CHARACTERIZATIONS OF RANDOM APPROXIMATIONS

ABDUL RAHIM KHAN AND NAWAB HUSSAIN

Abstract. Some characterizations of random approximations are obtained in a locally convex space through duality theory.

1. Introduction and preliminaries

Random approximation theory has received much attention after the publication of survey paper by Bharucha-Reid [3] in 1976. The interested reader is referred to recent papers in normed space framework by Tan and Yuan [11], Sehgal and Singh [10], Papageorgiou [7], Lin [5], Beg and Shahzad [2] and Beg [1]. The interplay between random approximation and random fixed point results is interesting and valuable (see for example [5], [7] and [11]). The applications of this closely related concept to random differential equations and integral equations in the context of Banach spaces may be found in Itoh [4] and O’Regan [6] respectively. So random approximations are needed in the study of random equations. Recently, Beg [1] obtained a characterization of random approximations in a normed space by employing the Hahn-Banach separation theorem. Characterization theorems of best approximation in the locally convex space setting have been considered in [8]. In this paper, we establish the characterizations concerning existence of random approximation in locally convex spaces by using the Hahn Banach extension theorem and a result of Tukey [13] about separation of convex sets; in particular Theorem 1 provides a random version of Theorem 2.1 of Rao and Elumalai [8] and Theorem 2 sets an analogue for metrizable locally convex spaces of the theorem due to Beg [1].

We now fix our terminology. Let $(\Omega, \Sigma)$ be a measurable space where $\Sigma$ is a sigma algebra of subsets of $\Omega$ and $M$ a subset of a locally convex space $E$ over the field $F$ of real or complex numbers. A map $T : \Omega \times M \to E$ is called a random operator if for each fixed $x \in M$, the map $T(\cdot, x) : \Omega \to E$ is measurable. Let $(E, d)$ be a metrizable locally convex space.

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(i) The ball with radius \( r \) and centre at \( x \) is defined as \( B_r(x) = \{ z \in E : d(z, x) \leq r \} \); in particular the ball \( B_r(0) \) has centre at 0.

(ii) \( d(x, M) = \inf_{u \in M} d(x, u) \).

(iii) \( P_M(x) = \{ y \in M : d(x, y) = d(x, M) \} \) (set of best approximations of \( x \) from \( M \)).

(iv) For a ball \( B_r(0) \) in \( (E, d) \), the set \( \{ z \in E : d(z, 0) = r \} \) is called metric boundary of \( B_r(0) \). In general, the topological boundary of \( B_r(0) \) is contained in its metric boundary. In case metric and topological boundaries of \( B_r(0) \) coincide, we say \( B_r(0) \) is bounding (cf. [12]).

In this note, \( \text{cl}, \text{int}, E^* \) and \( E \setminus M \) denote the closure, interior, dual of \( E \) and difference of sets \( E \) and \( M \), respectively.

2. Results

**Theorem 1.** Let \( E \) be a separable locally convex space with family \( P \) of seminorms and \( M \) a subspace of \( E \). Suppose \( T : \Omega \times M \to E \) is a random operator and \( \xi : \Omega \to M \) a measurable map such that \( T(\omega, \xi(\omega)) \in E \setminus M \). Then \( \xi \) is a random best approximation for \( T \) (i.e., \( p(\xi(\omega) - T(\omega, \xi(\omega))) = d_p(T(\omega, \xi(\omega)), M) \) for each \( p \in P \) if and only if for every \( p \in P \) there exists \( f^p \in E^* \) such that

\[
\begin{align*}
(a) & \quad f^p(0) = 0 \quad \text{for all} \quad g \in M. \\
(b) & \quad |f^p(T(\omega, \xi(\omega)) - \xi(\omega))| = p(T(\omega, \xi(\omega)) - \xi(\omega)). \\
(c) & \quad |f^p(T(\omega, \xi(\omega)) - g| \leq p(T(\omega, \xi(\omega)) - g) \quad \text{for all} \quad g \in M.
\end{align*}
\]

**Proof.** Suppose that \( \xi \) is a random approximation for \( T \). Then for each \( p \in P \) and \( g \in M \),

\[
p(T(\omega, \xi(\omega)) - \xi(\omega)) \leq p(T(\omega, \xi(\omega)) - g).
\]

In particular, for any \( 0 \neq \alpha \in F \) and \( g \in M \),

\[
\begin{align*}
(p) & \quad p(T(\omega, \xi(\omega)) - \xi(\omega)) \leq p \left( T(\omega, \xi(\omega)) - \left( \xi(\omega) - \frac{g}{\alpha} \right) \right).
\end{align*}
\]

Let \( B = \{ g + \alpha(T(\omega, \xi(\omega)) - \xi(\omega)) : \alpha \in F \} \).

Define \( f^p_0 \) on \( B \) by \( f^p_0(g + \alpha(T(\omega, \xi(\omega)) - \xi(\omega))) = \alpha p(T(\omega, \xi(\omega)) - \xi(\omega)) \) for all \( g \in M \). Then \( f^p_0(g) = 0 \) for all \( g \in M \) and

\[
f^p_0(T(\omega, \xi(\omega)) - \xi(\omega)) = p(T(\omega, \xi(\omega)) - \xi(\omega)).
\]

For any \( \alpha \neq 0 \) and \( g \in M \), we have

\[
|f^p_0(g + \alpha(T(\omega, \xi(\omega)) - \xi(\omega)))| = |\alpha| p(T(\omega, \xi(\omega)) - \xi(\omega)) \\
\leq |\alpha| p \left( T(\omega, \xi(\omega)) - \xi(\omega) + \frac{g}{\alpha} \right) \quad \text{(by (i))} \\
= p(g + \alpha(T(\omega, \xi(\omega)) - \xi(\omega))).
\]

For \( \alpha = 0 \) and \( g \in M \) this inequality obviously holds.

Hence for each \( z \in B \) and for each \( p \in P \),

\[
|f^p_0(z)| \leq p(z).
\]
Thus by the Hahn-Banach theorem, \( f_0^p \) can be extended to a continuous linear functional \( f^p \) on \( E \) such that \( |f^p(x)| \leq p(x) \) for every \( x \in E \) and

\[
|f^p(z)| = |f_0^p(z)| \quad \text{for each} \quad z \in M.
\]

The results (a)–(c) are now evident.

Conversely let the conditions (a)–(c) be satisfied. Then from (b) we get for all \( p \in P \) and \( g \in M \),

\[
p(T(\omega, \xi(\omega)) - \xi(\omega)) = |f^p(T(\omega, \xi(\omega)) - \xi(\omega))|
= |f^p(T(\omega, \xi(\omega)) - g) + f^p(g - \xi(\omega))|
= |f^p(T(\omega, \xi(\omega)) - g)| \quad \text{(by (a))}
\leq p(T(\omega, \xi(\omega)) - g) \quad \text{(by (c)).}
\]

Hence \( p(T(\omega, \xi(\omega)) - \xi(\omega)) = d_p(T(\omega, \xi(\omega)), M) \) for all \( p \in P \). \( \square \)

We shall follow the argument used in the proof of Theorem 2.3 of Thaheem [12] to prove the following:

**Theorem 2.** Let \((E, d)\) be a separable metrizable locally convex space with \(d\) as invariant metric. Assume that the ball \( B_r(0) \) is convex and bounding and \( M \) a convex subset of \( E \). Let \( T : \Omega \times M \to E \) be a random operator and \( \xi : \Omega \to M \) a measurable map such that \( T(\omega, \xi(\omega)) \notin \text{cl}(M) \). Then \( \xi \) is a random best approximation for \( T \) if and only if there exists a real continuous linear functional \( f \in E^*_R \) (\( R \) is the set of real numbers) such that

(a) \( f(T(\omega, \xi(\omega)) - \xi(\omega)) = d(T(\omega, \xi(\omega)), \xi(\omega)) = r(w) = r \) (say; for notational simplicity).

(b) \( f(y - \xi(\omega)) \leq 0 \) for all \( y \in M \).

(c) \( \|f\|_r = \sup\{|f(z)| : z \in B_r(0)\} = r \).

**Proof.** Assume that \( d(\xi(\omega), T(\omega, \xi(\omega))) = d(T(\omega, \xi(\omega)), M) \). Then \( M \) and \( \text{int}(B_r(T(\omega, \xi(\omega)))) \), where \( r = d(T(\omega, \xi(\omega)), M) \), are disjoint convex sets. By a result of Tukey [13] (see also Rudin [9]), there is a nonzero continuous linear real linear functional \( f_{\xi(\omega)} \in E_R^* \) and a real number \( c \) such that

(ii) \( f_{\xi(\omega)}(T(\omega, \xi(\omega)) - y) \geq c \quad \text{for all} \quad y \in M \),

and

\( f_{\xi(\omega)}(T(\omega, \xi(\omega)) - z) < c \quad \text{for all} \quad z \in \text{int}(B_r(T(\omega, \xi(\omega)))) \).

The continuity of \( f_{\xi(\omega)} \) implies that

\( f_{\xi(\omega)}(T(\omega, \xi(\omega)) - z) \leq c \quad \text{for all} \quad z \in B_r(T(\omega, \xi(\omega))) \).

Since \( \xi(\omega) \in M \cap B_r(T(\omega, \xi(\omega))) \), it follows that

(iii) \( f_{\xi(\omega)}(T(\omega, \xi(\omega)) - \xi(\omega)) = c \).

Obviously \( c \) is nonzero otherwise we get the contradiction that \( f_{\xi(\omega)} \) is identically zero.
Put $f = (1/c)rf_{\xi(\omega)}$. This implies by (iii) that
\[
f(T(\omega, \xi(\omega)) - \xi(\omega)) = (1/c)rf_{\xi(\omega)}(T(\omega, \xi(\omega)) - \xi(\omega)) = r
\]
\[
f(y - \xi(\omega)) = f(y - T(\omega, \xi(\omega))) + f(T(\omega, \xi(\omega)) - \xi(\omega)) \quad (y \in M)
\]
\[
= (1/c)rf_{\xi(\omega)}(y - T(\omega, \xi(\omega))) + (1/c)rf_{\xi(\omega)}(T(\omega, \xi(\omega)) - \xi(\omega))
\]
\[\leq 0 \quad \text{(by (ii) and (iii)).}
\]

It is easy to get by linearity of $f$ that $\|f\|_r = r$.

Conversely suppose that there is a real continuous linear functional $f$ satisfying the conditions (a)–(c).

If the conclusion is false, then for some $x$ in $M$, we have
\[
(iv) \quad d(T(\omega, \xi(\omega)), x) < d(T(\omega, \xi(\omega)), \xi(\omega)).
\]

The continuity of scalar multiplication implies that for any $\epsilon > 0$, there is $\beta > 0$ such that
\[
(v) \quad d(0, \beta T(\omega, \xi(\omega)) - \beta x) < \epsilon.
\]

Consider
\[
d(0, (1 + \beta)(T(\omega, \xi(\omega)) - x))
\]
\[\leq d(0, T(\omega, \xi(\omega)) - x) + d(T(\omega, \xi(\omega)) - x, (1 + \beta)(T(\omega, \xi(\omega)) - x))
\]
\[= d(0, T(\omega, \xi(\omega)) - x) + d(0, \beta T(\omega, \xi(\omega)) - \beta x) \quad \text{(by invariance of $d$)}
\]
\[< d(0, T(\omega, \xi(\omega)) - x) + \epsilon \quad \text{(by (v))}
\]
\[\leq d(T(\omega, \xi(\omega)), \xi(\omega)) \quad \text{(by (iv)).}
\]

The above inequality and the fact $f(\xi(\omega) - x) \geq 0$ lead to:
\[
f((1 + \beta)(T(\omega, \xi(\omega)) - x)) = (1 + \beta)f(T(\omega, \xi(\omega)) - x)
\]
\[\geq (1 + \beta)f(T(\omega, \xi(\omega)) - \xi(\omega)).
\]

This implies that $f(T(\omega, \xi(\omega)) - \xi(\omega))$ is not the supremum of $f$ over $B_r(0)$. This contradiction proves the result.

In case $M$ is a subspace we have the following:

**Corollary.** Let $(E, d)$ be a separable metrizable locally convex space with invariant metric $d$ and $M$ a subspace of $E$. Assume that the ball $B_r(0)$ is convex and bounding. Suppose that $T : \Omega \times M \to E$ is a random operator and $\xi : \Omega \to M$ a measurable map such that $T(\omega, \xi(\omega)) \notin \text{cl}(M)$. Then $\xi$ is a random best approximation for $T$ if and only if there exists a real continuous linear functional $f \in E_R^*$ such that
\[
(a) \quad f(T(\omega, \xi(\omega)) - \xi(\omega)) = d(T(\omega, \xi(\omega)), \xi(\omega)) = r(\omega) = r \quad \text{(say)}.
\]
\[
(b) \quad f(y) = 0 \text{ for all } y \text{ in } M.
\]
\[
(c) \quad \|f\|_r = r.
\]
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References


Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
E-mail: arahim@kfupm.edu.sa
(On leave from Bahauddin Zakaria University, Multan, Pakistan)

Centre for Advanced Studies in Pure and Applied Mathematics
Bahauddin Zakariya University
Multan, Pakistan
E-mail: mnawab2000@yahoo.com