

**A SINGULAR VERSION OF LEIGHTON'S COMPARISON  
THEOREM FOR FORCED QUASILINEAR  
SECOND ORDER DIFFERENTIAL EQUATIONS**

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ABSTRACT. We extend the classical Leighton comparison theorem to a class of quasilinear forced second order differential equations

$$(*) \quad (r(t)|x'|^{\alpha-2}x')' + c(t)|x|^{\beta-2}x = f(t), \quad 1 < \alpha \leq \beta, \quad t \in I = (a, b),$$

where the endpoints  $a, b$  of the interval  $I$  are allowed to be singular. Some applications of this statement in the oscillation theory of  $(*)$  are suggested.

1. INTRODUCTION

In this paper we deal with oscillatory properties of the quasilinear forced second order differential equation

$$(1.1) \quad L_{\alpha\beta}[x] := (r(t)|x'|^{\alpha-2}x')' + c(t)|x|^{\beta-2}x = f(t),$$

where  $t \in I = (a, b)$ ,  $-\infty \leq a < b \leq \infty$ ,  $r, c, f$  are real-valued continuous functions with  $r(t) > 0$ ,  $c(t) \geq 0$  for  $t \in I$  and  $1 < \alpha \leq \beta$  are real constants.

The classical Leighton comparison theorem [12] (see also [17]) concerns the pair of second order Sturm-Liouville equations

$$(1.2) \quad (r(t)x')' + c(t)x = 0,$$

$$(1.3) \quad (R(t)y')' + C(t)y = 0,$$

and reads as follows.

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**Proposition 1.1.** *Suppose that the interval  $I = [a, b]$  is compact, the functions  $r, R, c, C$  are continuous on  $I$ ,  $r(t) > 0, R(t) > 0$  in this interval and there exists a nontrivial solution  $\tilde{y}$  of (1.3) such that  $\tilde{y}(a) = 0 = \tilde{y}(b)$ . If*

$$\int_a^b [(r(t) - R(t))\tilde{y}'^2(t) - (c(t) - C(t))\tilde{y}^2(t)] dt \leq 0,$$

*Then every solution of (1.2) has a zero point in  $(a, b)$  or it is a multiple of  $\tilde{y}$ .*

Since 1962, when the original Leighton theorem was published, this statement has been extended in various directions, see [9-11,15,16] and the references given therein. Our investigation is mainly motivated by the papers [2,10]. In the first paper conjugacy criteria for the half-linear second order equation

$$(1.4) \quad (r(t)|x'|^{\alpha-2}x')' + c(t)|x|^{\alpha-2}x = 0$$

are established and the second one studies oscillation properties of equation (1.1) on a compact interval. In our treatment we combine ideas of these two papers and we prove a Leightone-type comparison theorem for (1.1) where the end points  $a, b$  of the interval  $I$  are allowed to be singular. The solution  $\tilde{y}$  from the classical Leighton theorem (i. e., the solution for which  $\tilde{y}(a) = 0 = \tilde{y}(b)$ ) is replaced by the principal solution at  $a$  and  $b$  of a certain half-linear equation. An important role in the proofs of our results is played by the recently established Picone's identity for half-linear equations, see [8].

## 2. PICONE'S IDENTITY AND PRINCIPAL SOLUTION OF HALF-LINEAR EQUATIONS

Denote, for convenience,  $\varphi(u) := |u|^{\alpha-2}u$  and rewrite half-linear equation (1.4) into the form

$$(2.1) \quad (r(t)\varphi(x'))' + c(t)\varphi(x) = 0.$$

It is well known that the classical Sturmian oscillation theory extends almost verbatim to half-linear equation (2.1), see e. g. [1,3,6,13]. In particular, the Riccati-type equation (related to (2.1) by the substitution  $w = r\varphi(x')/\varphi(x)$ )

$$(2.2) \quad w' + c(t) + (\alpha - 1)r^{\frac{1}{1-\alpha}}(t)|w|^{\frac{\alpha}{\alpha-1}} = 0,$$

and the  $\alpha$  degree functional

$$(2.3) \quad \mathcal{I}(y) := \int_a^b [r(t)|y'|^\alpha - c(t)|y|^\alpha] dt$$

play the same role as the classical Riccati equation and quadratic functionals in the linear oscillation theory. The proof of the relationship between (2.1), (2.2) and (2.3) is based on the recently established Picone's identity, see [8]. Here we present this identity in a modified form as used in [10].

**Lemma 2.1.** Let  $x, y, \varphi(y') \in C^1(I)$  and  $y(t) \neq 0$  for  $t \in I$ . Then

$$\left[ \frac{r(t)|x|^\alpha}{\varphi(y)} \varphi(y') \right]' = r(t)|x'|^\alpha - \left[ c(t)|y|^{\beta-\alpha} - \frac{f(t)}{\varphi(y)} \right] |x|^\alpha - r(t)P_\alpha(x', xy'/y) + \frac{|x|^\alpha}{\varphi(y)} \{L_{\alpha\beta}[y] - f(t)\},$$

where

$$P_\alpha(u, v) := |u|^\alpha - \alpha u \varphi(v) + (\alpha - 1)|v|^\alpha \geq 0$$

for every  $u, v \in \mathbb{R}$  with equality holding if and only if  $u = v$ .

We will need also the following statement proved in [10].

**Lemma 2.2.** Suppose that the interval  $I = [a, b]$  is compact, there exists a non-trivial piecewise differentiable function  $y \in C[a, b]$  such that  $y(a) = 0 = y(b)$  and

$$\mathcal{F}(y; a, b) := \int_a^b [r(t)|y'|^\alpha - M_{\alpha\beta}[c, f](t)|y|^\alpha] dt \leq 0,$$

where

$$(2.4) \quad M_{\alpha\beta}[c, f](t) := (\alpha - 1)^{-\frac{\alpha-1}{\beta-1}} (\beta - 1)(\beta - \alpha)^{\frac{\alpha-\beta}{\beta-1}} [c(t)]^{\frac{\alpha-1}{\beta-1}} |f(t)|^{\frac{\beta-\alpha}{\beta-1}}$$

(with the convention that  $0^0 = 1$  in case  $\alpha = \beta$ ). Then every solution of (1.1) defined on  $[a, b]$  and satisfying  $x(t)f(t) \leq 0$  in this interval has a zero in  $[a, b]$ .

Note that Lemma 2.2. is proved in [10] under the (stronger) assumption  $y \in C^1(I)$ , but the proof directly extends to the case when  $y \in C^1(I)$  only piecewise.

Now recall the concept of the principal solution of half-linear equation (2.1) as introduced by Mirzov [14]. Suppose that (2.1) is nonoscillatory, then among all proper solutions of the associated Riccati type equation (2.2) (i. e. solutions which are extendible up to  $\infty$ ) one can distinguish the so-called *eventually minimal* solution  $\tilde{w}$ , a solution with the property that any other proper solution  $w$  satisfies the inequality  $w(t) > \tilde{w}(t)$  eventually. The solution  $\tilde{x}$  of (2.1) given by the formula

$$x(t) = \exp \left\{ \int^t r^{\frac{1}{1-\alpha}}(s) \varphi^{-1}(\tilde{w}(s)) ds \right\}, \quad \text{i.e.} \quad \tilde{w}(t) = r(t) \frac{\varphi(\tilde{x}')}{\varphi(\tilde{x})},$$

where  $\varphi^{-1}$  is the inverse function of  $\varphi$ , is said to be the *principal* solution (at  $\infty$ ) of (2.1). Clearly, the principal solution is determined uniquely up to a multiple by a nonzero real constant. Observe that if  $b$  is a regular point of (2.1) (i. e. the initial value problem  $x(b) = x_0, r(b)\varphi(x'(b)) = x_1$  has the unique solution for any  $x_0, x_1 \in \mathbb{R}$ ) and  $x_b$  is a solution given by the initial condition  $x_b(b) = 0, r(b)\varphi(x'_b(b)) \neq 0$ , then the associated solution  $w_b = r\varphi(x'_b)/\varphi(x_b)$  of (2.2) satisfies  $w_b(b-) = -\infty$  and  $w_b$  is minimal solution of (2.2) in a left neighbourhood of  $b$ . Hence,  $x_b$  can be regarded as the principal solution at (the regular point)  $b$  and the condition  $\tilde{y}(a) = 0 = \tilde{y}(b)$  from Proposition 1.1 can be reformulated “ $\tilde{y}$  is the principal solution at  $a$  and  $b$ ”. From this point of view the main result of our paper can be regarded as a singular version of Leighton’s comparison theorem. For more details concerning principal solution of half-linear differential equations see [2, 4, 5, 7].

## 3. SINGULAR LEIGHTON'S THEOREM

The main result of our paper reads as follows.

**Theorem 3.1.** *Suppose that the half-linear equation*

$$(3.1) \quad (R(t)\varphi(y'))' + C(t)\varphi(y) = 0$$

*is nonoscillatory both at  $a$  and  $b$  and the principal solutions at  $a$  and  $b$  of this equation coincide, denote this solution by  $\tilde{y}$ . Further suppose that  $0 < r(t) \leq R(t)$  near  $a$  and  $b$  and*

$$(3.2) \quad \limsup_{t_1 \downarrow a, t_2 \uparrow b} \left\{ \int_{t_1}^{t_2} (r(t) - R(t)) |\tilde{y}'|^\alpha dt \right. \\ \left. - \inf_{s_1 \leq t_1, s_2 \geq t_2} \int_{s_1}^{s_2} (M_{\alpha\beta}[c, f](t) - C(t)) |\tilde{y}|^\alpha dt \right\} < 0.$$

*Then every solution of (1.1) satisfying  $x(t)f(t) \leq 0$  has a zero in  $I$ .*

**Proof.** According to Lemma 2.2 it suffices to find  $a_1, b_1 \in I$ ,  $a_1 < b_1$ , and a nontrivial piecewise  $C^1$  function  $y$  such that  $y(a_1) = 0 = y(b_1)$  and

$$\mathcal{F}(y; a_1, b_1) = \int_{a_1}^{b_1} [r(t)|y'|^\alpha - M_{\alpha\beta}[c, f](t)|y|^\alpha] dt \leq 0.$$

In our proof we borrow some ideas used in the proof of the conjugacy criterion of [2].

Let  $a < a_1 < t_1 < t_2 < b_1 < b$  (these values will be specified later) and let  $f, g$  be the solutions of (3.1) satisfying the boundary conditions

$$f(a_1) = 0, \quad f(t_1) = \tilde{y}(t_1), \quad g(t_2) = \tilde{y}(t_2), \quad g(b_1) = 0.$$

Note that such solutions exist if  $t_1$  and  $t_2$  are sufficiently close to  $a$  and  $b$  respectively (due to nonoscillation of (3.1) near  $a$  and  $b$  and the fact that the solution space of this equation is homogeneous). Define the function  $y$  as follows

$$y(t) = \begin{cases} 0 & t \in (a, a_1], \\ f(t) & t \in [a_1, t_1], \\ \tilde{y}(t) & t \in [t_1, t_2], \\ g(t), & t \in [t_2, b_1], \\ 0 & t \in [b_1, b]. \end{cases}$$

Then we have

$$\begin{aligned}
\mathcal{F}(y; a_1, b_1) &= \int_{a_1}^{b_1} [r(t)|y'|^\alpha - M_{\alpha\beta}[c, f](t)|y|^\alpha] dt \\
&= \int_{a_1}^{b_1} [R(t)|y'|^\alpha - C(t)|y|^\alpha] dt + \int_{a_1}^{b_1} [(r(t) - R(t))|y'|^\alpha - (M_{\alpha\beta}[c, f](t) - C(t))|y|^\alpha] dt \\
&= \int_{a_1}^{t_1} [R(t)|f'|^\alpha - C(t)|f|^\alpha] dt + \int_{a_1}^{t_1} [(r(t) - R(t))|f'|^\alpha - (M_{\alpha\beta}[c, f](t) - C(t))|f|^\alpha] dt \\
&\quad + \int_{t_1}^{t_2} [R(t)|\tilde{y}'|^\alpha - C(t)|\tilde{y}|^\alpha] dt + \int_{t_1}^{t_2} [(r(t) - R(t))|\tilde{y}'|^\alpha - (M_{\alpha\beta}[c, f](t) - C(t))|\tilde{y}|^\alpha] dt \\
&\quad + \int_{t_2}^{b_1} [R(t)|g'|^\alpha - C(t)|g|^\alpha] dt + \int_{t_2}^{b_1} [(r(t) - R(t))|g'|^\alpha - (M_{\alpha\beta}[c, f](t) - C(t))|g|^\alpha] dt.
\end{aligned}$$

Denote by  $v_f$ ,  $v_g$  and  $\tilde{v}$  the solutions of Riccati equation associated with (3.1)

$$(3.3) \quad v' + C(t) + (\alpha - 1)R^{\frac{1}{1-\alpha}}(t)|v|^{\frac{\alpha}{\alpha-1}} = 0$$

generated by  $f$ ,  $g$  and  $\tilde{y}$ , respectively, i.e.,

$$v_f = \frac{R\varphi(f')}{\varphi(f)}, \quad v_g = \frac{R\varphi(g')}{\varphi(g)}, \quad \tilde{v} = \frac{R\varphi(\tilde{y}')}{\varphi(\tilde{y})}.$$

Then using integration by parts

$$\begin{aligned}
\int_{a_1}^{t_1} [R(t)|f'|^\alpha - C(t)|f|^\alpha] dt &= Rf\varphi(f') \Big|_{a_1}^{t_1} - \int_{a_1}^{t_1} f [(R(t)\varphi(f'))' + C(t)\varphi(f)] dt \\
&= v_f |f|^\alpha \Big|_{a_1}^{t_1},
\end{aligned}$$

similarly,

$$\begin{aligned}
\int_{t_1}^{t_2} [R(t)|\tilde{y}'|^\alpha - C(t)|\tilde{y}|^\alpha] dt &= \tilde{v} |\tilde{y}|^\alpha \Big|_{t_1}^{t_2}, \\
\int_{t_2}^{b_1} [R(t)|g'|^\alpha - C(t)|g|^\alpha] dt &= v_g |g|^\alpha \Big|_{t_2}^{b_1}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_{a_1}^{b_1} [R(t)|y'|^\alpha - C(t)|y|^\alpha] dt &= v_f |f|^\alpha \Big|_{a_1}^{t_1} + \tilde{v} |\tilde{y}|^\alpha \Big|_{t_1}^{t_2} + v_g |g|^\alpha \Big|_{t_2}^{b_1} \\
&= |\tilde{y}(t_1)|^\alpha (v_f(t_1) - \tilde{v}(t_1)) + |\tilde{y}(t_2)|^\alpha (\tilde{v}(t_2) - v_g(t_2)).
\end{aligned}$$

Next we deal with the integral  $\int_{a_1}^{t_1} (M_{\alpha\beta}[c, f](t) - C(t)) |f|^\alpha dt$ . The function  $\frac{f}{y}$  is monotonically increasing in  $(a_1, t_1)$  since  $\frac{f}{y}(a_1) = 0$ ,  $\frac{f}{y}(t_1) = 1$  and  $\left(\frac{f}{y}\right)' =$

$\frac{f'\tilde{y}-f\tilde{y}'}{\tilde{y}^2} \neq 0$  in  $(a_1, t_1)$ . Indeed, if  $f'\tilde{y} - f\tilde{y}' = 0$  at some point  $\tilde{t} \in (a_1, t_1)$ , i.e.  $\frac{f'}{f}(\tilde{t}) = \frac{\tilde{y}'}{\tilde{y}}(\tilde{t})$  then  $v_f(\tilde{t}) = \tilde{v}(\tilde{t})$  which contradicts the unique solvability of (3.3). By the second mean value theorem of integral calculus now there exists  $\xi_1 \in (a_1, t_1)$  such that

$$\begin{aligned} \int_{a_1}^{t_1} (M_{\alpha\beta}[c, f](t) - C(t)) |f|^\alpha dt &= \int_{a_1}^{t_1} (M_{\alpha\beta}[c, f](t) - C(t)) |\tilde{y}|^\alpha \frac{|f|^\alpha}{|\tilde{y}|^\alpha} dt \\ &= \int_{\xi_1}^{t_1} (M_{\alpha\beta}[c, f](t) - C(t)) |\tilde{y}|^\alpha dt. \end{aligned}$$

By the same argument  $\frac{g}{\tilde{y}}$  is monotonically decreasing in  $(t_2, b_1)$  and

$$\int_{t_2}^{b_1} (M_{\alpha\beta}[c, f](t) - C(t)) |g|^\alpha dt = \int_{t_2}^{\xi_2} (M_{\alpha\beta}[c, f](t) - C(t)) |\tilde{y}|^\alpha dt$$

for some  $\xi_2 \in (t_2, b_1)$ .

Therefore

$$\int_{a_1}^{b_1} (M_{\alpha\beta}[c, f](t) - C(t)) |y|^\alpha dt = \int_{\xi_1}^{\xi_2} (M_{\alpha\beta}[c, f](t) - C(t)) |\tilde{y}|^\alpha dt.$$

Summarizing our computations and using the fact that  $r(t) - R(t) \leq 0$  on  $[a_1, t_1]$  and  $[t_2, b_1]$  if  $t_1, t_2$  are sufficiently close to  $a$  and  $b$ , respectively, i.e.,

$$\int_{a_1}^{t_1} (r(t) - R(t)) |f'|^\alpha dt \leq 0, \quad \int_{t_2}^{b_1} (r(t) - R(t)) |g'|^\alpha dt \leq 0,$$

we have

$$\begin{aligned} \mathcal{F}(y; a_1, b_1) &\leq |\tilde{y}(t_1)|^\alpha (v_f(t_1) - \tilde{v}(t_1)) + |\tilde{y}(t_2)|^\alpha (\tilde{v}(t_2) - v_g(t_2)) \\ &\quad + \int_{t_1}^{t_2} (r(t) - R(t)) |\tilde{y}'|^\alpha dt - \int_{\xi_1}^{\xi_2} (M_{\alpha\beta}[c, f](t) - C(t)) |\tilde{y}|^\alpha dt. \end{aligned}$$

According to (3.2) there exists  $\varepsilon > 0$  such that

$$\int_{t_1}^{t_2} (r(t) - R(t)) |\tilde{y}'|^\alpha dt - \int_{s_1}^{s_2} (M_{\alpha\beta}[c, f](t) - C(t)) |\tilde{y}|^\alpha dt < -\varepsilon$$

whenever  $s_1 \in (a, t_1)$ ,  $s_2 \in (t_2, b)$ , if  $t_1, t_2$  are sufficiently close to  $a$  and  $b$ , respectively. Further, since  $\tilde{v}$  is generated by the solution  $\tilde{y}$  of (3.1) which is principal both at  $t = a$  and  $t = b$ , according to the ‘‘Riccati equation’’ construction of the principal solution mentioned in the previous section, we have (for  $t_1, t_2$  fixed for a moment)

$$\lim_{a_1 \rightarrow a^+} [v_f(t_1) - \tilde{v}(t_1)] = 0, \quad \lim_{b_1 \rightarrow b^-} [v_g(t_2) - \tilde{v}(t_2)] = 0,$$

(see [2] for details) observe that solutions  $v_f, v_g$  actually depend also on  $a_1$  and  $b_1$ , respectively. Hence

$$|\tilde{y}(t_1)|^\alpha [v_f(t_1) - \tilde{v}(t_1)] < \frac{\varepsilon}{2}, \quad |\tilde{y}(t_2)|^\alpha [\tilde{v}(t_2) - v_g(t_2)] < \frac{\varepsilon}{2}$$

if  $a_1 < t_1, b_1 > t_2$  are sufficiently close to  $a$  and  $b$ , respectively.

Consequently, for the above specified choice of  $a_1 < t_1 < t_2 < b_1$  we have

$$\mathcal{F}(y; a_1, b_1) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} - \varepsilon = 0$$

what we needed to prove. The statement now follows from Lemma 2.2.  $\square$

If  $r(r) \leq R(t)$  in the *whole* interval  $I$  (not only near  $a$  and  $b$  as supposed in Theorem 3.1), assumptions of Theorem 3.1 can be modified as follows.

**Theorem 3.2.** *Suppose that  $r(t) \leq R(t)$  on  $I$ ,  $\tilde{y}$  is the same as in Theorem 3.1,*

$$(3.4) \quad \liminf_{s_1 \downarrow a, s_2 \uparrow b} \int_{s_1}^{s_2} \{M_{\alpha\beta}[c, f](t) - C(t)\} |\tilde{y}|^\alpha dt \geq 0$$

and

$$(3.5) \quad M_{\alpha\beta}[c, f](t) \not\equiv C(t) \quad \text{in } I.$$

*Then every solution of (1.1) satisfying  $x(t)f(t) \leq 0$  in  $I$  has a zero in this interval.*

**Proof.** We will proceed similarly as in the proof of the previous theorem. Continuity of the functions  $C, M_{\alpha\beta}[c, f]$  and (3.5) imply the existence of  $\bar{t} \in I$  and  $d, \varrho > 0$  such that  $(M_{\alpha\beta}[c, f](t) - C(t))|\tilde{y}(t)|^\alpha > d$  for  $(\bar{t} - \varrho, \bar{t} + \varrho)$ , and let  $\Delta$  be any positive differentiable function with the compact support in  $(\bar{t} - \varrho, \bar{t} + \varrho)$ . Further let  $a < t_1 < \bar{t} - \varrho < \bar{t} + \varrho < t_2 < b_1 < b$  and define the function  $y$  as follows

$$y(t) = \begin{cases} 0 & t \in (a, a_1), \\ f(t) & t \in [a_1, t_1], \\ \tilde{y}(t) & t \in [t_1, t_2] \setminus [\bar{t} - \varrho, \bar{t} + \varrho], \\ \tilde{y}(t)(1 + \delta\Delta(t)), & t \in [\bar{t} - \varrho, \bar{t} + \varrho], \\ g(t), & t \in [t_2, b_1], \\ 0 & t \in [b_1, b), \end{cases}$$

where  $\delta$  is a real parameter. We have

$$\begin{aligned} \mathcal{F}(y; a_1, b_1) &= \int_{a_1}^{b_1} [r(t)|y'|^\alpha - M_{\alpha\beta}[c, f](t)|y|^\alpha] dt \\ &= \int_{a_1}^{b_1} [R(t)|y'|^\alpha - C(t)|y|^\alpha] dt \\ &\quad + \int_{a_1}^{b_1} [(r(t) - R(t))|y'|^\alpha - (M_{\alpha\beta}[c, f](t) - C(t))|y|^\alpha] dt \\ &\leq \int_{a_1}^{b_1} [R(t)|y'|^\alpha - C(t)|y|^\alpha] dt - \int_{a_1}^{b_1} (M_{\alpha\beta}[c, f](t) - C(t))|y|^\alpha dt. \end{aligned}$$

Computation of the integrals over  $[a_1, t_1]$  and  $[t_2, b_1]$  is the same as in the proof of Theorem 3.1. Concerning the interval  $[t_1, t_2]$ , we have by Lemma 2.2 with  $\beta = \alpha$  and  $f \equiv 0$

$$\begin{aligned}
& \int_{t_1}^{t_2} [R(t)|y'|^\alpha - C(t)|y|^\alpha] dt \\
&= \tilde{v}|\tilde{y}|^\alpha \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[ R(t)|y'|^\alpha - \alpha\tilde{v}(t)y'\varphi(y) + (\alpha-1)R^{1-\frac{1}{\alpha}}(t)|\tilde{v}(t)|^{\frac{\alpha}{\alpha-1}}|y|^\alpha \right] dt \\
&= \tilde{v}|\tilde{y}|^\alpha \Big|_{t_1}^{t_2} + \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} \left\{ R(t)|\tilde{y}' + \delta(\Delta\tilde{y})'|^\alpha - \alpha R(t)\frac{\varphi(\tilde{y}')}{\varphi(\tilde{y})}(\tilde{y}' + \delta(\tilde{y}\Delta)')\tilde{y}^{\alpha-1} \right. \\
&\quad \left. \times (1 + \delta\Delta)^{\alpha-1} + (\alpha-1)R^{1-\frac{1}{\alpha}}(t)\left|\frac{R(t)\varphi(\tilde{y}')}{\varphi(\tilde{y})}\right|^{\frac{\alpha}{\alpha-1}}\tilde{y}^\alpha(1 + \delta\Delta)^\alpha \right\} dt \\
&= \tilde{v}|\tilde{y}|^\alpha \Big|_{t_1}^{t_2} + \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} R(t)\{|\tilde{y}'|^\alpha + \alpha\delta(\Delta\tilde{y})'\varphi(\tilde{y}') + o(\delta) - \alpha(\tilde{y}' + \delta(\Delta\tilde{y})')\varphi(\tilde{y}') \\
&\quad \times (1 + (\alpha-1)\delta\Delta + o(\delta)) + (\alpha-1)|\tilde{y}'|^\alpha(1 + \alpha\delta\Delta + o(\delta))\} dt \\
&= \tilde{v}|\tilde{y}|^\alpha \Big|_{t_1}^{t_2} + \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} R(t)\{|\tilde{y}'|^\alpha + \alpha\delta(\Delta\tilde{y})'\varphi(\tilde{y}') - \alpha|\tilde{y}'|^\alpha - \alpha\delta\varphi(\tilde{y}')(\Delta\tilde{y})' \\
&\quad - \alpha(\alpha-1)\delta\Delta|\tilde{y}'|^\alpha + (\alpha-1)|\tilde{y}'|^\alpha + (\alpha-1)\alpha\delta\Delta|\tilde{y}'|^\alpha + o(\delta)\} dt \\
&= \tilde{v}|\tilde{y}|^\alpha \Big|_{t_1}^{t_2} + o(\delta).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_{a_1}^{b_1} [R(t)|y'|^\alpha - C(t)|y|^\alpha] dt &= w_f|f|^\alpha \Big|_{a_1}^{t_1} + \tilde{v}|\tilde{y}|^\alpha \Big|_{t_1}^{t_2} + w_g|g|^\alpha \Big|_{t_2}^{b_1} + o(\delta) \\
&= |\tilde{y}(t_1)|^\alpha(w_f(t_1) - \tilde{v}(t_1)) + |\tilde{y}(t_2)|^\alpha(\tilde{v}(t_2) - w_g(t_2)) + o(\delta)
\end{aligned}$$

as  $\delta \rightarrow 0+$ . Further,

$$\begin{aligned}
\int_{t_1}^{t_2} (M_{\alpha\beta}[c, f](t) - C(t))|y|^\alpha dt &= \int_{t_1}^{\bar{t}-\varrho} (M_{\alpha\beta}[c, f](t) - C(t))|\tilde{y}|^\alpha \\
&\quad + \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} (M_{\alpha\beta}[c, f](t) - C(t))|\tilde{y}|^\alpha(1 + \delta\Delta)^\alpha dt \\
&\quad + \int_{\bar{t}+\varrho}^{t_2} (M_{\alpha\beta}[c, f](t) - C(t))|\tilde{y}|^\alpha dt \\
&= \int_{t_1}^{t_2} (M_{\alpha\beta}[c, f](t) - C(t))|\tilde{y}|^\alpha dt \\
&\quad + \delta\alpha \int_{\bar{t}-\varrho}^{\bar{t}+\varrho} (M_{\alpha\beta}[c, f](t) - C(t))|\tilde{y}|^\alpha\Delta(t) dt + o(\delta) \\
&\geq \int_{t_1}^{t_2} (M_{\alpha\beta}[c, f](t) - C(t))|\tilde{y}|^\alpha dt + \delta K + o(\delta),
\end{aligned}$$



where  $K = \alpha \int_{\xi-\rho}^{\xi+\rho} (M_{\alpha\beta} - C) |\tilde{y}|^\alpha dt > 0$ .

Summarizing the previous computations, we have

$$\begin{aligned} \mathcal{F}(y; a_1, b_1) &\leq |\tilde{y}(t_1)|^\alpha (v_f(t_1) - \tilde{v}(t_1)) + |\tilde{y}(t_2)|^\alpha (\tilde{v}(t_2) - v_g(t_2)) \\ &\quad - \int_{\xi_1}^{\xi_2} (M_{\alpha\beta}[c, f](t) - C(t)) |\tilde{y}|^\alpha dt - (K\delta + o(\delta)). \end{aligned}$$

Now, let  $\delta > 0$  (sufficiently small) be such that  $K\delta + o(\delta) =: \varepsilon > 0$ . According to (3.4) the points  $t_1, t_2$  can be chosen in such a way that

$$\int_{s_1}^{s_2} (M_{\alpha\beta}[c, f](t) - C(t)) |\tilde{y}|^\alpha dt > -\frac{\varepsilon}{4}$$

whenever  $s_1 \in (a, t_1)$ ,  $s_2 \in (t_2, b)$ . Consequently, for suitably chosen  $a_1 < t_1 < t_2 < b_1$ , using the estimates from the proof of Theorem 3.1 we have then

$$\mathcal{F}(y; a_1, b_1) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} - \varepsilon < 0$$

and the statement follows from Lemma 2.2.  $\square$

**Corollary 3.1.** *Suppose that  $I = \mathbb{R} = (-\infty, \infty)$ ,*

$$(3.6) \quad \int_{-\infty}^{\infty} r^{\frac{1}{1-\alpha}}(t) dt = \infty = \int_{-\infty}^{\infty} r^{\frac{1}{1-\alpha}}(t) dt,$$

and

$$\liminf_{s_1 \downarrow -\infty, s_2 \uparrow \infty} \int_{s_1}^{s_2} M_{\alpha\beta}[c, f](t) dt \geq 0, \quad M_{\alpha\beta}[c, f](t) \not\equiv 0 \quad \text{for } t \in \mathbb{R}.$$

Then every solution of (1.1) satisfying  $x(t)f(t) \leq 0$  for  $t \in \mathbb{R}$  has at least one zero in  $\mathbb{R}$ .

**Proof.** Let  $R(t) \equiv r(t)$  and  $C(t) \equiv 0$  in (3.1). Divergence of the integrals in (3.6) implies that  $\tilde{y} \equiv 1$  is the principal solution of  $(R(t)\varphi(y'))' = 0$  at  $\pm\infty$ , see e. g. [2]. The statement now follows from Theorem 3.2.  $\square$

**Corollary 3.2.** *Suppose that  $I = (0, \infty)$ ,*

$$\liminf_{s_1 \downarrow 0, s_2 \uparrow \infty} \int_{s_1}^{s_2} \left[ M_{\alpha\beta}[c, f](t) - \left( \frac{\alpha-1}{\alpha} \right)^\alpha t^{-\alpha} \right] t^{\alpha-1} dt \geq 0$$

and

$$M_{\alpha\beta}[c, f](t) - \left( \frac{\alpha-1}{\alpha} \right)^\alpha t^{-\alpha} \not\equiv 0 \quad \text{in } (0, \infty).$$

Then every solution of (1.1) with  $r(t) \equiv 1$  satisfying  $x(t)f(t) \leq 0$  for  $t \in (0, \infty)$  has a zero in this interval.

**Proof.** Let  $R(t) \equiv 1$  and  $C(t) = \left( \frac{\alpha-1}{\alpha} \right)^\alpha t^{-\alpha}$ . Then (3.1) is the generalized Euler equation

$$(\varphi(y'))' + \frac{\gamma_\alpha}{t^\alpha} \varphi(y) = 0, \quad \gamma_\alpha = \left( \frac{\alpha-1}{\alpha} \right)^\alpha.$$

This equation is disconjugate on  $I = (0, \infty)$  and  $\tilde{y} = t^{\frac{\alpha-1}{\alpha}}$  is its principal solution at  $t = 0$  and  $t = \infty$ , see [3]. The statement now again follows from Theorem 3.2.  $\square$

**Corollary 3.3.** *Suppose that  $r(t) \geq (1 + |t|^\alpha)^{\alpha-1}$  on  $\mathbb{R}$ ,*

$$\liminf_{t_1 \downarrow -\infty, t_2 \uparrow \infty} \int_{t_1}^{t_2} \left\{ M_{\alpha\beta}[c, f](t) - \frac{\alpha-1}{1+|t|^\alpha} \right\} \frac{1}{(1+|t|^\alpha)^{\alpha-1}} dt \geq 0$$

and

$$M_{\alpha\beta}[c, f](t) - \frac{\alpha-1}{1+|t|^\alpha} \neq 0 \quad \text{in } \mathbb{R}.$$

Then every solution of (1.1) satisfying  $x(t)f(t) \leq 0$  in  $\mathbb{R}$  has a zero point in  $\mathbb{R}$ .

**Proof.** Consider the equation

$$(3.7) \quad ((1 + |t|^\alpha)^{\alpha-1} \varphi(y'))' + \frac{\alpha-1}{1+|t|^\alpha} \varphi(y) = 0.$$

The transformation of independent variable  $x(t) = y(\arctan_\alpha t)$ , where

$$\arctan_\alpha t = \int_0^t \frac{ds}{1+|s|^\alpha},$$

transforms (3.7) into the equation

$$(\varphi(x'))' + (\alpha-1)\varphi(x) = 0.$$

The last equation has been extensively studied by Elbert [6]. Denote by  $\sin_\alpha t$  its unique solution given by the initial condition  $y(0) = 0, y'(0) = 1$ . Further denote  $\cos_\alpha t := (\sin_\alpha t)'$ ,  $\tan_\alpha t := \frac{\sin_\alpha t}{\cos_\alpha t}$ . The first positive zero of the function  $\sin_\alpha$  is  $\pi_\alpha := 2 \frac{\pi/\alpha}{\sin(\pi/\alpha)}$ ,  $\cos_\alpha t = 0$  for  $t = \pm \frac{\pi_\alpha}{2}$  and the function  $\tan_\alpha$  (which is the inverse function of  $\arctan_\alpha$ ) maps the interval  $(-\frac{\pi_\alpha}{2}, \frac{\pi_\alpha}{2})$  onto  $\mathbb{R}$ . This implies that  $\tilde{y}(t) = \cos_\alpha(\arctan_\alpha t) = (1 + |t|^\alpha)^{-1}$  is the principal solution of (3.7) at  $\pm\infty$ . The statement now follows from Theorem 3.2.  $\square$

#### REFERENCES

- [1] Došlý, O., *Methods of oscillation theory of half-linear second order differential equations*, Czech. Math. J. **125** (2000), 657–671.
- [2] Došlý, O., *A remark on conjugacy of half-linear second order differential equations*, Math. Slovaca, **50** (2000), 67–79.
- [3] Došlý, O., *Half-linear oscillation theory*, Stud. Univ. Žilina, Ser. Math. Phys. **13** (2001), 65–73.
- [4] Došlý, O., Elbert, Á., *Integral characterization of principal solution of half-linear second order differential equations*, Studia Sci. Math. Hungar. **36** (2000), 455–469.
- [5] Došlý, O., Řezíčková, J., *Regular half-linear second order differential equations*, Arch. Math. (Brno) **39** (2003), 233–245.

- [6] Elbert, Á., *A half-linear second order differential equation*, Colloq. Math. Soc. János Bolyai **30** (1979), 153–180.
- [7] Elbert, Á. and Kusano, T., *Principal solutions of nonoscillatory half-linear differential equations*, Advances in Math. Sci. Appl. **18** (1998), 745–759.
- [8] Jaroš, J., Kusano, T., *A Picone type identity for half-linear differential equations*, Acta Math. Univ. Comenianae **68** (1999), 127–151.
- [9] Jaroš, J., Kusano, T., Yoshida, N., *Forced superlinear oscillations via Picone's identity*, Acta Math. Univ. Comenianae **LXIX** (2000), 107–113.
- [10] Jaroš, J., Kusano, T., Yoshida, N., *Generalized Picone's formula and forced oscillation in quasilinear differential equations of the second order*, Arch. Math. (Brno) **38** (2002), 53–59.
- [11] Komkov, V., *A generalization of Leighton's variational theorem*, Appl. Anal. **2** (1973), 377–383.
- [12] Leighton, W., *Comparison theorems for linear differential equations of second order*, Proc. Amer. Math. Soc. **13** (1962), 603–610.
- [13] Mirzov, J. D., *On some analogs of Sturm's and Kneser's theorems for nonlinear systems*, J. Math. Anal. Appl. **53** (1976), 418–425.
- [14] Mirzov, J. D., *Principal and nonprincipal solutions of a nonoscillatory system*, Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy **31** (1988), 100–117.
- [15] Müller-Pfeiffer, E., *Comparison theorems for Sturm-Liouville equations*, Arch. Math. **22** (1986), 65–73.
- [16] Müller-Pfeiffer, E., *Sturm comparison theorems for non-selfadjoint differential equations on non-compact intervals*, Math. Nachr. **159** (1992), 291–298.
- [17] Swanson, C. A., *Comparison and Oscillation Theory of Linear Differential Equation*, Acad. Press, New York, 1968.

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