ON THE POWERFULL PART OF $n^2 + 1$

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Abstract. We show that $n^2 + 1$ is powerfull for $O(x^{2/5+\epsilon})$ integers $n \leq x$ at most, thus answering a question of P. Ribenboim.

The distribution of powerfull integers, i.e. integers such that every prime factor occurs at least twice, is quiet obscure. In [4], P. Ribenboim posed the following problem: Show that for almost all $m$, $m^4 - 1$ is not powerfull. In his review, D. R. Heath-Brown [2] pointed out that this and the more general statement, that for every polynomial $f$, not powerfull as a polynomial, $f(m)$ is not powerfull for almost all $m$, can be obtained using a simple sieve. In fact, if $n$ is powerfull and $p$ prime, $n \mod p^2$ is restricted to $p^2 - p + 1$ residue classes. By a standard application of the arithmetic large sieve one gets that the number $N$ of $m$ such that $f(m)$ is powerfull is $N \ll \frac{\log x}{\log 2}$. In this note we will use a diferent approach to this problem to prove the following theorem. For an integer $n$ we write $P(n)$ for the powerfull part of $n$, i.e. the product of all $p^k$ with $k \geq 2$, where $p^k|n$, but $p^{k+1} \nmid n$, $\omega(n)$ for the number of distinct prime divisors of $n$, and $d^+(n)$ for the number of squarefree divisors of $n$.

Theorem 1. Let $A$ and $x$ be real numbers. Then there are at most $cx^{2/5}A^{4/5}\log C$ integers $n \leq x$, such that $P(n^2 + 1) > n^2 A^{-1}$ where $C = 18730$.

Choosing $A = 2$ resp. $A = x^{2/3-\epsilon}$ we obtain the following statements.

Corollary 2. For almost all $n$ we have $P(n^2 + 1) < n^{4/3+\epsilon}$.

Corollary 3. There are $\ll x^{2/5}\log C$ integers $m \leq x$ such that $m^2 + 1$ is powerfull or twice a powerfull integer.

Note that $\lim \sup \frac{P(n^2 + 1)}{n} = \infty$, thus the exponent 4/3 is not too bad. It seems that the gap stems from the fact that the equation $x^2 + 1 = D \cdot z^3$ considered in Lemma 5 may very well have no integral solutions at all for many values of $D$.

To prove our theorem, we need some Lemmata. First we have to count solutions of diophantine equations.

Received June 14, 2001.
Lemma 4. For any $D$, the equation $x^2 - Dy^2 = -1$ has $\leq 4$ solutions with $x, y$ integers and $X \leq x \leq 2X$, $X$ arbitrary real.

Proof. We may assume that $D$ is not a perfect square, since for $D = 1$ there are only the solutions $x = 0, y = \pm 1$, and for $D > 1$, $x + \sqrt{D} y$ would be a rational integral divisor of $-1$. The solutions of the equation correspond to units in $\mathbb{Q}(\sqrt{D})$. If $(x_1, y_1)$ is a minimal solution, all solutions are obtained by the recursion $x_{n+1} = x_n x_1 + D y_n y_1, y_{n+1} = x_1 y_n + y_1 x_n$. We may assume that $x_1, y_1$ are positive, thus $x_{n+1} > x_n x_1$. Further we trivially have $x_1 \geq 2$, thus in every interval of the form $[X, 2X]$, there is at most one solution with both variables positive. Taking signs into account, the total number of solutions with $x_n \leq X$ is therefore $\leq 4$.

Lemma 5. For any $D$, the equation $x^2 + 1 = Dz^3$ has $c \cdot d^+(D)^{c_0}$ solutions at most, where $c_0 = \frac{2 \log 17 + 4 \log 3}{\log 2} \leq 14.6$.

Proof. This is a special case of theorem 1 in [1], proven by J. H. Evertse and J. H. Silverman. In their notation we have $n = 3, d = 2, m = 1, L = \mathbb{Q}(i), M = 2$ and $K_3(L) = 0$. We consider the equation $\frac{x^2 + 1}{x+1} = y^3$, which is integral at all but $\omega(D)$ places, thus $s = \omega(D) + 1$. Applying their theorem we obtain for the number $N$ of solutions the bound $N \leq 17^{14 + 2\omega(D)} 3^{\omega(D)} \ll (17^2 3^4)^{\omega(D)}$. Since $d^+(D) = 2^{\omega(D)}$, we get $N \ll d^+(n)^{c_0}$, where $c_0 = \frac{2 \log 17 + 4 \log 3}{\log 2} \leq 14.6$.

Note that the actual value of $c_0$ is of lesser importance, since only the exponent of the logarithm is concerned. In fact, we have $C = 2^{c_0}$. Note further that we can prove theorem 1 with a bound of $x^{2/3} A^{2/3}$ without appealing to the very deep theorem of Evertse and Silverman.

Lemma 6. We have for any positive real number $c$ the bound $\sum_{n \leq x} d(n)^c \ll c x \log^{2c-1} x$.

This was proven by C. Mardjanichvili [3].

Now we can prove theorem 1. Every integer $k \geq 2$ can be written as a nonnegative integral linear combination of 2 and 3, thus every powerfull number $n$ can be written as $n = y^2 z^3$ with $y, z$ integral. Thus every integer $n$ can be written as $n = ay^2 z^3$ with $y, z$ integral and $a = \frac{n}{(n)}$. Thus to prove theorem 1, it suffices to show that the equation

$$n^2 + 1 = ay^2 z^3$$

has $\ll x^{2/5} A^{2/5} \log^C x$ integral solutions with $n \leq x$ and $a \leq A$. Now we count the solutions within the range $Y \leq y < 2Y$, $B \leq a < 2B$ and $Z \leq z < 2Z$.

Fix $a$ and $z$, and set $D = az^3$. Now $n$ is restricted to an interval of the form $[x, 8x]$, thus by lemma 4 there are $\ll 1$ solutions of the equation $n^2 - Dy^2 = -1$ with these restrictions. Thus the total number of solutions is $\ll BZ$.

Now we fix $a$ and $y$, and set $D = ay^2$. Then by lemma 5 the equation $n^2 + 1 = Dz^3$ has $\ll d^+(D)^{c_0}$ solutions, where $c_0$ is defined as above. We set $c_1 = 2^{c_0}$.
23709. Thus the total number of solutions in this range is therefore bounded by
\[ \ll \sum_{B \leq a \leq 2B} \sum_{Y \leq y \leq 2Y} d^+(ay^2)^c \leq \sum_{B \leq a \leq 2B} d(a)^c \sum_{Y \leq y \leq 2Y} d(y)^c. \]
Using Lemma 6 and replacing the occurring log-factors by \( \log x \), these sums are \( \ll BY \log^{2c_1 - 2} x \). With these two estimates we obtain for the total number \( N \) of solutions the estimate
\[ N \ll \log^3 x \max_{Y, Z \geq 1} \min_{\mathfrak{B} \leq A} \left( BY \log^{2c_1 - 2} x, BZ \right) \]
\[ \ll \log^3 x \max_{Y, Z \geq 1} \min_{\mathfrak{B} \leq A} \left( AY \log^{2c_1 - 2} x, A \left( \frac{x^2}{A^2 Y^2} \right)^{1/3} \right) \]
\[ \ll A^{4/5} x^{2/5} \log^{\frac{4}{5}(c_1 - 1) + 3} x \]
which gives the bound of theorem 1, since \( \frac{4}{5}(c_1 - 1) + 3 = 18729.4 \).

References


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