = D'(\mathcal{R}^n)$ is the space of all continuous linear functionals on D , where continuity means that $\varphi_\alpha \rightarrow \varphi$ in D as $\lambda \rightarrow \lambda_0$ implies $\langle T, \varphi_\lambda \rangle \rightarrow \langle T, \varphi \rangle$ as $\lambda \rightarrow \lambda_0$, $T \in D'$.

Note. $\langle T, \varphi \rangle$ denotes the value of the functional T , when it acts on the function φ .

D' is called the space of distributions.

$S = S(\mathcal{R}^n)$ denotes the space of all infinitely differentiable complex valued function φ on \mathcal{R}^n satisfying

$$\sup_{t \in \mathcal{R}^n} |t^\beta D^\alpha \varphi(t)| < \infty$$

for all n -tuple α and β of nonnegative integers. Convergence in S is defined in the following way: a sequence $\{\varphi_\lambda\}$ of functions $\varphi_\lambda \in S$ converges to $\varphi \in S$ in S as $\lambda \rightarrow \lambda_0$ if and only if

$$\lim_{\lambda \rightarrow \lambda_0} \sup_{t \in \mathcal{R}^n} |t^\beta D^\alpha [\varphi_\lambda(t) - \varphi(t)]| = 0$$

for all n -tuple α and β of nonnegative integers.

Again, S' is the space of all continuous, linear functionals on S , called the space of tempered distributions.

Let φ be an element of one of the above function spaces D or S , and f be a function for which

$$\langle T_f, \varphi \rangle = \int_{\mathcal{R}^n} f(t)\varphi(t) dt, \quad \varphi \in D \quad (\varphi \in S)$$

exists and is finite. Then T_f is regular distribution on D (or S) generated by f .

Now, let $B(z)$ be the Blaschke product, $z = x + iy \in \Pi^+$, with zeros z_n that belong to the upper half plane. In [7] it is proven that $\langle B^+, \varphi \rangle$, where

$$(0.2.1) \quad \langle B^+, \varphi \rangle = \lim_{y \rightarrow 0^+} \int_{-\infty}^{\infty} B(z)\varphi(x) dx, \quad z = x + iy \in \Pi^+, \quad \varphi \in D(\mathcal{R}),$$

is distribution on D , named upper Blaschke distribution on D .

Note. This is a new notion in the theory of distributions and has useful application in the problems of approximation. The introduced Blaschke distributions in [7] were used for representing some distributions in D' as a limit of sequence of Blaschke distributions.

The following theorem gives another application of the Blaschke distribution.

1. MAIN RESULT

Theorem 1.1. Let $f(z) \in H^\infty(\Pi^+)$ and $\|f\|_{H^\infty} \leq 1$. Let T_{f^*} be the distribution in $S'(\mathcal{R})$ generated with the boundary value f^* of the function $f(z)$. Then

for every $\varepsilon > 0$, and for every $\varphi \in S(\mathcal{R})$ there are upper Blaschke distributions $B_1^+, B_2^+, \dots, B_n^+$ and positive numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, $\sum_{k=1}^n \lambda_k = 1$ such that

$$(1.1) \quad \left| \langle T_{f^*}, \varphi \rangle - \sum_{k=1}^n \lambda_k \langle B_k^+, \varphi \rangle \right| < \varepsilon.$$

Proof. Let $f(z) \in H^\infty(\Pi^+)$, $\|f\|_{H^\infty} \leq 1$. Let $\varepsilon > 0$ and $\varphi \in S(\mathcal{R})$ be arbitrary chosen. Because $S(\mathcal{R}) \subset L^1(\mathcal{R})$, it follows that $\varphi \in L^1(\mathcal{R})$.

Let $\varepsilon_1 = \frac{\varepsilon}{\|\varphi\|_{L^1}} > 0$. Then because of the Marshal theorem, there are Blaschke products $B_1(z), B_2(z), \dots, B_n(z)$ with zeros in the upper half plane Π^+ , and positive numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, $\sum_{k=1}^n \lambda_k = 1$ such that

$$(1.2) \quad \left\| f(z) - \sum_{k=1}^n \lambda_k B_k(z) \right\|_{H^\infty} < \varepsilon_1.$$

From (1.2), we have that for $B_k(z)$, λ_k , $k \in \{1, 2, \dots, n\}$ hold

$$(1.3) \quad \left| f(z) - \sum_{k=1}^n \lambda_k B_k(z) \right| < \varepsilon_1, \quad \forall z \in \Pi^+.$$

Because the Blaschke products $B_1(z), B_2(z), \dots, B_n(z)$ have zeros in Π^+ , they define upper Blaschke distributions $B_1^+, B_2^+, \dots, B_n^+$ respectively, as in [7]. Now, let

$$(1.4) \quad \langle B_k^+, \varphi \rangle = \lim_{y \rightarrow 0^+} \int_{-\infty}^{\infty} B_k(z) \varphi(x) dx, \quad z = x + iy \in \Pi^+, \quad \varphi \in S(\mathcal{R}).$$

We will prove that $B_k^+ \in S'(\mathcal{R})$, for $k \in \{1, 2, \dots, n\}$. Because of the theorem of characterization of tempered distributions given in [8], it is enough to prove that $B_k^+ * \alpha$ are continuous and bounded functions on \mathcal{R} , for every $\alpha \in D(\mathcal{R})$. So, let $\alpha \in D(\mathcal{R})$, $\text{supp}(\alpha) = K$, $t \in \mathcal{R}$ and $K_1 = t - K$. Then

$$\begin{aligned} (B_k^+ * \alpha)(t) &= \langle B_{kx}^+, \alpha(t-x) \rangle \\ &= \lim_{y \rightarrow 0^+} \int_{-\infty}^{\infty} B_k(x+iy) \alpha(t-x) dx \\ &= \lim_{y \rightarrow 0^+} \int_{K_1} B_k(x+iy) \alpha(t-x) dx. \end{aligned}$$

First, we will show that $B_k^+ * \alpha$ is bounded function on \mathcal{R} :

$$\begin{aligned} |(B_k^+ * \alpha)(t)| &= \left| \lim_{y \rightarrow 0^+} \int_{K_1} B_k(x+iy)\alpha(t-x) dx \right| \stackrel{|B_k(x+iy)| \leq 1}{\leq} \int_{K_1} |\alpha(t-x)| dx \\ &\leq M \cdot m(K) < \infty, \end{aligned}$$

where $m(K)$ is the Lebesgue measure of K .

Now, we will prove the continuity of $B_k^+ * \alpha$ on \mathcal{R} . Let $\varepsilon > 0$, $t_0 \in \mathcal{R}$ and let $K_0 = t_0 - K$. Since α is continuous, there exists $\delta > 0$, so that $|t - t_0| < \delta$ implies $|\alpha(t) - \alpha(t_0)| < \varepsilon$ i.e. if $x \in \mathcal{R}$ is any real number, the last is equivalent with: there exists $\delta > 0$, so that $|(t-x) - (t_0-x)| < \delta$ implies $|\alpha(t-x) - \alpha(t_0-x)| < \varepsilon$. Now

$$\begin{aligned} |(B_k^+ * \alpha)(t) - (B_k^+ * \alpha)(t_0)| &= \left| \lim_{y \rightarrow 0^+} \int_{-\infty}^{\infty} B_k(x+iy)\alpha(t-x) dx - \lim_{y \rightarrow 0^+} \int_{-\infty}^{\infty} B_k(x+iy)\alpha(t_0-x) dx \right| \\ &\leq \left| \lim_{y \rightarrow 0^+} \int_{K_2} B_k(x+iy)[\alpha(t-x) - \alpha(t_0-x)] dx \right| \stackrel{|B_k(x+iy)| \leq 1}{\leq} \\ &\leq \int_{K_2} |\alpha(t-x) - \alpha(t_0-x)| dx \\ &\leq \varepsilon(2m(K) + \delta) = \varepsilon_1 \quad \text{when} \quad |t - t_0| < \delta \end{aligned}$$

(K_2 is a compact set that contains K_0 i K_1 .)

On the other hand, using the properties of the space H^∞ , it is clear that the boundary function f^* of the function $f(z)$ exists, $f^* \in L^\infty$ and $f(x+iy) \rightarrow f^*(x)$, in L^∞ , as $y \rightarrow 0^+$, $x+iy \in \Pi^+$.

Even more, theorem 5.3 in [3] claims that $f(x+iy) \rightarrow f^*(x)$ in $S'(\mathcal{R})$, as $y \rightarrow 0^+$, $x+iy \in \Pi^+$ i.e.

$$(1.5) \quad \lim_{y \rightarrow 0^+} \int_{-\infty}^{\infty} f(x+iy)\varphi(x) dx = \langle T_{f^*}, \varphi \rangle, \quad x+iy \in \Pi^+, \quad \varphi \in S(\mathcal{R}).$$

Now, we get that

$$\begin{aligned} &\left| \langle T_{f^*}, \varphi \rangle - \sum_{k=1}^n \lambda_k \langle B_k^+, \varphi \rangle \right| \\ &\stackrel{(1.4)}{=} \stackrel{(1.5)}{=} \left| \lim_{y \rightarrow 0^+} \int_{-\infty}^{\infty} f(x+iy)\varphi(x) dx - \sum_{k=1}^n \lambda_k \lim_{y \rightarrow 0^+} \int_{-\infty}^{\infty} B_k(x+iy)\varphi(x) dx \right| \end{aligned}$$

$$\begin{aligned}
&= \left| \lim_{y \rightarrow 0^+} \int_{-\infty}^{\infty} f(x + iy) \varphi(x) dx - \lim_{y \rightarrow 0^+} \sum_{k=1}^n \lambda_k \int_{-\infty}^{\infty} B_k(x + iy) \varphi(x) dx \right| \\
&= \left| \lim_{y \rightarrow 0^+} \int_{-\infty}^{\infty} f(x + iy) \varphi(x) dx - \lim_{y \rightarrow 0^+} \int_{-\infty}^{\infty} \left[\sum_{k=1}^n \lambda_k B_k(x + iy) \right] \varphi(x) dx \right| \\
&= \left| \lim_{y \rightarrow 0^+} \int_{-\infty}^{\infty} \left[f(x + iy) - \sum_{k=1}^n \lambda_k B_k(x + iy) \right] \varphi(x) dx \right| \\
&\leq \lim_{y \rightarrow 0^+} \int_{-\infty}^{\infty} \left| f(x + iy) - \sum_{k=1}^n \lambda_k B_k(x + iy) \right| |\varphi(x)| dx \\
&\stackrel{(1.3)}{\leq} \lim_{y \rightarrow 0^+} \int_{-\infty}^{\infty} \varepsilon_1 |\varphi(x)| dx = \varepsilon_1 \int_{-\infty}^{\infty} |\varphi(x)| dx \\
&= \varepsilon_1 \|\varphi\|_{L^1} = \frac{\varepsilon}{\|\varphi\|_{L^1}} \|\varphi\|_{L^1} = \varepsilon.
\end{aligned}$$

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