THE CANONICAL TENSOR FIELDS
OF TYPE (1, 1) ON $(J^r(\odot^2 T^*))^*$

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Abstract. We prove that every natural affinor on $(J^r(\odot^2 T^*))^*(M)$ is proportional to the identity affinor if $\dim M \geq 3$.

0. Introduction

For every $n$-dimensional manifold $M$ we have the vector bundle

$$J^r(\odot^2 T^*)(M) = \{ j^r_x \tau | \tau \text{ is a symmetric tensor field type } (0, 2) \text{ on } M, x \in M \}.$$

Every local diffeomorphism $\varphi : M \rightarrow N$ between $n$-manifolds gives a vector bundle homomorphism $J^r(\odot^2 T^*)(\varphi) : J^r(\odot^2 T^*)(M) \rightarrow J^r(\odot^2 T^*)(N)$, $j^r_x \tau \rightarrow j^r_{\varphi(x)}(\varphi_* \tau)$. Functor $J^r(\odot^2 T^*) : \mathcal{M}f_n \rightarrow \mathcal{VB}$ is a vector natural bundle over $n$-manifolds in the sense of [5]. Let $(J^r(\odot^2 T^*))^* : \mathcal{M}f_n \rightarrow \mathcal{VB}$ be the dual vector bundle, $(J^r(\odot^2 T^*))^*(M) = (J^r(\odot^2 T^*)(M))^*$, $(J^r(\odot^2 T^*))^*(\varphi) = (J^r(\odot^2 T^*)(\varphi^{-1}))^*$ for $M$ and $\varphi$ as above.

An affinor on a manifold $M$ is a tensor field of type $(1, 1)$ on $M$. A natural affinor $Q$ on $(J^r(\odot^2 T^*))^*$ is a system of affinors

$$Q : T(J^r(\odot^2 T^*))^*(M) \rightarrow T(J^r(\odot^2 T^*))^*(M)$$

on $(J^r(\odot^2 T^*))^*(M)$ for every $n$-manifold $M$ satisfying the naturality condition $T(J^r(\odot^2 T^*))^*(\varphi) \circ Q = Q \circ T(J^r(\odot^2 T^*))^*(\varphi)$ for every local diffeomorphism $\varphi : M \rightarrow N$ between $n$-manifolds.

In this paper we prove, that every natural affinor $Q$ on $(J^r(\odot^2 T^*))^*$ over $n$-manifolds is proportional to the identity affinor if $n \geq 3$.

The proof of the classification theorem is based on the method from paper [7], where there are determined the natural affinors on $(J^r(\Lambda^2 T^*))^*$. However the proof is different, because the tensor field $dx^1 \odot dx^1$ on $\mathbb{R}^n$ is non-zero, in contrast to $dx^1 \wedge dx^1$.

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Natural affinors on some natural bundle $F$ can be used to study torsions $[Q,\Gamma]$ of a connection $\Gamma$ of $F$. That is why, the natural affinors have been study in many papers, [11], e.t.c.

The usual coordinates on $\mathbb{R}^n$ are denoted by $x^i$. The canonical vector fields on $\mathbb{R}^n$ are denoted by $\partial_i = \frac{\partial}{\partial x^i}$.

All manifolds are assumed to be finite dimensional and smooth, i.e. of class $C^\infty$. Mappings between manifolds are assumed to be smooth.

1. **The Linear Natural Transformations** $T(J'(\mathbb{R}^2T^*))^* \to (J'(\mathbb{R}^2T^*))^*$

A natural transformation $T(J'(\mathbb{R}^2T^*))^* \to (J'(\mathbb{R}^2T^*))^*$ over $n$-manifolds is a system of fibred maps

$$A : T(J'(\mathbb{R}^2T^*))^*(M) \to (J'(\mathbb{R}^2T^*))^*(M)$$

over $\text{id}_M$ for every $n$-manifold $M$ such that

$$(J'(\mathbb{R}^2T^*))^*(f) \circ A = A \circ T(J'(\mathbb{R}^2T^*))^*(f)$$

for every local diffeomorphism $f : M \to N$ between $n$-manifolds.

A natural transformation $A : T(J'(\mathbb{R}^2T^*))^* \to (J'(\mathbb{R}^2T^*))^*$ is called linear if $A$ gives a linear map $T_y((J'(\mathbb{R}^2T^*))^*(M) \to ((J'(\mathbb{R}^2T^*))^*(M))_x$ for any $y \in ((J'(\mathbb{R}^2T^*))^*(M))$, $x \in M$.

**Theorem 1.** If $n \geq 3$ and $r$ are natural numbers, then every linear natural transformation $A : T(J'(\mathbb{R}^2T^*))^* \to (J'(\mathbb{R}^2T^*))^*$ over $n$-manifolds is equal to 0.

The proof of Theorem 1 will occupy Sections 2 – 6.

2. **The Reducibility Propositions**

Every element from the fibre $((J'(\mathbb{R}^2T^*))^*(\mathbb{R}^n))_0$ is a linear combination of all elements $(j^0_\alpha(x^\alpha \, dx^i \odot dx^j))^*$, where $\alpha \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| \leq r$, $i \leq j$, $i, j = 1, \ldots, n$. The elements $(j^0_\alpha(x^\alpha \, dx^i \odot dx^j))^*$ are dual basis to the basis $j^0_\alpha(x^\alpha \, dx^i \odot dx^j)$ of $(J'(\mathbb{R}^2T^*)(\mathbb{R}^n))_0$.

Consider a linear natural transformation $A : T(J'(\mathbb{R}^2T^*))^* \to (J'(\mathbb{R}^2T^*))^*$.

**Lemma 1.** Suppose $A$ satisfies

$$(A(u), j^0_\alpha(x^\alpha \, dx^i \odot dx^j)) = 0$$

for every $u \in T(J'(\mathbb{R}^2T^*))^*(\mathbb{R}^n))$, $\alpha \in (\mathbb{N} \cup \{0\})^n$, $|\alpha| \leq r$, $i \leq j$, $i, j = 1, \ldots, n$. Then $A = 0$.

**Proof.** If assumptions of Lemma 1 meet, then $A(u) = 0$ for every $u \in (T(J'(\mathbb{R}^2T^*))^*(\mathbb{R}^n))_0$. Let $w \in T(J'(\mathbb{R}^2T^*))^*(M)_x$, $x \in M$. There exists a chart $\varphi : M \supset U \to \mathbb{R}^n$ such that $\varphi(x) = 0$ and $U$ is open subset including $x$. Since $A$ is invariant with respect to $\varphi$, we have $A(w) = T(J'(\mathbb{R}^2T^*))^*(\varphi^{-1})(A(u))$, where $u = T(J'(\mathbb{R}^2T^*))^*(\varphi)(w) \in T(J'(\mathbb{R}^2T^*))^*(\mathbb{R}^n))_0$. Then $A(w) = 0$, because $A(u) = 0$. That is why $A = 0$. The lemma is proved. \qed
Lemma 2. Suppose that
\[ \langle A(u), j_0^\alpha(x^\alpha dx^1 \circ dx^2) \rangle = \langle A(u), j_0^\alpha(x^\alpha dx^1 \circ dx^2) \rangle = 0 \]
for every \( u \in (T(J^r(\odot^2 T^*))^*(\mathbb{R}^n))_0 \), \( \alpha \in (\mathbb{N} \cup \{0\})^n \), \( |\alpha| \leq r \), \( i \leq j, i, j = 1, \ldots, n \). Then \( A = 0 \).

Proof. Let \( u \in (T(J^r(\odot^2 T^*))^*(\mathbb{R}^n))_0 \), \( \alpha \in (\mathbb{N} \cup \{0\})^n \), \( |\alpha| \leq r \), \( i \leq j, i, j = 1, \ldots, n \). It is enough to prove that \( \langle A(u), j_0^\alpha(x^\alpha dx^1 \circ dx^2) \rangle = 0 \).

Consider two cases

a) \( i = j \). Let \( \varphi : \mathbb{R}^n \to \mathbb{R}^n \) be a diffeomorphism transforming \( x^i \) into \( x^j \) and \( x^\alpha \) into \( x^\alpha \) for some \( \tilde{\alpha} \in (\mathbb{N} \cup \{0\})^n \), \( |\tilde{\alpha}| \leq r \). From the invariance of \( A \) with respect to \( \varphi \) and the assumption of Lemma 2, we have \( \langle A(u), j_0^\alpha(x^\alpha dx^1 \circ dx^2) \rangle = \langle A(u), j_0^\alpha(x^\alpha dx^1 \circ dx^2) \rangle = 0 \), where \( \tilde{u} = T(J^r(\odot^2 T^*))^*(\varphi)(u) \).

b) \( i \neq j \). Let \( \varphi : \mathbb{R}^n \to \mathbb{R}^n \) be a diffeomorphism transforming \( x^i \) in \( x^j \) in \( x^2 \) and \( x^\alpha \) in \( x^\alpha \) for some \( \tilde{\alpha} \in (\mathbb{N} \cup \{0\})^n \), \( |\tilde{\alpha}| \leq r \). From the invariance of \( A \) with respect to \( \varphi \) and the assumption of Lemma 2, we have \( \langle A(u), j_0^\alpha(x^\alpha dx^1 \circ dx^2) \rangle = \langle A(u), j_0^\alpha(x^\alpha dx^1 \circ dx^2) \rangle = 0 \), where \( \tilde{u} = T(J^r(\odot^2 T^*))^*(\varphi)(u) \).

Lemma 3. Suppose \( A \) satisfies
\[ \langle A(u), j_0^\alpha(dx^1 \circ dx^1) \rangle = \langle A(u), j_0^\alpha(dx^1 \circ dx^1) \rangle = \langle A(u), j_0^\alpha(dx^1 \circ dx^2) \rangle = 0 \]
for every \( u \in (T(J^r(\odot^2 T^*))^*(\mathbb{R}^n))_0 \), \( \alpha \in (\mathbb{N} \cup \{0\})^n \), \( |\alpha| \leq r \), \( i \leq j, i, j = 1, \ldots, n \). Then \( A = 0 \).

Proof. Let \( \alpha \in (\mathbb{N} \cup \{0\})^n \), \( |\alpha| \leq r \), \( u \in (T(J^r(\odot^2 T^*))^*(\mathbb{R}^n))_0 \), \( \alpha \neq e_3 = (0, 0, 1, 0, \ldots, 0) \in (\mathbb{N} \cup \{0\})^n \).

On the strength of Lemma 2 it is enough to prove that
\[ \langle A(u), j_0^\alpha(x^\alpha dx^1 \circ dx^1) \rangle = \langle A(u), j_0^\alpha(x^\alpha dx^1 \circ dx^2) \rangle = 0. \]

We can set that \( \alpha \neq 0 \). Let \( \varphi : \mathbb{R}^n \to \mathbb{R}^n \) be a diffeomorphism transforming \( x^1 \) in \( x^1 \), \( x^2 \) in \( x^2 \) and \( x^3 + x^\alpha \) in \( x^3 \). From the invariance of \( A \) with respect to \( \varphi \) and the assumption of Lemma 3, we have
\[ \langle A(u), j_0^\alpha(x^\alpha dx^1 \circ dx^2) \rangle = \langle A(u), j_0^\alpha(x^\alpha dx^1 \circ dx^1) \rangle + \langle A(u), j_0^\alpha(x^\alpha dx^1 \circ dx^1) \rangle = \langle A(u), j_0^\alpha((x^3 + x^\alpha) dx^1 \circ dx^1) \rangle = \langle A(\tilde{u}), j_0^\alpha(x^3 dx^1 \circ dx^1) \rangle = 0 \]
where \( \tilde{u} = T(J^r(\odot^2 T^*))^*(\varphi)(u) \).
Similarly \( \langle A(u), j_0^\alpha(x^\alpha dx^1 \circ dx^2) \rangle = 0. \)

Lemma 4. Suppose that
\[ \langle A(u), dx^1 \circ dx^2 \rangle = \langle A(u), j_0^\alpha(x^\alpha dx^1 \circ dx^2) \rangle = 0 \]
for every \( u \in (T(J^r(\odot^2 T^*))^*(\mathbb{R}^n))_0 \). Then \( A = 0 \).
**Proof.** By Lemma 3 it is sufficient to show that
\[ \langle A(u), j_0^2(dx^1 \otimes dx^1) \rangle = \langle A(u), j_0^2(x^3 dx^1 \otimes dx^1) \rangle = 0 \]
for every \( u \in (T(J^r(\odot^2 T^*))^*(R^n))_0 \).

Let \( u \in (T(J^r(\odot^2 T^*))^*(R^n))_0 \). Consider a diffeomorphism \( \varphi : R^n \to R^n \)
transforming \( x^1 \) in \( x^1 \), \( x^2 \) in \( x^1 + x^2 \) and \( x^3 \) in \( x^3 \). Then from the invariance of \( A \)
with respect to \( \varphi \) and the assumption of lemma, we have
\[ 0 = \langle A(\tilde{u}), j_0^2(dx^1 \otimes dx^2) \rangle \\
= \langle A(u), j_0^2(dx^1 \otimes (dx^1 + dx^2)) \rangle \\
= \langle A(u), j_0^2(dx^1 \otimes dx^1) \rangle + \langle A(u), j_0^2(dx^1 \otimes dx^2) \rangle, \]
where \( \tilde{u} = T(J^r(\odot^2 T^*))^*(\varphi^{-1})(u) \). So \( \langle A(u), j_0^2(dx^1 \otimes dx^1) \rangle = 0 \).
Similarly \( \langle A(u), j_0^2(x^3 dx^1 \otimes dx^1) \rangle = 0 \). \( \square \)

Using Lemma 4 we see that Theorem 1 will be proved after proving the following
two propositions.

**Proposition 1.** We have
\[ \langle A(u), j_0^2(dx^1 \otimes dx^2) \rangle = 0 \]
for every \( u \in (T(J^r(\odot^2 T^*))^*(R^n))_0 \).

**Proposition 2.** We have
\[ \langle A(u), j_0^2(x^3 dx^1 \otimes dx^2) \rangle = 0 \]
for every \( u \in (T(J^r(\odot^2 T^*))^*(R^n))_0 \).

3. Some notations

We have the obvious trivialization
\[ (T(J^r(\odot^2 T^*))^*(R^n))_0 \cong R^n \times ((J^r(\odot^2 T^*))^*(R^n))_0 \times ((J^r(\odot^2 T^*))^*(R^n))_0 \]
given by \( (u_1, u_2, u_3) \rightarrow (\tilde{u}_1)^C(u_2) + \frac{d}{dt}|_{t=0}(u_2 + tu_3) \), where \( \tilde{u}_1 \) is the constant
vector field on \( R^n \) such that \( \tilde{u}_1^0 = u_1 \in R^n \cong T_0R^n \) and \( (\tilde{u}_1)^C \) is the complete
lift of \( \tilde{u}_1 \) to \( (J^r(\odot^2 T^*))^* \).

Each \( u_\tau \in ((J^r(\odot^2 T^*))^*(R^n))_0 \), \( \tau = 2, 3 \) can be expressed in the form
\[ u_\tau = \sum u_{\tau, \alpha, i, j} (j_0^\alpha(x^i dx^j \otimes dx^j))^* \],
where the sum is over all \( \alpha \in (N \cup \{0\})^n \), \( |\alpha| \leq r, i \leq j, i, j = 1, \ldots, n \).
It defines \( u_{\tau, \alpha, i, j} \) for each \( u_\tau \) as above.

4. Proof of Proposition 1

We start with the following lemma.

**Lemma 5.** There exists the number \( \lambda \in R \) such that
\[ \langle A(u), j_0^2(dx^1 \otimes dx^2) \rangle = \lambda u_{3,(0),1,2} \]
for every \( u = (u_1, u_2, u_3) \in (T(J^r(\odot^2 T^*))^*(R^n))_0 \).
Proof. Let $\Phi : \mathbb{R}^n \times ((J^{r}(\odot 2T^{*}))^*(\mathbb{R}^n))_0 \times ((J^{r}(\odot 2T^{*}))^*(\mathbb{R}^n))_0 \rightarrow \mathbb{R}$ be such that

$$\Phi(u_1, u_2, u_3) = \langle A(u), J_0^r(dx^1 \odot dx^2) \rangle,$$

where $u = (u_1, u_2, u_3)$, $u_1 = (u_1^i) \in \mathbb{R}^n$, $i = 1, \ldots, n$, $u_2 \in ((J^{r}(\odot 2T^{*}))^*(\mathbb{R}^n))_0$, $u_3 \in ((J^{r}(\odot 2T^{*}))^*(\mathbb{R}^n))_0$.

The invariance of $A$ with respect to the homotheties $a_t = (t^1x^1, \ldots, t^n x^n)$ for $t = (t^1, \ldots, t^n) \in \mathbb{R}^n_+$ gives the homogeneous condition

$$\Phi(T(J^{r}(\odot 2T^{*}))^*(a_1)(u)) = t^{1}t^{2}\Phi(u).$$

Then from the homogeneous function theorem, [5], it follows that $\Phi(u)$ is the linear combination of monomials in $u_1^i$, $u_{r,a,i,j}$ of weight $t^1t^2$. Moreover $\Phi(u_1, u_2, u_3)$ is linear in $u_1$, $u_3$ for $u_2$, since $A$ is linear. It implies the lemma.

In particular from Lemma 5 it follows that

$$(*) \quad \langle A(\partial^C_1|w), J_0^r(dx^1 \odot dx^2) \rangle = \langle A(e_1, w, 0), J_0^r(dx^1 \odot dx^2) \rangle = 0$$

for every $w \in ((J^{r}(\odot 2T^{*}))^*(\mathbb{R}^n))_0$, where $\partial_1 = \frac{d}{dx^1}$ and $(J^C)$ is the complete lift to $(J^r(\odot 2T^{*}))^*$.

We are now in position to prove Proposition 1. Let $\lambda$ be from Lemma 5. It is enough to prove that $\lambda$ is equal to 0.

We see that $\lambda = \langle A(0, 0, (J_0^r(dx^1 \odot dx^2))^*), J_0^r(dx^1 \odot dx^2) \rangle$.

We have

$$(** ) \quad 0 = \langle A((x^1)^{r+1}\partial_1)^C|w), J_0^r(dx^1 \odot dx^2) \rangle$$

$$= (r+1)\langle A(0, w, (J_0^r(dx^1 \odot dx^2))^* + \ldots), J_0^r(dx^1 \odot dx^2) \rangle$$

$$= (r+1)\langle A(0, 0, (J_0^r(dx^1 \odot dx^2))^*), J_0^r(dx^1 \odot dx^2) \rangle,$$

where $w = (J_0^r((x^1)^r dx^1 \odot dx^2))^*$ and the dots is a linear combination of the $(J_0^r(x^a dx^i \odot dx^j))^*$ with $(J_0^r(x^a dx^i \odot dx^j))^* \neq (J_0^r(dx^1 \odot dx^2))^*$.

It remains to explain (**).

At first we show the second equality in (**). Let $\varphi_t$ be the flow of $(x^1)^{r+1}\partial_1$. We have the following sequences of equalities

$$\langle ((x^1)^{r+1}\partial_1)^C|w), J_0^r(dx^1 \odot dx^2) \rangle = \frac{d}{dt}|_{t=0} \langle (J^r(\odot 2T^{*}))^*|_{t=0}(\varphi_t)(w), J_0^r(dx^1 \odot dx^2) \rangle$$

$$= \frac{d}{dt}|_{t=0} \langle (J^r(\odot 2T^{*}))^*|_{t=0}(\varphi_t)(w), J_0^r(dx^1 \odot dx^2) \rangle$$

$$= \frac{d}{dt}|_{t=0} \langle w, J_0^r((\varphi_{-t})^* dx^1 \odot dx^2) \rangle$$

$$= \langle w, J_0^r(d\frac{d}{dt}|_{t=0} ((\varphi_{-t})^* dx^1 \odot dx^2)) \rangle$$

$$= \langle w, J_0^r((L_{(x^1)^{r+1}\partial_1}(dx^1 \odot dx^2)) \rangle$$

$$= (r+1)\langle w, J_0^r((x^1)^r dx^1 \odot dx^2) \rangle = r+1.$$
Lemma 6. with the following lemma. where

Then, from the homogeneous function theorem, \( \text{Theorem 5,} \) it follows that \((**\))

The last equality in (**\) is clear because of Lemma 5.

We can prove the first equality in (**\) as follows. Vector fields \( \partial_1 + (x^1)^{r+1} \partial_1 \) and \( \partial_2 \) have the same \( r \)-jets at 0 \( \in \mathbb{R}^n \). Then, by [12], there exists a diffeomorphism \( \varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that \( j_0^{r+1} \varphi = \text{id} \) and \( \varphi \cdot \partial_1 = \partial_1 + (x^1)^{r+1} \partial_1 \) in a certain neighborhood of 0. Obviously, \( \varphi \) preserves \( j_0^{r+1} (dx^1 \otimes dx^2) = j_0^{r+1} (\otimes^2 T^*) (\varphi)(j_0^{r+1} (dx^1 \otimes dx^2)) \) because \( j_0^{r+1} \varphi = \text{id} \). Then, using the invariance of \( A \) with respect to \( \varphi \), from (*) it follows that \( \langle A(\partial_1 + (x^1)^{r+1} \partial_1)_{\mid_w}, j_0^r (dx^1 \otimes dx^2) \rangle = \langle A(\partial_1)_{\mid_w}, j_0^r (dx^1 \otimes dx^2) \rangle = 0 \) for every \( w \in ((J^r(\otimes^2 T^*))^n) \). Now, using the linearity of \( A \), we end the proof of the first equality of (**\).

The proof of Proposition 1 is complete. \( \Box \)

5. PROOF OF PROPOSITION 2

The proof of Proposition 2 is similar to the proof of Proposition 1. We start with the following lemma.

Lemma 6. For every \( u = (u^1, u^2, u^3) \in (T(J^r(\otimes^2 T^*))^n) \), we have

\[
\langle A(u), j_0^r (x^1 dx^1 \otimes dx^2) \rangle = au_1 u_2(0),2,3 + bu_1^2 u_2(0,1,3) + cu_1^3 u_2(0,1,2)
+ eu_3 r,2,3 + fu_3 r,1,3 + gu_3 r,1,2
\]

where \( e_i = (0,0,\ldots,1,0,\ldots,0) \in (\mathbb{N} \cup \{0\})^n, 1 \) in \( i \)-position.

Proof. We will use the similar arguments as in the proof of Lemma 5.

Let \( \Phi : \mathbb{R}^n \times ((J^r(\otimes^2 T^*))^n) \rightarrow \mathbb{R} \) such that

\[
\Phi(u_1, u_2, u_3) = \langle A(u), j_0^r (x^3 dx^1 \otimes dx^2) \rangle,
\]

\( u = (u_1, u_2, u_3), u_1 = (u_1^t) \in \mathbb{R}^n, t = 1, \ldots, n, u_2 \in ((J^r(\otimes^2 T^*))^n), u_3 \in ((J^r(\otimes^2 T^*))^n) \). The invariance of \( A \) with respect to the homotheties \( a_t = (t^1 x^1, \ldots, t^n x^n) \) for \( t = (t^1, \ldots, t^n) \in \mathbb{R}_+^n \) gives the homogeneous condition

\[
\Phi(T(J^r(\otimes^2 T^*))^n)(a_t)(u)) = t^1 t^2 t^3 \Phi(u).
\]

Then from the homogeneous function theorem, [5], it follows that \( \Phi(u) \) is the linear combination of monomials in \( u_1^t, u_r, a, i, j \) of weight \( t^1 t^2 t^3 \). Moreover \( \Phi(u_1, u_2, u_3) \) is linear in \( u_1 \) and \( u_3 \) for \( u_2 \), since \( A \) is linear. It implies the lemma. \( \Box \)

To prove Proposition 2 we have to show that \( a = b = c = e = f = g = 0 \). We need the following lemmas.

Lemma 7. For every \( u \in (T(J^r(\otimes^2 T^*))^n) \), we have

\[
\langle A(u), j_0^r (x^3 dx^1 \otimes dx^2) \rangle = -\langle A(u'), j_0^r (x^3 dx^1 \otimes dx^2) \rangle,
\]

where \( u' \) is the image of \( u \) by \( (x^2, x^3, x^1) \times \text{id}_{\mathbb{R}^{n-3}} \).
Proof. Consider \( u \in \left( T(J^*\otimes T^*)^* (\mathbb{R}^n) \right)_0 \). Let \( \tilde{u} \) be the image of \( u \) by \( \varphi = (x^1 + x^1x^3, x^2, \ldots, x^n) \). From Proposition 1 we have

\[
\langle A(\tilde{u}), j^*_0(dx^1 \otimes dx^2) \rangle = \langle A(u), j^*_0(dx^1 \otimes dx^2) \rangle = 0.
\]

Using the invariance of \( A \) with respect to \( \varphi^{-1} \) we have

\[
0 = \langle A(u), j^*_0(dx^1 \otimes dx^2) \rangle = \langle A(u), j^*_0(x^3 dx^1 \otimes dx^2) \rangle + \langle A(u), j^*_0(x^1 dx^2 \otimes dx^3) \rangle
\]

because \( \varphi^{-1} \) preserves \( A \), it transforms \( \tilde{u} \) in \( u \) and \( j^*_0(dx^1 \otimes dx^2) \) in \( j^*_0(dx^1 \otimes dx^2) + j^*_0(x^3 dx^1 \otimes dx^2) + j^*_0(x^1 dx^2 \otimes dx^3) \). So, \( \langle A(u), j^*_0(x^3 dx^1 \otimes dx^2) \rangle = -\langle A(u), j^*_0(x^1 dx^2 \otimes dx^3) \rangle \). Hence we have the lemma because \( (x^2, x^3, x^1) \times \mathbb{R}^{n-3} \) sends \( u \) in \( u' \) and \( j^*_0(x^1 dx^2 \otimes dx^3) \) in \( j^*_0(x^3 dx^1 \otimes dx^2) \).

Lemma 8. We have \( g = f = e = 0 \).

Proof. Obviously

\[
g = \langle A(0,0), (j^*_0(x^3 dx^1 \otimes dx^2))^*, j^*_0(x^3 dx^1 \otimes dx^2) \rangle
\]

by Lemma 6. Similarly

\[
f = \langle A(0,0), (j^*_0(x^2 dx^1 \otimes dx^3))^*, j^*_0(x^3 dx^1 \otimes dx^2) \rangle,
\]

\[
e = \langle A(0,0), (j^*_0(x^1 dx^2 \otimes dx^3))^*, j^*_0(x^3 dx^1 \otimes dx^2) \rangle.
\]

So, to prove Lemma 8 we have to show

\[
\langle A(0,0), (j^*_0(x^3 dx^1 \otimes dx^2))^*, j^*_0(x^3 dx^1 \otimes dx^2) \rangle
\]

\[
= \langle A(0,0), (j^*_0(x^2 dx^1 \otimes dx^3))^*, j^*_0(x^3 dx^1 \otimes dx^2) \rangle
\]

\[
= \langle A(0,0), (j^*_0(x^1 dx^2 \otimes dx^3))^*, j^*_0(x^3 dx^1 \otimes dx^2) \rangle = 0.
\]

We can see that \( (x^2, x^3, x^1) \times \text{id}_{\mathbb{R}^{n-3}} \) sends \( (j^*_0(x^3 dx^1 \otimes dx^2))^* \) in \( (j^*_0(x^2 dx^1 \otimes dx^3))^* \) and \( (j^*_0(x^1 dx^2 \otimes dx^3))^* \) in \( (j^*_0(x^2 dx^1 \otimes dx^3))^* \). Then using Lemma 7 it is enough to verify that \( \langle A(0,0), (j^*_0(x^3 dx^1 \otimes dx^2))^*, j^*_0(x^3 dx^1 \otimes dx^2) \rangle = 0 \). So, it is enough to prove the sequence of equalities:

\[
0 = \langle A((x^1)\varphi(-1))_{\tilde{w}}, j^*_0(x^3 dx^1 \otimes dx^2) \rangle
\]

\[
= r(\langle A(0,0, w), (j^*_0(x^3 dx^1 \otimes dx^2))^* \rangle, j^*_0(x^3 dx^1 \otimes dx^2))
\]

\[
= r(\langle A(0,0, (j^*_0(x^3 dx^1 \otimes dx^2))^*), j^*_0(x^3 dx^1 \otimes dx^2) \rangle,
\]

where \( w = (j^*_0(x^3 dx^1 \otimes dx^2))^* \in \left( (J^*\otimes T^*)^* (\mathbb{R}^n) \right)_0 \).

The third equality in \((***)\) is clear on the basis of Lemma 6.

Let us explain the first equality in \((***)\). Vector fields \( \partial_1 + (x^1)^r \partial_1 \) and \( \partial_1 \) have the same \((r - 1)\)-jets at \( 0 \in \mathbb{R}^n \). Then, by \([12]\) there exist a diffeomorphism \( \varphi = \varphi_1 \times \text{id}_{\mathbb{R}^{n-1}} : \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} \) such that \( \varphi_1 : \mathbb{R} \rightarrow \mathbb{R}, j^*_0\varphi = \text{id} \) and \( \varphi_1(\partial_1) = \partial_1 + (x^1)^r \partial_1 \) in a certain neighborhood of \( 0 \in \mathbb{R}^n \). Let \( \varphi^{-1} \) sends \( \omega \) in \( \tilde{\omega} \). Then \( \tilde{\omega} \) is a linear combination of the elements \( (j^*_0(x^j dx^i \otimes dx^j))^* \in \left( (J^*\otimes T^*)^* (\mathbb{R}^n) \right)_0 \) for \( r \geq |\alpha| \geq 1, i, j = 1, \ldots, n, i \leq j \). (For \( \omega, j^*_0(dx^i \otimes dx^j) = (\omega, j^*_0(d(x^i \circ \varphi^{-1})) = 0 \). Then, by Lemma 6, \( \langle A(\partial_1^*, \omega), j^*_0(x^3 dx^1 \otimes dx^2) \rangle = \langle A(\omega, \tilde{\omega}, 0), j^*_0(x^3 dx^1 \otimes dx^2) \rangle = 0 \).
\( dx^2 \) = 0 (as \( j_0^* \varphi = \text{id} \)). Then from naturality of \( A \) with respect to \( \varphi \) we obtain
\[
\langle A((\partial_1 + (x^1)^r \partial_h)^\mathbb{C}_h), j_0^*(x^3 \, dx^1 \circ dx^2) \rangle = 0. \]
Now, using the linearity of \( A \) we have
\[
\langle A(((x^1)^r \partial_h)^\mathbb{C}_h), j_0^*(x^3 \, dx^1 \circ dx^2) \rangle = 0. \]
This ends the proof of the first equality in (**
\( \star \)).

Let us explain the second equality in (**
\( \star \)). Analysing the flow of vector field
\( (x^1)^r \partial_1 \) and taking \( \omega = (j_0^*(x^3(x^1)^r \, dx^1 \circ dx^2))^* \in ((J^r(\mathbb{C}^2 T^*)^*)(\mathbb{R}^n))_0 \) we have
(similarly as in the justification of the second equality of (**
\( \star \)))
\[
\langle (x^1)^r \partial_1 \rangle^\mathbb{C}_{\omega}j_0^*(\alpha \, dx^1 \circ dx^2) = \langle \omega, j_0^*(L_{(x^1)^r \partial_1}(x^\alpha \, dx^1 \circ dx^2)) \rangle
\]
\[
= \langle \omega, \alpha_1 j_0^*((x^1)^{r-1}x^\alpha \, dx^1 \circ dx^2) \rangle
\]
\[
+ \langle \omega, j_0^*(x^\alpha \delta^1_i r(x^1)^{r-1} \, dx^1 \circ dx^2) \rangle,
\]
where \( \delta^1_i \) is the Kronecker delta.
Since \( \omega = (j_0^*(x^3(x^1)^r \, dx^1 \circ dx^2))^* \) the last sum is equal to \( r \) if \( \alpha = e_3 \) and
\( (i, j) = (1, 2) \), and 0 in the other cases. Then \( (x^1)^r \partial_1 \rangle^\mathbb{C}_{\omega} = r(j_0^*(x^3 \, dx^1 \circ dx^2))^* \).
This ends the proof of the second equality of (**
\( \star \)).
The proof of Lemma 8 is complete. \( \square \)

**Lemma 9.** We have \( a = b = c = 0 \).

**Proof.** Using Lemma 7 (similarly as for \( g = f = e \)) it is sufficient to prove that
\( c = 0 \), i.e. \( \langle A(\partial_0^\mathbb{C}_h j_0^*(dx^1 \circ dx^2)), j_0^*(x^3 \, dx^1 \circ dx^2) \rangle = 0 \).
But we have
\[
0 = \langle A(\partial_0^\mathbb{C}_h j_0^*(dx^1 \circ dx^2)), j_0^*(x^3 \, dx^1 \circ dx^2) \rangle
\]
\[
= \langle A(\partial_0^\mathbb{C}_h j_0^*(dx^1 \circ dx^2)), j_0^*(x^3 \, dx^1 \circ dx^2) \rangle
\]
\[
= \langle A(\partial_0^\mathbb{C}_h j_0^*(dx^1 \circ dx^2)), j_0^*(x^3 \, dx^1 \circ dx^2) \rangle,
\]
where the dots is the linear combination of elements \( (j_0^*(dx^1 \circ dx^2))^* \neq (j_0^*(dx^1 \circ dx^2))^* \), \( \alpha \in (\mathbb{N} \cup \{0\})^n, |\alpha| \leq r, i \leq j, i, j = 1, \ldots, n \).

Equalities first and third are clear because of Lemma 6.

Let us explain the second equality. Consider the local diffeomorphism \( \varphi =
(x^1 + \frac{1}{r+1}(x^1)^{r+1}, x^2, \ldots, x^n)^{-1} \). We see that \( \varphi^{-1} \) preserves
\( j_0^*(x^3 \, dx^1 \circ dx^2) \) and \( \partial_0 \). Moreover \( \varphi^{-1} \) sends
\( (j_0^*(dx^1 \circ dx^2))^* \) in \( (j_0^*(dx^1 \circ dx^2))^* + \ldots \), where
the dots is as above. Now, by the invariance of \( A \) with respect to \( \varphi^{-1} \) we get the second equality in (**
\( \star \)).
The proof of Lemma 9 is complete. \( \square \)
The proof of Proposition 2 is complete.
The proof of Theorem 1 is complete. \( \square \)

7. **The natural affinors on \((J^r(\mathbb{C}^2 T^*))^* \) of vertical type**

A natural affinor \( Q : T(J^r(\mathbb{C}^2 T^*))^* \to T(J^r(\mathbb{C}^2 T^*))^* \) on \((J^r(\mathbb{C}^2 T^*))^* \) is of
vertical type if the image of \( Q \) is in the vertical space \( V(J^r(\mathbb{C}^2 T^*))^*(M) \) for every
\( n \)-manifolds \( M \).
We have the natural isomorphism

\[ V(J^r(\mathbb{O}^2T^*))^*(M) \cong (J^r(\mathbb{O}^2T^*))^*(M) \times (J^r(\mathbb{O}^2T^*))^*(M) \]

given by \((u, v) = \frac{d}{dt}|_{t=0}(u + tv), u, v \in (J^r(\mathbb{O}^2T^*))_x^*(M), x \in M, \) and the natural projection \(pr_2 : V(J^r(\mathbb{O}^2T^*))^*M \to (J^r(\mathbb{O}^2T^*))^*M \) on the second factor.

Let \(Q : T(J^r(\mathbb{O}^2T^*))^* \to T(J^r(\mathbb{O}^2T^*))^* \) on \((J^r(\mathbb{O}^2T^*))^* \) be a natural affinor of vertical type. Composing \(Q \) with \(pr_2 \) we get a natural linear transformation \(pr_2 \circ Q : T(J^r(\mathbb{O}^2T^*))^* \to (J^r(\mathbb{O}^2T^*))^* \) over \(n\)-manifolds. It is equal to 0 because of Theorem 1. So, we have the following corollary.

**Corollary 1.** Let \(n \geq 3\), \(r \) be natural numbers. Every natural affinor \(Q \) of vertical type on \((J^r(\mathbb{O}^2T^*))^* \) over \(n\)-manifolds is equal to 0.

**8. The Linear Natural Transformations**

\(T(J^r(\mathbb{O}^2T^*))^* \to T\)

Let \(\pi \) be the projection of natural bundle \((J^r(\mathbb{O}^2T^*))^*\). Then the tangent map \(T\pi : T(J^r(\mathbb{O}^2T^*))^* \to TM \) defines a linear natural transformation \(T\pi : T(J^r(\mathbb{O}^2T^*))^* \to T. \) (The definition of a linear natural transformation \(T(J^r(\mathbb{O}^2T^*))^* \to T \) over \(n\)-manifolds is similar to the one in Section 1.)

**Theorem 2.** Let \(n \) and \(r \) be natural numbers. Every linear natural transformation \(B : T(J^r(\mathbb{O}^2T^*))^* \to T \) over \(n\)-manifolds is proportional to \(T\pi. \)

**9. Proof of Theorem 2**

Consider a linear natural transformation \(B : T(J^r(\mathbb{O}^2T^*))^* \to T. \) We have

**Lemma 10.** If \(\langle B(u), \varrho_0x^1 \rangle = 0 \) for every \(u \in (T(J^r(\mathbb{O}^2T^*))^*)(\mathbb{R}^n)\) \(_0\), then \(B = 0. \)

**Proof.** The proof of Lemma 10 is similar to the proofs of Lemmas 1 – 4. From the invariance of \(B \) with respect to the coordinate permutation we see that \(\langle B(u), \varrho_0x^i \rangle = 0 \) for \(i = 1, \ldots, n\) and \(u \in (T(J^r(\mathbb{O}^2T^*))^*)(\mathbb{R}^n)\) \(_0\). So \(B(u) = 0 \) for every \(u \in (T(J^r(\mathbb{O}^2T^*))^*)(\mathbb{R}^n)\) \(_0\). Then using the invariance of \(B \) with respect to the charts we obtain that \(B = 0. \)

**Lemma 11.** We have \(\langle B(u), \varrho_0x^1 \rangle = \lambda u_1^1 \) for some \(\lambda \in \mathbb{R}, \) where \(u = (u_1, u_2, u_3), \)

\(u_1 = (u_1^1) \in \mathbb{R}^n, \ i = 1, \ldots, n, \) and \(u_2, u_3 \in ((J^r(\mathbb{O}^2T^*))^*)(\mathbb{R}^n)\) \(_0. \)

**Proof.** The proof of Lemma 11 is similar to the proof of Lemma 5.

Lemma 11 shows that \(\langle (B - \lambda T\pi)(u), \varrho_0x^1 \rangle = 0 \) for every \(u \in (T(J^r(\mathbb{O}^2T^*))^*)(\mathbb{R}^n)\) \(_0. \) Then \(B - \lambda T\pi = 0 \) by Lemma 10, i.e. \(B = \lambda T\pi. \)

The proof of Theorem 2 is complete.
10. The main result

The main result of the present paper is the following theorem.

**Theorem 3.** Let \( n \geq 3 \) and \( r \) be natural numbers. Every natural affinor \( Q : T(J^r(\odot^2 T^*))^* \to T(J^r(\odot^2 T^*))^* \) on \( (J^r(\odot^2 T^*))^* \) over \( n \)-manifolds is proportional to the identity affinor.

**Proof.** The composition \( T\pi \circ Q : (J^r(\odot^2 T^*))^* \to T \) is a linear natural transformation. Hence, by Theorem 2, \( T\pi \circ Q = \lambda T\pi \) for some \( \lambda \in \mathbb{R} \). Then \( Q - \lambda \text{id} : (J^r(\odot^2 T^*))^* \to (J^r(\odot^2 T^*))^* \) is a natural affinor of vertical type, because \( T\pi \circ (Q - \lambda \text{id}) = T\pi \circ Q - \lambda T\pi = 0 \). From Corollary 1 we obtain that \( Q - \lambda \text{id} = 0 \). Thus \( Q = \lambda \text{id} \). The proof of Theorem 3 is complete. \( \square \)

References


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