

CHARACTERIZATIONS OF RANDOM APPROXIMATIONS

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ABSTRACT. Some characterizations of random approximations are obtained in a locally convex space through duality theory.

1. INTRODUCTION AND PRELIMINARIES

Random approximation theory has received much attention after the publication of survey paper by Bharucha-Reid [3] in 1976. The interested reader is referred to recent papers in normed space framework by Tan and Yaun [11], Sehgal and Singh [10], Papageorgiou [7], Lin [5], Beg and Shahzad [2] and Beg [1]. The interplay between random approximation and random fixed point results is interesting and valuable (see for example [5], [7] and [11]). The applications of this closely related concept to random differential equations and integral equations in the context of Banach spaces may be found in Itoh [4] and O'Regan [6] respectively. So random approximations are needed in the study of random equations. Recently, Beg [1] obtained a characterization of random approximations in a normed space by employing the Hahn-Banach separation theorem. Characterization theorems of best approximation in the locally convex space setting have been considered in [8]. In this paper, we establish the characterizations concerning existence of random approximation in locally convex spaces by using the Hahn Banach extension theorem and a result of Tukey [13] about separation of convex sets; in particular Theorem 1 provides a random version of Theorem 2.1 of Rao and Elumalai [8] and Theorem 2 sets an analogue for metrizable locally convex spaces of the theorem due to Beg [1].

We now fix our terminology. Let (Ω, Σ) be a measurable space where Σ is a sigma algebra of subsets of Ω and M a subset of a locally convex space E over the field F of real or complex numbers. A map $T : \Omega \times M \rightarrow E$ is called a random operator if for each fixed $x \in M$, the map $T(\cdot, x) : \Omega \rightarrow E$ is measurable. Let (E, d) be a metrizable locally convex space.

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- (i) The ball with radius r and centre at x is defined as $B_r(x) = \{z \in E : d(z, x) \leq r\}$; in particular the ball $B_r(0)$ has centre at 0.
- (ii) $d(x, M) = \inf_{u \in M} d(x, u)$.
- (iii) $P_M(x) = \{y \in M : d(x, y) = d(x, M)\}$ (set of best approximations of x from M).
- (iv) For a ball $B_r(0)$ in (E, d) , the set $\{z \in E : d(z, 0) = r\}$ is called metric boundary of $B_r(0)$. In general, the topological boundary of $B_r(0)$ is contained in its metric boundary. In case metric and topological boundaries of $B_r(0)$ coincide, we say $B_r(0)$ is bounding (cf. [12]).

In this note, cl , int , E^* and $E \setminus M$ denote the closure, interior, dual of E and difference of sets E and M , respectively.

2. RESULTS

Theorem 1. *Let E be a separable locally convex space with family P of seminorms and M a subspace of E . Suppose $T : \Omega \times M \rightarrow E$ is a random operator and $\xi : \Omega \rightarrow M$ a measurable map such that $T(\omega, \xi(\omega)) \in E \setminus M$. Then ξ is a random best approximation for T (i.e., $p(\xi(\omega) - T(\omega, \xi(\omega))) = d_p(T(\omega, \xi(\omega)), M)$ for each $p \in P$) if and only if for every $p \in P$ there exists $f^p \in E^*$ such that*

- (a) $f^p(g) = 0$ for all $g \in M$.
- (b) $|f^p(T(\omega, \xi(\omega)) - \xi(\omega))| = p(T(\omega, \xi(\omega)) - \xi(\omega))$.
- (c) $|f^p(T(\omega, \xi(\omega)) - g)| \leq p(T(\omega, \xi(\omega)) - g)$ for all $g \in M$.

Proof. Suppose that ξ is a random approximation for T . Then for each $p \in P$ and $g \in M$,

$$p(T(\omega, \xi(\omega)) - \xi(\omega)) \leq p(T(\omega, \xi(\omega)) - g).$$

In particular, for any $0 \neq \alpha \in F$ and $g \in M$,

$$(i) \quad p(T(\omega, \xi(\omega)) - \xi(\omega)) \leq p\left(T(\omega, \xi(\omega)) - \left(\xi(\omega) - \frac{g}{\alpha}\right)\right).$$

Let $B = \{g + \alpha(T(\omega, \xi(\omega)) - \xi(\omega)) : \alpha \in F\}$.

Define f_0^p on B by $f_0^p(g + \alpha[T(\omega, \xi(\omega)) - \xi(\omega)]) = \alpha p(T(\omega, \xi(\omega)) - \xi(\omega))$ for all $g \in M$. Then $f_0^p(g) = 0$ for all $g \in M$ and

$$f_0^p(T(\omega, \xi(\omega)) - \xi(\omega)) = p(T(\omega, \xi(\omega)) - \xi(\omega)).$$

For any $\alpha \neq 0$ and $g \in M$, we have

$$\begin{aligned} |f_0^p(g + \alpha[T(\omega, \xi(\omega)) - \xi(\omega)])| &= |\alpha|p(T(\omega, \xi(\omega)) - \xi(\omega)) \\ &\leq |\alpha|p\left(T(\omega, \xi(\omega)) - \xi(\omega) + \frac{g}{\alpha}\right) \quad (\text{by (i)}) \\ &= p(g + \alpha[T(\omega, \xi(\omega)) - \xi(\omega)]). \end{aligned}$$

For $\alpha = 0$ and $g \in M$ this inequality obviously holds.

Hence for each $z \in B$ and for each $p \in P$,

$$|f_0^p(z)| \leq p(z).$$

Thus by the Hahn-Banach theorem, f_0^p can be extended to a continuous linear functional f^p on E such that $|f^p(x)| \leq p(x)$ for every $x \in E$ and

$$|f^p(z)| = |f_0^p(z)| \quad \text{for each } z \in M.$$

The results (a)–(c) are now evident.

Conversely let the conditions (a)–(c) be satisfied. Then from (b) we get for all $p \in P$ and $g \in M$,

$$\begin{aligned} p(T(\omega, \xi(\omega)) - \xi(\omega)) &= |f^p(T(\omega, \xi(\omega)) - \xi(\omega))| \\ &= |f^p(T(\omega, \xi(\omega)) - g) + f^p(g - \xi(\omega))| \\ &= |f^p(T(\omega, \xi(\omega)) - g)| \quad \text{(by (a))} \\ &\leq p(T(\omega, \xi(\omega)) - g) \quad \text{(by (c)).} \end{aligned}$$

Hence $p(T(\omega, \xi(\omega)) - \xi(\omega)) = d_p(T(\omega, \xi(\omega)), M)$ for all $p \in P$. □

We shall follow the argument used in the proof of Theorem 2.3 of Thaheem [12] to prove the following:

Theorem 2. *Let (E, d) be a separable metrizable locally convex space with d as invariant metric. Assume that the ball $B_r(0)$ is convex and bounding and M a convex subset of E . Let $T : \Omega \times M \rightarrow E$ be a random operator and $\xi : \Omega \rightarrow M$ a measurable map such that $T(\omega, \xi(\omega)) \notin \text{cl}(M)$. Then ξ is a random best approximation for T if and only if there exists a real continuous linear functional $f \in E_{\mathbf{R}}^*$ (\mathbf{R} is the set of real numbers) such that*

- (a) $f(T(\omega, \xi(\omega)) - \xi(\omega)) = d(T(\omega, \xi(\omega)), \xi(\omega)) = r(\omega) = r$ (say; for notational simplicity).
- (b) $f(y - \xi(\omega)) \leq 0$ for all y in M .
- (c) $\|f\|_r = \sup\{|f(z)| : z \in B_r(0)\} = r$.

Proof. Assume that $d(\xi(\omega), T(\omega, \xi(\omega))) = d(T(\omega, \xi(\omega)), M)$. Then M and $\text{int}(B_r(T(\omega, \xi(\omega))))$, where $r = d(T(\omega, \xi(\omega)), M)$, are disjoint convex sets. By a result of Tukey [13] (see also Rudin [9]), there is a nonzero continuous real linear functional $f_{\xi(\omega)} \in E_{\mathbf{R}}^*$ and a real number c such that

(ii) $f_{\xi(\omega)}(T(\omega, \xi(\omega)) - y) \geq c \quad \text{for all } y \in M,$

and

$$f_{\xi(\omega)}(T(\omega, \xi(\omega)) - z) < c \quad \text{for all } z \in \text{int}(B_r(T(\omega, \xi(\omega)))).$$

The continuity of $f_{\xi(\omega)}$ implies that

$$f_{\xi(\omega)}(T(\omega, \xi(\omega)) - z) \leq c \quad \text{for all } z \in B_r(T(\omega, \xi(\omega))).$$

Since $\xi(\omega) \in M \cap B_r(T(\omega, \xi(\omega)))$, it follows that

(iii) $f_{\xi(\omega)}(T(\omega, \xi(\omega)) - \xi(\omega)) = c.$

Obviously c is nonzero otherwise we get the contradiction that $f_{\xi(\omega)}$ is identically zero.

Put $f = (1/c)rf_{\xi(\omega)}$. This implies by (iii) that

$$\begin{aligned} f(T(\omega, \xi(\omega)) - \xi(\omega)) &= (1/c)rf_{\xi(\omega)}(T(\omega, \xi(\omega)) - \xi(\omega)) = r \\ f(y - \xi(\omega)) &= f(y - T(\omega, \xi(\omega))) + f(T(\omega, \xi(\omega)) - \xi(\omega)) \quad (y \in M) \\ &= (1/c)rf_{\xi(\omega)}(y - T(\omega, \xi(\omega))) + (1/c)rf_{\xi(\omega)}(T(\omega, \xi(\omega)) - \xi(\omega)) \\ &\leq 0 \quad (\text{by (ii) and (iii)}). \end{aligned}$$

It is easy to get by linearity of f that $\|f\|_r = r$.

Conversely suppose that there is a real continuous linear functional f satisfying the conditions (a)–(c).

If the conclusion is false, then for some x in M , we have

$$(iv) \quad d(T(\omega, \xi(\omega)), x) < d(T(\omega, \xi(\omega)), \xi(\omega)).$$

The continuity of scalar multiplication implies that for any $\epsilon > 0$, there is $\beta > 0$ such that

$$(v) \quad d(0, \beta T(\omega, \xi(\omega)) - \beta x) < \epsilon.$$

Consider

$$\begin{aligned} &d(0, (1 + \beta)(T(\omega, \xi(\omega)) - x)) \\ &\leq d(0, T(\omega, \xi(\omega)) - x) + d(T(\omega, \xi(\omega)) - x, (1 + \beta)(T(\omega, \xi(\omega)) - x)) \\ &= d(0, T(\omega, \xi(\omega)) - x) + d(0, \beta T(\omega, \xi(\omega)) - \beta x) \quad (\text{by invariance of } d) \\ &< d(0, T(\omega, \xi(\omega)) - x) + \epsilon \quad (\text{by (v)}) \\ &\leq d(T(\omega, \xi(\omega)), \xi(\omega)) \quad (\text{by (iv)}). \end{aligned}$$

The above inequality and the fact $f(\xi(\omega) - x) \geq 0$ lead to:

$$\begin{aligned} f((1 + \beta)(T(\omega, \xi(\omega)) - x)) &= (1 + \beta)f(T(\omega, \xi(\omega)) - x) \\ &\geq (1 + \beta)f(T(\omega, \xi(\omega)) - \xi(\omega)). \end{aligned}$$

This implies that $f(T(\omega, \xi(\omega)) - \xi(\omega))$ is not the supremum of f over $B_r(0)$. This contradiction proves the result. \square

In case M is a subspace we have the following:

Corollary. *Let (E, d) be a separable metrizable locally convex space with invariant metric d and M a subspace of E . Assume that the ball $B_r(0)$ is convex and bounding. Suppose that $T : \Omega \times M \rightarrow E$ is a random operator and $\xi : \Omega \rightarrow M$ a measurable map such that $T(\omega, \xi(\omega)) \notin \text{cl}(M)$. Then ξ is a random best approximation for T if and only if there exists a real continuous linear functional $f \in E_R^*$ such that*

- (a) $f(T(\omega, \xi(\omega)) - \xi(\omega)) = d(T(\omega, \xi(\omega)), \xi(\omega)) = r(\omega) = r$ (say).
- (b) $f(y) = 0$ for all y in M .
- (c) $\|f\|_r = r$.

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