A counterexample to a conjecture on linear systems on \( \mathbb{P}^3 \)

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Abstract. In his paper [1] Ciliberto proposes a conjecture in order to characterize special linear systems of \( \mathbb{P}^n \) through multiple base points. In this note we give a counterexample to this conjecture by showing that there is a substantial difference between the speciality of linear systems on \( \mathbb{P}^2 \) and those of \( \mathbb{P}^3 \).

Let us take the projective space \( \mathbb{P}^n \) and let us consider the linear system of hypersurfaces of degree \( d \) having some points of fixed multiplicity. The virtual dimension of such systems is the dimension of the space of degree \( d \) polynomials minus the conditions imposed by the multiple points and the expected dimension is the maximum between the virtual one and \(-1\). The systems whose dimension is bigger than the expected one are called special systems.

There exists a conjecture due to Hirschowitz (see [5]), characterizing special linear systems on \( \mathbb{P}^2 \), which has been proved in some special cases [2], [3], [7], [6].

Concerning linear systems on \( \mathbb{P}^n \), in [1] Ciliberto gives a conjecture based on the classification of special linear systems through double points. In this note we describe a linear system on \( \mathbb{P}^3 \) that we found in a list of special systems generated with the help of Singular [4] and which turns out to be a counterexample to that conjecture.

The paper is organized as follows: in Section 1 we fix some notation and state Ciliberto’s conjecture, while Section 2 is devoted to the counterexample. In Section 3 we try to explain speciality of some systems by the Riemann–Roch formula, and we conclude the note with an appendix containing some computations.

1 Preliminaries

We start by fixing some notation.

Notation 1.1. Let us denote by \( \mathbb{I}_n(d,m_1^{a_1}, \ldots, m_r^{a_r}) \) the linear system of hypersurfaces of \( \mathbb{P}^n \) of degree \( d \), passing through \( a_i \) points with multiplicity \( m_i \), for \( i = 1, \ldots, r \). Let \( \mathcal{I}_Z \) be the ideal of the zero-dimensional scheme of multiple points. We denote by \( \mathcal{L}_n(d,m_1^{a_1}, \ldots, m_r^{a_r}) \) the sheaf \( \mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_Z \). Given the system \( \mathbb{I} = \mathbb{I}_n(d,m_1^{a_1}, \ldots, m_r^{a_r}) \), its virtual dimension is
and the expected dimension is

\[ e(\mathbb{L}) = \max(v(\mathbb{L}), -1). \]

A linear system will be called special if its expected dimension is strictly smaller than the effective one.

**Remark 1.2.** Throughout the paper, if no confusion arises, we will use sometimes the same letter to denote a linear system and the general divisor in the system.

We recall the following definition, see [1].

**Definition 1.3.** Let \( X \) be a smooth, projective variety of dimension \( n \), let \( C \) be a smooth, irreducible curve on \( X \) and let \( N^j_X \) be the normal bundle of \( C \) in \( X \). We will say that \( C \) is a negative curve if there is a line bundle \( N \) of negative degree and a surjective map \( N^j_X \to N \). The curve \( C \) is called a \((\mathcal{C}_0^1)\)-curve of size \( a \), with \( 1 \leq a \leq n - 1 \), on \( X \) if \( C \cong \mathbb{P}^1 \) and \( N^j_X \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus a} \oplus N \), where \( N \) has no summands of negative degree.

The main conjecture stated in [1] is the following.

**Conjecture 1.4.** Let \( X \) be the blow-up of \( \mathbb{P}^n \) at general points \( p_1, \ldots, p_r \) and let \( \mathbb{L} = \mathbb{L}_n(d, m_1, \ldots, m_r) \) be a linear system with multiple base points at \( p_1, \ldots, p_r \). Then:

(i) the only negative curves on \( X \) are \((\mathcal{C}_0^1)\)-curves;

(ii) \( \mathbb{L} \) is special if and only if there is a \((\mathcal{C}_0^1)\)-curve \( C \) on \( X \) corresponding to a curve \( \Gamma \) on \( \mathbb{P}^n \) containing \( p_1, \ldots, p_r \) such that the general member \( D \in \mathbb{L} \) is singular along \( \Gamma \);

(iii) if \( \mathbb{L} \) is special, let \( B \) be the component of the base locus of \( \mathbb{L} \) containing \( \Gamma \) according to Bertini’s theorem. Then the codimension of \( B \) in \( \mathbb{P}^n \) is equal to the size of \( C \) and \( B \) appears multiply in the base locus scheme of \( \mathbb{L} \).

In this note we give a counterexample to points (ii) and (iii) of this conjecture.

## 2 Counterexample

Let us consider the linear system of surfaces of degree nine with one point of degree six and eight points of degree four in \( \mathbb{P}^3 \), i.e. the system \( \mathbb{L} = \mathbb{L}_3(9, 6, 4^8) \). In this section we are going to study this system, showing in particular that it is special but its
general member is not singular along a rational curve. If we denote by \( Q = \mathbb{I}_3(2, 1, 1^8) \) the quadric through the nine simple points, we have the following:

**Claim 1.** \( \mathbb{I}_3(9, 6, 4^8) = Q + \mathbb{I}_3(7, 5, 3^8) \).

If we denote by \( H_1, H_2 \) two generators of Pic(\( Q \)), considering the restriction \( \mathbb{I}_3|_Q \) we get the system of curves in \(|9H_1 + 9H_2|\), with one point of multiplicity 6 and eight points of multiplicity 4. We denote for short this system by \( |9H_1 + 9H_2| - 6p_0 - \sum 4p_i \).

Looking at Appendix 4.1, we can see that \(|9H_1 + 9H_2| - 6p_0 - \sum 4p_i \) corresponds to the planar system \( \mathbb{I}_3(12, 3^2, 4^8) \). This last system cannot be \((-1)\)-special (see the Appendix 4.2) and \( v(\mathbb{I}_3(12, 3^2, 4^8)) = -2 \). Therefore, by \([7]\) we may conclude that it is empty.

In particular, also \( \mathbb{I}_3|_Q = \emptyset \), and hence \( \mathbb{I}_3 \) must contain \( Q \) as a fixed component. By subtracting \( Q \) from \( \mathbb{I}_3 \) we get \( \mathbb{I}_3(7, 5, 3^8) \), which proves our claim.

This means that the free part of \( \mathbb{I}_3 \) is contained in \( \mathbb{I}_3(7, 5, 3^8) \) which has virtual dimension 4. So \( \mathbb{I}_3 \) is a special system.

In order to show that \( \mathbb{I}_3 \) gives a counterexample to Conjecture 1.4 we are now going to prove that the general member of \( \mathbb{I}_3 \) is singular only along the curve \( C \), intersection of \( Q \) and \( \mathbb{I}_3(7, 5, 3^8) \), and that \( C \) does not contain rational components.

We can consider \( C \) as the restriction \( \mathbb{I}_3(7, 5, 3^8)|_Q \). This is equal to \(|7H_1 + 7H_2| - 5p_0 - \sum 3p_i \) on the quadric \( Q \), which corresponds to \( \mathbb{I}_3(9, 2^2, 3^8) \) on \( \mathbb{P}^2 \). This system is not special of dimension 0 and it does not contain rational components (see Appendix 4.3).

Clearly the curve \( C \) is contained in \( \mathbb{I}_3|_{\text{sing}} \) (i.e. the singular locus of \( \mathbb{I}_3 \)). We are going to show that in fact \( C = \mathbb{I}_3|_{\text{sing}} \). First of all, let us denote by \( \mathbb{I}_3(7, 5, 3^8, 1_Q) \) the subsystem of \( \mathbb{I}_3(7, 5, 3^8) \) obtained by imposing one general simple point on the quadric. Since \( \mathbb{I}_3(7, 5, 3^8, 1_Q)|_Q = \emptyset \), \( Q \) is a fixed component of this system and the residual part is given by \( \mathbb{I}_3(5, 4, 2^8) \). Now \( \mathbb{I}_3(5, 4, 2^8)|_Q \) is the system \(|5H_1 + 5H_2| - 4p_0 - \sum 2p_i \) which corresponds to the non-special system \( \mathbb{I}_3(6, 1^2, 2^8) \), of dimension 1. Therefore, imposing two general simple points on \( Q \) and restricting we get that the system \( \mathbb{I}_3(5, 4, 2^8, 1_Q)|_Q \) is empty, which implies that \( \mathbb{I}_3(5, 4, 2^8, 1_Q) \) has \( Q \) as a fixed component. The residual system \( \mathbb{I}_3(3, 3, 1^8) \) is non-special of dimension 1 (because each surface of this system is a cone over a plane cubic through eight fixed points). This implies that the effective dimension of \( \mathbb{I}_3(5, 4, 2^8) \) cannot be greater than 3. Therefore it must be 3 since the virtual dimension is 3. By the same argument one shows that the effective dimension of \( \mathbb{I}_3(7, 5, 3^8) \) is 4.

Observe that \( \text{Bs}(\mathbb{I}_3(7, 5, 3^8)) \subseteq \text{Bs}(2Q + \mathbb{I}_3(3, 3, 1^8)) \) since \( 2Q + \mathbb{I}_3(3, 3, 1^8) \subseteq \mathbb{I}_3(7, 5, 3^8) \). So \( \text{Bs}(\mathbb{I}_3(7, 5, 3^8)) \) could have only \( Q \) as fixed component, but this is not the case since \( \dim \mathbb{I}_3(7, 5, 3^8) = \dim \mathbb{I}_3(5, 4, 2^8) + 1 \). The only curves that may belong to \( \text{Bs}(\mathbb{I}_3(7, 5, 3^8)) \) are the genus 2 curve \( C = \mathbb{I}_3(7, 5, 3^8)|_Q \) and the nine lines of \( \text{Bs}(\mathbb{I}_3(3, 3, 1^8)) \) through the vertex of the cone and each one of the nine base points of the pencil of plane cubics.

We can then conclude that the singular locus \( \mathbb{I}_3|_{\text{sing}} \) consists only of the curve \( C \), since the subsystem \( 3Q + \mathbb{I}_3(3, 3, 1^8) \) is not singular along the nine fixed lines.
Let $Z$ be a zero-dimensional scheme of $\mathbb{P}^3$ and $\mathcal{I}_Z$ be its ideal sheaf. We put $\mathcal{L} = \mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_Z$ and consider the exact sequence

$$0 \to H^0(\mathcal{L}) \to H^0(\mathcal{O}_{\mathbb{P}^3}(d)) \to H^0(\mathcal{O}_Z) \to H^1(\mathcal{L}) \to H^1(\mathcal{O}_{\mathbb{P}^3}(d)) \to H^1(\mathcal{O}_Z) \to H^2(\mathcal{L}) \to H^2(\mathcal{O}_{\mathbb{P}^3}(d)) \to H^2(\mathcal{O}_Z) \to H^3(\mathcal{L}) \to H^3(\mathcal{O}_{\mathbb{P}^3}(d)) \to H^3(\mathcal{O}_Z),$$

obtained tensoring by $\mathcal{O}_{\mathbb{P}^3}(d)$ the sequence defining $Z$ and taking cohomology. From this sequence we obtain that $h^i(\mathcal{L}) = h^i(\mathcal{O}_{\mathbb{P}^3}(d)) = 0$ for $i = 2, 3$ since $h^i(\mathcal{O}_Z) = 0$ for $i = 1, 2, 3$. We also obtain that the virtual dimension of $\mathbb{I}_L$, $h^0(\mathcal{O}_{\mathbb{P}^3}(d)) - h^0(\mathcal{O}_Z) - 1$ is equal to $h^0(\mathcal{L}) - h^1(\mathcal{L}) - 1$ and hence to $\chi(\mathcal{L}) - 1$.

If $Z = \sum m_ip_i$ is a scheme of fat points, then on the blow-up $X \to \mathbb{P}^3$ along these points we may consider the divisor $\mathcal{L} = \pi^*\mathcal{O}_{\mathbb{P}^3}(d) - \sum m_iE_i$ and the associated sheaf $\mathcal{L} = \mathcal{O}_X(\mathcal{L})$. Since $h^i(X, \mathcal{L}) = h^i(\mathbb{P}^3, \mathcal{L})$, the virtual dimension of $\mathbb{I}_L$ is equal to $\chi(\mathcal{L}) - 1$. By the Riemann–Roch formula (see [4]) for a divisor $\mathcal{L}$ on the threefold $X$,

$$\chi(\mathcal{L}) = \frac{\mathcal{L}(\mathcal{L} - K_X)(2\mathcal{L} - K_X) + c_2(X) \cdot \mathcal{L}}{12} + \chi(\mathcal{O}_X),$$

we obtain the following formula for the virtual dimension of $\mathbb{I}_L$:

$$v(\mathbb{I}_L) = \frac{\mathcal{L}(\mathcal{L} - K_X)(2\mathcal{L} - K_X) + c_2(X) \cdot \mathcal{L}}{12}$$

since $\chi(\mathcal{O}_X) = 1$.

If the linear system $\mathbb{I}_L$ can be written as $\mathcal{F} + \mathbb{M}$, where $\mathcal{F}$ is the fixed divisor and $\mathbb{M}$ is a free part, then on $X$ we have $|\mathcal{L}| = |\mathcal{F}| + |\mathcal{M}|$. Therefore the above formula says that

$$v(\mathcal{L}) = v(\mathcal{F}) + v(\mathcal{M}) + \frac{\mathcal{F}\mathcal{M}(\mathcal{L} - K_X)}{2}.$$

Let us suppose that the residual system $\mathbb{M}$ is non-special. The system $\mathbb{I}_L$ has the same effective dimension as $\mathbb{M}$, while their virtual dimensions differ by $v(\mathcal{F}) + \frac{\mathcal{F}\mathcal{M}(\mathcal{L} - K_X)}{2}$. Therefore we can conclude that $\mathbb{I}_L$ is special if $v(\mathcal{F}) + \frac{\mathcal{F}\mathcal{M}(\mathcal{L} - K_X)}{2}$ is smaller than zero.

**Example 3.1.** For instance, let us consider the system $\mathbb{I}_L := \mathbb{I}_L(4,2^9)$. It is special because its virtual dimension is $-2$ while it is not empty since it is equal to $2\mathcal{Q}$, where $\mathcal{Q}$ is the quadric through the nine simple points. In this case $\mathcal{F} = 2\mathcal{Q}$ and $\mathbb{M} = \mathcal{C}$, so $v(\mathcal{F}) = -2$ and $\frac{\mathcal{F}\mathcal{M}(\mathcal{L} - K_X)}{2} = 0$.

**Example 3.2.** Let us consider now the example we described in the previous section,
i.e. the system $\mathbb{L}_3(9, 6, 4^8)$. We have seen that it can be written as $Q + \mathbb{M}$, where $Q$ is the quadric through the nine points, while $\mathbb{M} = \mathbb{L}_3(7, 5, 3^8)$ is the residual free part. The Chow ring $A^\ast(X)$ (where $X$ is the blow-up of $\mathbb{P}^3$ along the nine simple points) is generated by $\langle H, E_0, E_1, \ldots, E_8 \rangle$, where $H$ is the pull-back of the hyperplane divisor of $\mathbb{P}^3$ and the $E_i$’s are the exceptional divisors. The second Chow group $A^2(X)$ is generated by $\langle h, e_0, e_1, \ldots, e_8 \rangle$, where $h = H^2$ is the pull-back of a line, while $e_i = -E_i^2$ is the class of a line inside $E_i$, for $i = 0, 1, \ldots, 8$. Clearly $H \cdot E_i = E_i \cdot E_j = 0$ for $i \neq j$. With this notation we can write:

\[
\begin{align*}
\mathbb{L} &= 9H - 6E_0 - \sum 4E_i \\
\mathbb{M} &= 7H - 5E_0 - \sum 3E_i \\
Q &= 2H - E_0 - \sum E_i \\
K_X &= -4H + 2E_0 + \sum 2E_i.
\end{align*}
\]

Therefore $Q \cdot \mathbb{M} = 14h - 5e_0 - \sum 3e_i$, $Q - K_X = 13H - 8E_0 - \sum 6E_i$ and hence $QM(\mathcal{L} - K_X)/2 = -1$ (while $v(Q) = 0$), which implies the speciality of $\mathbb{L}$.

4 Appendix

4.1 Linear systems on a quadric. In order to study linear systems on a quadric $Q$ it may be helpful to transform them into planar systems by means of a birational transformation $Q \to \mathbb{P}^2$ obtained by blowing up a point and contracting the strict transforms of the two lines through it. Such a transformation gives rise to a $1 : 1$ correspondence between linear systems on $Q$ with one multiple point on the quadric and linear systems with two multiple points on $\mathbb{P}^2$.

In fact, let us consider a linear system $|aH_1 + bH_2| - mp$ (i.e. a system of curves of kind $(a, b)$ through one point $p$ of multiplicity $m$). Blowing up at $p$, one obtains the complete system $|a\pi^*H_1 + b\pi^*H_2 - mE|$ which may be written as $|(a + b - m) \cdot (\pi^*H_1 + \pi^*H_2 - E) - (b - m)(\pi^*H_1 - E) - (a - m)(\pi^*H_2 - E)|$. Since the divisors $\pi^*H_i - E$ ($i = 1, 2$) are $(1)$-curves, they may be contracted giving a linear system on $\mathbb{P}^2$ of degree $a + b - m$ through two points of multiplicity $b - m$ and $a - m$ and hence

$|aH_1 + bH_2| - mp \to \mathbb{L}_2(a + b - m, b - m, a - m)$.

4.2 $(−1)$-curves. In order to study the speciality of the systems $\mathbb{L}_2(12, 3^2, 4^8)$, $\mathbb{L}_2(9, 2^2, 3^8)$ and $\mathbb{L}_2(6, 1^2, 2^8)$, we need to produce a complete list of all the $(−1)$-curves of $\mathbb{P}^2$ of kind $\mathbb{L}_2(d, m_1, m_2, m_3, \ldots, m_{10})$ which may have an intersection less than $−1$ with some of these systems. Clearly it is enough to consider the system $\mathbb{L}_2(12, 3^2, 4^8)$, whose degree and multiplicities are the biggest. From the condition of
being contained twice in this system we deduce the following inequalities: \( d \leq 6, \)
\( 0 \leq m_1, m_2 \leq 1 \) and \( 0 \leq m_3, \ldots, m_{10} \leq 2. \) Moreover let us see that \( m_3 = \cdots = m_{10} = m. \) Otherwise the system would contain twice the compound \(-1\)-curve given by the union of all the simple \(-1\)-curves obtained by permuting the points \( p_3, \ldots, p_{10}. \) In this case the multiplicities of the compound curve at these points would be too big.

An explicit calculation shows that the only \(-1\)-curve of the form \( \mathbb{L}_2(d, m_1, m_2, m^8) \) satisfying the preceding conditions is \( \mathbb{L}_2(1, 1, 1, 0^8) \), but this has non-negative intersection with any of these systems.

### 4.3 \( \mathbb{L}_2(9, 2^2, 3^8) \) does not contain rational components.

Let \( S \) be the blow up of \( \mathbb{P}^2 \) along the ten points and let \( C \) be the strict transform of the curve given by \( \mathbb{L} = \mathbb{L}_2(9, 2^2, 3^8). \) Suppose that there exists an irreducible rational component \( C_1 \) of \( C. \)

We are going to see that if this is the case, then \( C \cdot C_1 = -1. \) Let us take the following exact sequence:

\[
0 \rightarrow \mathcal{O}_S(C - C_1) \rightarrow \mathcal{O}_S(C) \rightarrow \mathcal{O}_{\mathbb{P}^1}(C \cdot C_1) \rightarrow 0.
\]

By the subsection above, \( h^1(\mathcal{O}_S(C)) = 0. \) Let us see that also \( h^1(\mathcal{O}_S(C - C_1)) = 0. \) Otherwise the system \( |C - C_1| \) would be special and in particular, by [7] there would exist a \(-1\)-curve \( C_2 \) such that \( C_2 \cdot (C - C_1) \leq -2. \) Since \( \mathbb{L} \) is non-special, \( C \cdot C_2 \geq -1 \) and hence \( C_1 \cdot C_2 \geq 1. \) This implies that \( |C_1 + C_2| \) has dimension at least 1, which is impossible since \( C_1 + C_2 \) is contained in the fixed locus of \( \mathbb{L}. \) Since \( h^0(\mathcal{O}_S(C)) = h^0(\mathcal{O}_{\mathbb{P}^1}(C \cdot C_1)) = h^1(\mathcal{O}_S(C - C_1)) = 0, \) which means that \( C \cdot C_1 = -1 \) as claimed before.

Arguing as in the previous subsection, we get \( |C_1| = \mathbb{L}_2(d, m_1, m_2, m^8) \) with \( d \leq 9, \)
\( 0 \leq m_1, m_2 \leq 2, \) and \( 0 \leq m \leq 3. \) An easy computation shows that the only \(-1\)-curve of this form is \( \mathbb{L}_2(1, 1, 1, 0^8) \) and in this case \( C \cdot C_1 = 5. \)

### References


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