AN EXTENSION OF THE METHOD OF QUASILINEARIZATION

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Abstract. The method of quasilinearization is a well-known technique for obtaining approximate solutions of nonlinear differential equations. This method has recently been generalized and extended using less restrictive assumptions so as to apply to a larger class of differential equations. In this paper, we use this technique to nonlinear differential problems.

1. Introduction

Let $y_0, z_0 \in C^1(J, \mathbb{R})$ with $y_0(t) \leq z_0(t)$ on $J$ and define the following sets

$$\Omega = \{(t, u) : y_0(t) \leq u \leq z_0(t), \ t \in J\},$$

$$\bar{\Omega} = \{(t, u) : y_0(t) \leq u \leq z_0(t), \ t \in J\}.$$

In this paper, we consider the following initial value problem

(1) \hspace{1cm} x'(t) = f(t, x(t)), \quad t \in J = [0, b], \ x(0) = k_0,

where $f \in C(\Omega, \mathbb{R}), k_0 \in \mathbb{R}$ are given. If we replace $f$ by the sum $[f = g_1 + g_2]$ of convex and concave functions, then corresponding monotone sequences converge quadratically to the unique solution of problem (1) (see [6,8]). In this paper we will generalize this result. Assume that $f$ has the splitting $f(t, x) = F(t, x, x)$, where $F \in C(\Omega, \mathbb{R})$. Then problem (1) takes the form

(2) \hspace{1cm} x'(t) = F(t, x(t), x(t)), \quad t \in J, \ x(0) = k_0.

2000 Mathematics Subject Classification: 34A45.
Key words and phrases: quasilinearization, monotone iterations, quadratic convergence.
Received August 15, 2001.
2. Main results

A function \( v \in C^1(J, \mathbb{R}) \) is said to be a lower solution of problem (2) if
\[
v'(t) \leq F(t, v(t), v(t)) , \quad t \in J , \quad v(0) \leq k_0 ,
\]
and an upper solution of (2) if the inequalities are reversed.

**Theorem 1.** Assume that:
1° \( y_0, z_0 \in C^1(J, \mathbb{R}) \) are lower and upper solutions of problem (2), respectively, such that \( y_0(t) \leq z_0(t) \) on \( J \).
2° \( F, F_x, F_y, F_{xx}, F_{xy}, F_{yy} \in C(\Omega, \mathbb{R}) \) and
\[
F_{xx}(t, x, y) \geq 0 , \quad F_{xy}(t, x, y) \leq 0 , \quad F_{yy}(t, x, y) \leq 0 \quad \text{for} \quad (t, x, y) \in \Omega.
\]

Then there exist monotone sequences \( \{y_n\}, \{z_n\} \) which converge uniformly to the unique solution \( x \) of (2) on \( J \), and the convergence is quadratic.

**Proof.** The above assumptions guarantee that (2) has exactly one solution on \( \Omega \).

Observe that 2° implies that \( F_x \) is nondecreasing in the second variable, \( F_x \) is nonincreasing in the third variable and \( F_y \) is nonincreasing in the last two variables. Denote this property by (A).

Let us construct the elements of sequences \( \{y_n\}, \{z_n\} \) by
\[
y_{n+1}^1(t) = F(t, y_n, y_n) + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)](y_{n+1}(t) - y_n(t)) ,
\]
\[
y_{n+1}(0) = k_0 ,
\]
\[
z_{n+1}^1(t) = F(t, z_n, z_n) + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)](z_{n+1}(t) - z_n(t)) ,
\]
\[
z_{n+1}(0) = k_0
\]
for \( n = 0, 1, \cdots \). Note that the above sequences are well defined.

Indeed, \( y_0(t) \leq z_0(t) \) on \( J \), by 1°. We shall show that
\[
y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t) \quad \text{on} \quad J.
\]

Put \( p = y_0 - y_1 \) on \( J \). Then
\[
p'(t) \leq F(t, y_0, y_0) - F(t, y_0, y_0) - [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)][y_1(t) - y_0(t)]
\]
\[
= [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)]p(t) .
\]

Hence \( p(t) \leq 0 \) on \( J \), since \( p(0) \leq 0 \), showing that \( y_0(t) \leq y_1(t) \) on \( J \). Note that if we put \( p = z_1 - z_0 \) on \( J \), then
\[
p'(t) \leq F(t, z_0, z_0) + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)][z_1(t) - z_0(t)] - F(t, z_0, z_0)
\]
\[
= [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)]p(t) , \quad \text{and} \quad p(0) \leq 0 ,
\]

respectively,
so $z_1(t) \leq z_0(t)$ on $J$. Next, we let $p = y_1 - z_1$ on $J$, so $p(0) = 0$. By the mean value theorem and property (A), we have

$$p'(t) = F(t, y_0, y_0) - F(t, z_0, y_0) + F(t, y_1, y_0) - F(t, z_1, z_0)$$

$$+ [F_x(t, y_0, y_0) + F_y(t, z_0, z_0)]y_1(t) - y_0(t) - z_1(t) + z_0(t)$$

$$+ [F_x(t, y_0, y_0) + F_y(t, z_0, z_0)][y_0(t) - z_0(t)]$$

$$F_x(t, y_0, y_0) - F_x(t, z_0, y_0)[z_0(t) - y_0(t)]$$

$$+ [F_x(t, y_0, z_0) + F_y(t, y_0, z_0)]p(t)$$

$$\leq [F_x(t, y_0, y_0) + F_y(t, z_0, z_0)]p(t),$$

where $y_0(t) < \xi(t), \sigma(t) < z_0(t)$ on $J$. As the result we get $p(t) \leq 0$ on $J$, so $y_1(y) \leq z_1(t)$ on $J$. It proves that (3) holds.

Now we prove that $y_1, z_1$ are lower and upper solutions of (2), respectively. The mean value theorem and property (A) yield

$$y_1'(t) = F(t, y_0, y_0) - F(t, y_1, y_0) + F(t, y_1, y_1) - F(t, y_1, y_1)$$

$$+ [F_x(t, y_0, y_0) + F_y(t, z_0, z_0)][y_1(t) - y_0(t)]$$

$$= [F_x(t, \xi_1, y_0) + F_y(t, y_1, \sigma_1)][y_0(t) - y_1(t)] + F(t, y_1, y_1)$$

$$+ [F_x(t, y_0, y_0) + F_y(t, z_0, z_0)][y_1(t) - y_0(t)]$$

$$\leq [F_x(t, y_0, z_0) - F_x(t, y_0, y_0) + F_y(t, z_0, z_0) - F_y(t, y_1, y_1)][y_1(t) - y_0(t)]$$

$$+ F(t, y_1, y_1) \leq F(t, y_1, y_1),$$

where $y_0(t) < \xi_1(t), \sigma_1(t) < y_1(t)$ on $J$. Similarly, we get

$$z_1'(t) = F(t, z_1, z_1) + F(t, z_0, z_0) - F(t, z_1, z_0) - F(t, z_1, z_1)$$

$$+ [F_x(t, y_0, y_0) + F_y(t, z_0, z_0)][z_1(t) - z_0(t)]$$

$$= F(t, z_1, z_1) + [F_x(t, \xi_2, z_0) + F_y(t, z_1, \sigma_2)][z_0(t) - z_1(t)]$$

$$+ [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)][z_1(t) - z_0(t)]$$

$$\geq F(t, z_1, z_1) + [F_x(t, z_1, z_0) - F_x(t, y_0, z_0) + F_y(t, z_1, z_0)$$

$$- F_y(t, z_0, z_0)][z_0(t) - z_1(t)] \geq F(t, z_1, z_1),$$

where $z_1(t) < \xi_2(t), \sigma_2(t) < z_0(t)$ on $J$. The above proves that $y_1, z_1$ are lower and upper solutions of (2).

Let us assume that

$$y_0(t) \leq y_1(t) \leq \cdots \leq y_{k-1}(t) \leq y_k(t) \leq z_k(t) \leq z_{k-1}(t) \leq \cdots \leq z_1(t) \leq z_0(t),$$

$$t \in J,$$

and let $y_k, z_k$ be lower and upper solutions of problem (2) for some $k \geq 1$. We shall prove that:

$$y_k(t) \leq y_{k+1}(t) \leq z_{k+1}(t) \leq z_k(t), \quad t \in J.$$
Let \( p = y_k - y_{k+1} \) on \( J \), so \( p(0) = 0 \). Using the mean value theorem, property (A) and the fact that \( y_k \) is a lower solution of problem (2), we obtain

\[
p'(t) \leq F(t, y_k, y_k) - F(t, y_k, y_k) - [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)][y_{k+1}(t) - y_k(t)] \\
= [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)]p(t) .
\]

Hence \( p(t) \leq 0 \), so \( y_k(t) \leq y_{k+1}(t) \) on \( J \). Similarly, we can show that \( z_{k+1}(t) \leq z_k(t) \) on \( J \).

Now, if \( p = y_{k+1} - z_{k+1} \) on \( J \), then

\[
p'(t) = F(t, y_k, y_k) - F(t, z_k, y_k) + F(t, z_k, y_k) - F(t, z_k, z_k) \\
+ [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)][y_k(t) - z_{k+1}(t) + z_k(t)] \\
= [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)][y_k(t) - z_k(t)] \\
+ [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)][y_k(t) - z_{k+1}(t) + z_k(t)] \\
\leq [F_x(t, y_k, z_k) - F_x(t, y_k, y_k)][z_k(t) - y_k(t)] \\
+ [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)]p(t) \\
\leq [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)]p(t)
\]

with \( y_k(t) < \xi(t) \), \( \bar{\sigma}(t) < z_k(t) \). It proves that \( y_{k+1}(t) \leq z_{k+1}(t) \) on \( J \), so relation (4) holds.

Hence, by induction, we have

\[
y_0(t) \leq y_1(t) \leq \cdots \leq y_n(t) \leq z_n(t) \leq \cdots \leq z_1(t) \leq y_0(t) , \quad t \in J ,
\]

for all \( n \). Employing standard techniques [5], it can be shown that the sequences \( \{y_n\} \), \( \{z_n\} \) converge uniformly and monotonically to the unique solution \( x \) of problem (2).

We shall next show the convergence of \( y_n \), \( z_n \) to the unique solution \( x \) of problem (2) is quadratic. For this purpose, we consider

\[
p_{n+1} = x - y_{n+1} \geq 0 , \quad q_{n+1} = z_{n+1} - x \geq 0 \quad \text{on} \quad J ,
\]

and note that \( p_{n+1}(0) = q_{n+1}(0) = 0 \) for \( n \geq 0 \). Using the mean value theorem and property (A), we get

\[
p'_{n+1}(t) = F(t, x, x) - F(t, y_n, x) + F(t, y_n, x) - F(t, y_n, y_n) \\
- [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][y_{n+1}(t) - x(t) + x(t) - y_n(t)] \\
= [F_x(t, \xi_1, x) + F_y(t, y_n, \bar{\sigma}_1)]p_n(t) \\
+ [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][p_{n+1}(t) - p_n(t)] \\
\leq [F_x(t, x, x) - F_x(t, y_n, y_n) + F_x(t, y_n, x) - F_x(t, y_n, z_n) \\
+ F_y(t, y_n, y_n) - F_y(t, z_n, y_n) + F_y(t, z_n, y_n) - F_y(t, z_n, z_n)]p_n(t) \\
+ [F_y(t, y_n, z_n) + F_y(t, z_n, z_n)]p_{n+1}(t) \\
= \{F_{xx}(t, \xi_2, x)p_n(t) - F_{xy}(t, y_n, \bar{\sigma}2)q_n(t) - F_{yx}(t, \xi_3, y_n)[y_n(t) - y_n(t)] \\
- F_{yy}(t, z_n, \bar{\sigma}_3)[z_n(t) - y_n(t)]\}p_n(t) \\
+ [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)]p_{n+1}(t) ,
\]
where \( y_n(t) < \tilde{x}_1(t), \tilde{x}_2(t), \tilde{\sigma}_1(t) < x(t), x(t) < \tilde{\sigma}_2(t) < z_n(t), y_n(t) < \tilde{x}_3(t), \tilde{\sigma}_3(t) < z_n(t) \) on \( J \). Thus we obtain

\[
p'_{n+1}(t) \leq \{A_1p_n(t) + A_2q_n(t) + [A_2 + A_3][q_n(t) + p_n(t)]\}p_n(t) + M_{p_{n+1}}(t)
\]

\[
\leq M_{p_{n+1}}(t) + B_1p_n^2(t) + B_2q_n^2(t),
\]

where

\[
|F_{xx}(t, u, v)| \leq A_1, \quad |F_{xy}(t, u, v)| \leq A_2, \quad |F_{yy}(t, u, v)| \leq A_3, \quad |F_x(t, u, v)| \leq M_1,
\]

\[
|F_y(t, u, v)| \leq M_2 \quad \text{on} \quad \Omega \quad \text{with} \quad M = M_1 + M_2, \quad B_1 = A_1 + 2A_2 + \frac{3}{2}A_3,
\]

\[
B_2 = A_2 + \frac{1}{2}A_3.
\]

Now, the differential inequality implies

\[
0 \leq p_{n+1}(t) \leq \int_0^t [B_1p_n^2(s) + B_2q_n^2(s)]\exp[M(t - s)]\,ds.
\]

This yields the following relation

\[
\max_{t \in J}|x(t) - y_{n+1}(t)| \leq a_1 \max_{t \in J}|x(t) - y_n(t)|^2 + a_2 \max_{t \in J}|x(t) - z_n(t)|^2,
\]

where \( a_i = B_iS, \ i = 1, 2 \) with

\[
S = \begin{cases} \frac{b}{M \exp(Mb) - 1} & \text{if} \quad M > 0, \\ b & \text{if} \quad M = 0. \end{cases}
\]

Similarly, we find that

\[
q'_{n+1}(t) = F(t, z_n, z_n) - F(t, x, z_n) + F(t, x, z_n) - F(t, x, x)
\]

\[
+ [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][z_{n+1}(t) - x(t) + x(t) - z_n(t)]
\]

\[
= [F_x(t, \tilde{x}_4, z_n) + F_y(t, x, \tilde{\sigma}_4)]q_n(t)
\]

\[
+ [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][q_{n+1}(t) - q_n(t)]
\]

\[
\leq [F_x(t, z_n, z_n) - F_x(t, y_n, z_n) + F_y(t, x, x) - F_y(t, z_n, x)
\]

\[
+ F_y(t, z_n, x) - F_y(t, z_n, z_n)]q_n(t) + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)]q_{n+1}(t)
\]

\[
= [F_{xx}(t, \tilde{x}_5, z_n) + F_{xy}(t, \tilde{\sigma}_6, x) + F_{yy}(t, z_n, \tilde{\sigma}_5)]q_n(t)
\]

\[
+ [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)]q_{n+1}(t),
\]

where \( x(t) < \tilde{x}_4(t), \tilde{x}_6(t), \tilde{\sigma}_4(t), \tilde{\sigma}_5(t) < z_n(t), y_n(t) < \tilde{x}_5(t) < z_n(t) \) on \( J \). Hence, we get

\[
q'_{n+1}(t) \leq \{A_1[q_n(t) + p_n(t)] + A_2q_n(t) + A_3[q_n(t) + p_n(t)]\}q_n(t) + M_{q_{n+1}}(t),
\]

\[
\leq M_{q_{n+1}}(t) + \tilde{B}_1p_n^2(t) + \tilde{B}_2q_n^2(t),
\]
where
\[ \bar{B}_1 = \frac{1}{2} A_1, \quad \bar{B}_2 = \frac{3}{2} A_1 + A_2 + A_3. \]

Now, the last differential inequality implies
\[ q_{n+1}(t) \leq [\bar{B}_1 \max_{s \in J} p_n^2(s) + \bar{B}_2 \max_{s \in J} q_n^2(s)]S, \quad t \in J \]
or
\[ \max_{t \in J} |x(t) - z_{n+1}(t)| \leq \bar{a}_1 \max_{t \in J} |x(t) - y_n(t)|^2 + \bar{a}_2 \max_{t \in J} |x(t) - z_n(t)|^2 \]
with \( \bar{a}_i = \bar{B}_i S, \ i = 1, 2. \)

The proof is complete. \( \square \)

**Remark 1.** Let \( f = h + g, \) and \( h, h, h_{xx}, g, g_{xx}, g_{xxx} \in C(\Omega_1, \mathbb{R}) \) for \( \Omega_1 = \{(t, u) : t \in J, y_0(t) \leq u \leq z_0(t)\} \). Put \( F(t, x, y) = h(t, x) + g(t, y) \). Indeed, \( F(t, x, y) = f(t, x) \) and \( F_{xx}(t, x, y) = h_{xx}(t, x) \), \( F_{xy}(t, x, y) = F_{yx}(t, x, y) = F_{yy}(t, x, y) = 0 \). Then the conclusion of Theorem 1 remains valid. (see also a result of [6] for \( \Phi = 0, \Phi_g \geq 0 \)).

**Remark 2.** Let \( f, h, g \) be as in Remark 1 and moreover let \( \Phi, \Phi_x, \Phi_{xx}, \Psi, \Psi_x, \Psi_{xx} \in C(\Omega_1, \mathbb{R}) \). Put \( F(t, x, y) = H(t, x) + G(t, y) - \Phi(t, y) - \Psi(t, x) \) for \( H = h + \Phi, \ G = g + \Psi \). Indeed, \( F(t, x, y) = f(t, x) \) and \( F_{xx}(t, x, y) = H_{xx}(t, x) - \Psi_{xx}(t, x), F_{xy}(t, x, y) = F_{yx}(t, x, y) = 0 \), \( F_{yy}(t, x, y) = G_{yy}(t, y) - \Phi_g(t, y) \). If assumptions of Theorem 1.4.3[8] hold \( (H_{xx} \geq 0, \ \Psi_{xx} \leq 0, \ G_{yy} \leq 0, \ \Phi_g \geq 0) \) then Theorem 1 is satisfied (see also a result of [6] for \( \Phi = 0, \Phi_g \geq 0 \)).

**Theorem 2.** Assume that
(i) condition 1° of Theorem 1 holds,
(ii) \( F, F_x, F_y, F_{xx}, F_{xy}, F_{yx}, F_{yy} \in C(\Omega, \mathbb{R}) \) and
\[ F_{xx}(t, x, y) \geq 0, \quad F_{xy}(t, x, y) \geq 0, \quad F_{yy}(t, x, y) \leq 0 \quad \text{for } (t, x, y) \in \Omega. \]

Then the conclusion of Theorem 1 remains valid.

**Proof.** Note that, in view of (ii), \( F_x \) is nondecreasing in the last two variables, \( F_y \) is nondecreasing in the second variable, and \( F_{yy} \) is nonincreasing in the third one. Denote this property by (B).
We construct the monotone sequences \( \{y_n\}, \{z_n\} \) by formulas:
\[ y_{n+1}(t) = F(t, y_n, y_n) + [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)]|y_{n+1}(t) - y_n(t)|, \quad y_{n+1}(0) = k_0, \]
\[ z_{n+1}(t) = F(t, z_n, z_n) + [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)]|z_{n+1}(t) - z_n(t)|, \quad z_{n+1}(0) = k_0 \]
for \( n = 0, 1, \ldots \).
Let \( p = y_0 - y_1 \) on \( J \). Then
\[
p'(t) \leq F(t, y_0, y_0) - F(t, y_0, y_0) - [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)][y_1(t) - y_0(t)]
\]
\[
= [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)]p(t), \quad \text{and} \quad p(0) \leq 0.
\]
Hence \( p(t) \leq 0 \) on \( J \), showing that \( y_0(t) \leq y_1(t) \) on \( J \). Similarly, we can show that \( z_1(t) \leq z_0(t) \) on \( J \). If we now put \( p = y_1 - z_1 \) on \( J \), then the mean value theorem and property (B), we have
\[
p'(t) = F(t, y_0, y_0) - F(t, z_0, y_0) + F(t, z_0, y_0) - F(t, z_0, z_0)
+ [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)][y_1(t) - y_0(t) - z_1(t) + z_0(t)]
= [F_x(t, \xi, y_0) + F_y(t, z_0, \sigma)][y_0(t) - z_0(t)]
+ [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)][p(t) - z_1(t) + z_0(t)]
\leq |F_y(t, y_0, z_0) - F_y(t, y_0, z_0)][z_0(t) - y_0(t)]
+ [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)]p(t)
\]
\[
\leq [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)]p(t), \quad \text{as} \quad p(0) = 0
\]
with \( y_0(t) < \xi(t), \sigma(t) < z_0(t) \) on \( J \). Hence \( y_1(t) \leq z_1(t) \) on \( J \), and as a result, we obtain
\[
y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t) \quad \text{on} \quad J.
\]
Continuing this process successively, by induction, we get
\[
y_0(t) \leq y_1(t) \leq \cdots \leq y_n(t) \leq z_n(t) \leq \cdots \leq z_1(t) \leq z_0(t), \quad t \in J,
\]
for all \( n \). Indeed, the sequences \( \{y_n\}, \{z_n\} \) converge uniformly and monotonically to the unique solution \( x \) of problem (2). Now, we are in a position to show that this convergence is quadratic.

Let
\[
p_{n+1} = x - y_{n+1} \geq 0, \quad q_{n+1} = z_{n+1} - x \geq 0 \quad \text{on} \quad J.
\]
Hence \( p_{n+1}(0) = q_{n+1}(0) = 0 \). The mean value theorem and property (B) yield
\[
p'_{n+1}(t) = F(t, x, x) - F(t, y_n, x) + F(t, y_n, x) - F(t, y_n, y_n)
- [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)][y_{n+1}(t) - x(t) + x(t) - y_n(t)]
= [F_x(t, \xi_1, x) + F_y(t, y_n, \sigma_1)]p_n(t)
+ [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)]p_{n+1}(t) - p_n(t)]
\leq [F_x(t, x, x) - F_x(t, y_n, x) + F_x(t, y_n, x) - F_x(t, y_n, y_n)
+ F_y(t, y_n, y_n) - F_y(t, y_n, z_n)]p_n(t)
+ [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)]p_{n+1}(t)
\]
\[
\leq \{F_{xx}(t, \xi_2, x)p_n(t) + F_{xy}(t, y_n, \sigma_2)p_n(t)
- F_{yy}(t, y_n, \sigma_3)[z_n(t) - y_n(t)]\}p_n(t)
+ [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)]p_{n+1}(t),
\]
where \( y_n(t) < \xi_1(t), \xi_2(t), \sigma_1(t), \sigma_2(t) < x(t), y_n(t) < \sigma_3(t) < z_n(t) \) on \( J \). Thus we obtain

\[
p_{n+1}'(t) \leq \{(A_1 + 2A_2)p_n(t) + A_3[q_n(t) + p_n(t)]\}p_n(t) + Mp_{n+1}(t)
\]

\[
\leq Mp_{n+1}(t) + D_1p_n(t) + D_2q_n^2(t),
\]

where \( D_1 = A_1 + A_2 + \frac{3}{2}A_3, \quad D_2 = \frac{1}{2}A_3 \). Hence, we get

\[
0 \leq p_{n+1}(t) \leq \int_0^t [D_1p_n^2(s) + D_2q_n^2(s)] \exp[M(t-s)] ds,
\]

and it yields the relation

\[
\max_{t \in J} |x(t) - y_{n+1}(t)| \leq d_1 \max_{t \in J} |x(t) - y_n(t)|^2 + d_2 \max_{t \in J} |x(t) - z_n(t)|^2,
\]

where \( d_i = D_i S, \ i = 1, 2 \).

By the similar argument, we can show that

\[
\max_{t \in J} |x(t) - z_{n+1}(t)| \leq \tilde{d}_1 \max_{t \in J} |x(t) - y_n(t)|^2 + \tilde{d}_2 \max_{t \in J} |x(t) - z_n(t)|^2,
\]

with \( \tilde{d}_i = \tilde{D}_i S, \ i = 1, 2 \), for \( \tilde{D}_1 = \frac{1}{2}A_1 + A_2, \quad \tilde{D}_2 = \frac{3}{4}A_1 + 2A_2 + A_3 \).

This ends the proof. \( \square \)

References


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