A geometrical construction of the oval(s) associated with an \( \alpha \)-flock

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Abstract. It is known, via algebraic methods, that a flock of a quadratic cone in \( \text{PG}(3,q) \) gives rise to a family of \( q+1 \) ovals of \( \text{PG}(2,q) \) and similarly that a flock of a cone over a translation oval that is not a conic gives rise to an oval of \( \text{PG}(2,q) \). In this paper we give a geometrical construction of these ovals and provide an elementary geometrical proof of the construction. Further we also give a geometrical construction of a spread of the \( \text{GQ}_{T_2}(\mathcal{O}) \) for \( \mathcal{O} \) an oval corresponding to a flock of a translation oval cone in \( \text{PG}(3,q) \), previously constructed algebraically.

1 Introduction and definitions

The essence of this paper is a geometrical construction of an oval \( \mathcal{O} \) of \( \text{PG}(2,q) \), \( q \) even, from a flock of a translation oval cone in \( \text{PG}(3,q) \) and a spread of the corresponding \( \text{GQ}_{T_2}(\mathcal{O}) \). This construction, along with a geometrical proof that it does indeed give an oval \( \mathcal{O} \) and a spread of \( T_2(\mathcal{O}) \), can be found in Section 3 and preliminary results required can be found in Section 2. Much of this introduction gives the known algebraic constructions of these objects while in Section 4 it is shown that the geometrical construction we present here is the same as the algebraic one.

An oval \( \mathcal{O} \) of \( \text{PG}(2,q) \) is a set of \( q+1 \) points no three of which are collinear. A line of \( \text{PG}(2,q) \) is called an external line, a tangent line or a secant line of \( \mathcal{O} \) depending on whether it is incident with zero, one or two points of \( \mathcal{O} \), respectively. From this point we assume that \( q \) is even. In the case where \( q \) is even the tangents to \( \mathcal{O} \) are concurrent in a point \( N \) called the nucleus of \( \mathcal{O} \). A hyperoval of \( \text{PG}(2,q) \) is a set of \( q+2 \) points no three collinear. An oval together with its nucleus forms a hyperoval of \( \text{PG}(2,q) \). If an oval \( \mathcal{O} \) has a tangent line \( \ell \) such that there exists a group of \( q \) elations of \( \text{PG}(2,q) \) each element of which has axis \( \ell \) and fixes \( \mathcal{O} \), then \( \mathcal{O} \) is called a translation oval. The line \( \ell \) is called an axis of \( \mathcal{O} \). It was proved by Payne in [5] that each translation oval is of the form \( \mathcal{O}(x) = \{(1,t,t^2) : t \in \text{GF}(q)\} \cup \{(0,0,1)\} \), for some generator \( x \) of \( \text{Aut} \left( \text{GF}(q) \right) \). Note that in the case where \( x : x \mapsto x^2 \), or abusing notation \( x = 2 \), that the translation oval is the classical oval, the non-degenerate conic.

Let \( \mathcal{K} \) be a quadratic cone in \( \text{PG}(3,q) \) with vertex \( V \). A flock \( \mathcal{F} \) of \( \mathcal{K} \) is a set of \( q \)
planes of $\text{PG}(3, q)$ partitioning the points of $\mathcal{H}\backslash\{V\}$. If we suppose that $\mathcal{H}$ is defined by the equation $x_0x_2 = x_1^2$, then following Thas in [7] we may write the flock in the form $\mathcal{F} = \{ \pi_t : t \in \text{GF}(q) \}$ where

$$\pi_t : a_1x_0 + b_1x_1 + c_1x_2 + x_3 = 0.$$  

It follows that $t \mapsto a_t, t \mapsto b_t$ and $t \mapsto c_t$ are permutations of $\text{GF}(q)$. Without loss of generality the elements of the flock may be normalised to

$$\pi_t : f(t)x_0 + t^{1/2}x_1 + ag(t)x_2 + x_3 = 0,$$

for permutations $f$ and $g$ of $\text{GF}(q)$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$ and trace$(a) = 1$. In [3] the authors prove the following theorem concerning flocks of the above form.

**Theorem 1.1.** Each of the sets

$$\left\{ \left( 1, t, \frac{f(x) + asg(x) + s^{1/2}x^{1/2}}{1 + as + s^{1/2}} \right) : x \in \text{GF}(q) \right\} \cup \{(0, 1, 0), (0, 0, 1)\}$$

for $s \in \text{GF}(q)$ and

$$\{(1, t, g(t)) : t \in \text{GF}(q)\} \cup \{(0, 1, 0), (0, 0, 1)\}$$

is a hyperoval of $\text{PG}(2, q)$.

In [3] the set of $q + 1$ functions defining the hyperovals as above is called a herd. In [8] Thas gave a geometrical construction of these hyperovals from the flock (although not a geometrical proof of the construction).

Let $x$ be a generator of $\text{Aut}(\text{GF}(q))$, $q = 2^e$. Following Cherowitzo in [2] define an $x$-cone $\mathcal{K}_x$ of $\text{PG}(3, q)$ to be a cone with point vertex $V$ and base an oval equivalent to $\mathcal{O}(x)$. If $X$ is a point of the base oval on an axis, then the line $\langle X, V \rangle$ is called an axial line of $\mathcal{K}_x$. A flock of $\mathcal{K}_x$, also known as an $x$-flock, is a set of $q$ planes of $\text{PG}(3, q)$ partitioning the points of $\mathcal{K}_x \backslash\{V\}$. If $\mathcal{K}_x$ is defined by the equation $x_1^2 = x_0x_2^{2^{-1}}$ and $\mathcal{F}_x$ a flock of $\mathcal{K}_x$, then similarly to the case of a flock of a quadratic cone we may write the elements of $\mathcal{F}_x$ as

$$\pi_t : f(t)x_0 + t^{1/2}x_1 + ag(t)x_2 + x_3 = 0 \quad \text{for } t \in \text{GF}(q),$$

where $f$ and $g$ are permutations of $\text{GF}(q)$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$ and trace$(a) = 1$. Then Cherowitzo ([2]) proves the following result concerning $x$-flocks.

**Theorem 1.2.** The set $\{(1, t, f(t)) : t \in \text{GF}(q)\} \cup \{(0, 1, 0), (0, 0, 1)\}$ is a hyperoval of $\text{PG}(2, q)$. 
In Section 3 we give a generalisation of a construction in [8] that each pair (axial line of $\mathcal{K}_z$, flock of $\mathcal{K}_z$) gives rise to an oval of $\text{PG}(2, q)$. In the case where $\mathcal{K}_z$ is a quadratic cone it was shown in [8] that in this way a flock gives rise to the $q + 1$ hyperovals of the corresponding herd, while in Section 4 we show that for a general $z$-flock the oval completes to the hyperoval of Theorem 1.2. In this way we have a geometric proof of Theorem 1.1 and Theorem 1.2.

We now consider the Generalized Quadrangle (GQ) $T_2(\mathcal{O})$ of Tits; see [4]. Let $\mathcal{O}$ be an oval in $\text{PG}(2, q)$ and embed $\text{PG}(2, q)$ in $\text{PG}(3, q)$, then $T_2(\mathcal{O})$ is a GQ of order $q$ and is constructed in the following manner. Points are (i) the points of $\text{PG}(3, q) \setminus \text{PG}(2, q)$, (ii) the planes of $\text{PG}(3, q)$ which meet $\text{PG}(2, q)$ in a single point of $\mathcal{O}$ and (iii) a symbol $(\infty)$; lines are (a) the lines of $\text{PG}(3, q)$, not in $\text{PG}(2, q)$, which meet $\text{PG}(2, q)$ in a single point of $\mathcal{O}$, and (b) the points of $\mathcal{O}$; with incidence inherited from $\text{PG}(3, q)$ plus $(\infty)$ is incident with all lines of type (b). Note that $T_2(\mathcal{O})$ is the classical GQ $Q(4, q)$ if and only if $\mathcal{O}$ is a conic; see [6, 3.2.2]. A spread $\mathcal{S}$ of $T_2(\mathcal{O})$ is a set of lines such that each point of $T_2(\mathcal{O})$ is incident with a unique element of $\mathcal{S}$. It follows that $\mathcal{S}$ has size $q^2 + 1$. In [1] the authors show that $\mathcal{S}$ must consist of a point $P$ of $\mathcal{O}$ and the $q^2$ lines not in $\text{PG}(2, q)$ of $q$ oval cones, $\mathcal{K}_X$, $X \in \mathcal{O}\setminus\{P\}$; where $\mathcal{K}_X$ has vertex $X$, contains $P$ and has nuclear line $\langle X, N \rangle$, with $N$ the nucleus of the oval $\mathcal{O}$. The following theorem, in an equivalent form, also appears in [1].

**Theorem 1.3.** Let $\mathcal{O} = \{(t, 1, f(t)) : t \in GF(q)\} \cup \{(0, 0, 1)\}$, with $f(0) = 0$ and $f(1) = 1$, be an oval of $\text{PG}(2, q)$, $q$ even. Embed $\text{PG}(2, q)$ in $\text{PG}(3, q)$ as $x_2 = 0$ and let $z$ be a generator of $\text{Aut}(GF(q))$. Let $\mathcal{K}_t$ be the cone with vertex $(t, 1, 0, f(t))$ and base \{(s^2 + a^2g(t)^2, 0, 1, s) : s \in GF(q)\} \cup \{(0, 0, 0, 1)\}, with trace(a) = 1. Then $\langle 0, 0, 0, 1 \rangle$ plus the $q^2$ lines not in $\text{PG}(2, q)$ of the cones $\mathcal{K}_t$ form a spread of $T_2(\mathcal{O})$ if and only if \{\begin{align*} f(t)x_0 + t^{1/2}x_1 + ag(t)x_2 + x_3 = 0 : t \in GF(q) \end{align*}\} is an $z$-flock of $\mathcal{K}_z : x_1^2 = x_0x_2^{q-1}$, with $g(0) = 0$ and $g(1) = 1$.

In this way the ovals corresponding to an $z$-flock, as in Theorem 1.2, are characterised as those for which the corresponding Tits GQ admits a spread of the form above. Our geometrical construction in Section 3 characterises these ovals in the same way and by attaching coordinates in Section 4 we see that it gives a non-algebraic proof of Theorem 1.3.

Now we state our main theorem.

**Theorem 1.4.** For $z$ a generator of $GF(q)$, $q$ even, let $\mathcal{K}_z$ be a cone in $\text{PG}(3, q)$ over a translation oval equivalent to $\mathcal{D}(z) = \{(1, t, t^2) : t \in GF(q)\} \cup \{(0, 0, 1)\}$. If $\mathcal{F}_z$ is a flock of $\mathcal{K}_z$, then to each pair $(F_2, a)$, where $a$ is an axial line of $\mathcal{K}_z$, there corresponds an oval $\mathcal{O}$ of $\text{PG}(2, q)$. Further, there also corresponds a spread $\mathcal{S}$ of the generalized quadrangle $T_2(\mathcal{O})$ which consists of one point $Y$ of $\mathcal{O}$ and the $q^2$ lines not in $\text{PG}(2, q)$ of $q$ $z$-cones $\mathcal{K}_X$, where $\mathcal{K}_X$ has vertex $X \in \mathcal{O}\setminus\{Y\}$, base oval equivalent to $\mathcal{D}(z)$ and is tangent to $\text{PG}(2, q)$ at the axial line $\langle Y, X \rangle$.

Conversely, if a GQ $T_2(\mathcal{O})$ has such a spread $\mathcal{S}$, then there corresponds an $z$-flock giving rise to the oval $\mathcal{O}$. 
In Section 2 we shall state some basic properties of translation ovals and flocks of translation oval cones which shall be used in the proof of Theorem 1.4 in Section 3. In Section 4 we apply coordinates to the construction in the proof of Theorem 1.4 to show that it gives both Theorem 1.2 and Theorem 1.3.

2 Preliminaries

In this section we give some basic results on translation ovals and $x$-flocks to be used in the proof of our main theorem.

By Payne ([5]) we know that any translation oval of $\text{PG}(2, 2^h)$ is equivalent to an oval of the form $\mathcal{D}(z) = \{(1, t, t^z) : t \in \text{GF}(q)\} \cup \{(0, 0, 1)\}$ with nucleus $(0, 1, 0)$, where $z$ is a generator of Aut($\text{GF}(q)$). From this form it is clear that $\mathcal{D}(z)$ is a conic if and only if $z = 2$. In the case where $\mathcal{D}(z)$ is a conic each tangent to $\mathcal{D}(z)$ is an axis of $\mathcal{D}(z)$ and the group of the conic is transitive on the axes. In the case where $\mathcal{D}(z)$ is not a conic then $\mathcal{D}(z)$ has a unique axis $[1, 0, 0]$. From the canonical form of a translation oval it is also straightforward to see that for a given line $\ell$ of $\text{PG}(2, q)$ and distinct points $P, N$ incident with $\ell$ that there are exactly $q(q - 1)$ ovals equivalent to $\mathcal{D}(z)$ containing $P$ and with nucleus $N$, such that $\ell$ is an axis of the oval. If $R$ is a fixed point of $\text{PG}(2, q) \setminus \ell$, then there are $q(q - 1)$ ovals equivalent to $\mathcal{D}(z)$ with axis $\ell$ and containing the points $P$ and $R$.

Another notion that we shall need is that of compatibility of ovals. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two ovals of $\text{PG}(2, q)$ and let $P$ be a point of $\text{PG}(2, q)$ not on either of the ovals and distinct from their nuclei. Then $\mathcal{O}_1$ and $\mathcal{O}_2$ are compatible at $P$ if they have the same nucleus, they have a point $Q$ in common, the line $\langle P, Q \rangle$ is a tangent to both $\mathcal{O}_1$ and $\mathcal{O}_2$ and every secant line to $\mathcal{O}_1$ on $P$ is external to $\mathcal{O}_2$. As a consequence every external line to $\mathcal{O}_1$ on $P$ is a secant line to $\mathcal{O}_2$. In particular we will need information regarding points of compatibility in the case where $\mathcal{O}_1$ and $\mathcal{O}_2$ are both ovals equivalent to $\mathcal{D}(z)$ with a common axis $\ell$, common nucleus $N$, $\ell \cap \mathcal{O}_1 = \ell \cap \mathcal{O}_2 = \{Q\}$ and such that $\mathcal{O}_2$ is the image of $\mathcal{O}_1$ under an elation with axis $\ell$ and centre $Q$. Without loss of generality we may assume that $\mathcal{O}_1 = \{(1, u, u^z) : u \in \text{GF}(q)\} \cup \{(0, 0, 1)\}$ and that $\mathcal{O}_2 = \{(1, t, t^2 + B) : t \in \text{GF}(q)\} \cup \{(0, 0, 1)\}$ for $B \in \text{GF}(q)$. A point $(0, 1, s)$, $s \neq 0$, on the common axis of $\mathcal{O}_1$ and $\mathcal{O}_2$ is a point of compatibility of $\mathcal{O}_1$ and $\mathcal{O}_2$ if and only if $\text{trace}(B/s^{2z/(z-1)}) = 1$, which has $q/2$ solutions for $s \in \text{GF}(q)$. Hence $\mathcal{O}_1$ and $\mathcal{O}_2$ have $q/2$ points of compatibility on the common axis.

Now consider a cone $\mathcal{K}_z$ in $\text{PG}(3, q)$ with vertex $V$ and base an oval equivalent to $\mathcal{D}(z)$. Let $\ell$ be an axial line of the cone and let $P$ be any point incident with $\ell$ distinct from $V$ and let $\pi$ be any plane of $\text{PG}(3, q)$ not containing $P$. If we project the $q^3 - q^2$ oval sections of $\mathcal{K}_z$ not containing $P$, from $P$ onto $\pi$, then we obtain a one-to-one correspondence between this set and the $q^2(q - 1)$ ovals of $\pi$ equivalent to $\mathcal{D}(z)$ that contain the point $Y = \ell \cap \pi$ and have axis $n = \pi_i \cap \pi$, where $\pi_i$ is the plane tangent to $\mathcal{K}_z$ at $\ell$. Similarly, the $q^2$ oval sections of $\mathcal{K}_z$ containing $P$ are in one-to-one correspondence with the $q^2$ lines of $\pi$ not incident with $Y$. This correspondence is the planar representation of $\mathcal{K}_z$. If we consider a set of $q$ oval sections of $\mathcal{K}_z$ that are mutually tangent at a point of $\langle P, V \rangle \setminus \{P, V\}$, then in the planar representation this set of $q$ ovals is called an axial linear pencil of ovals. Equivalently, such a set of ovals
may be described as the images of an oval equivalent to \( D(\alpha) \) under the group of elations with axis the axis of the oval and centre the point of the oval on the axis.

Now consider a flock \( F = \{\pi_1, \ldots, \pi_q\} \) of \( \mathcal{K}_2 \). Without loss of generality suppose that \( P \in \pi_q \). For \( i = 1, \ldots, q - 1 \) let the projection of the oval \( \pi_i \cap \mathcal{K}_2 \) from \( \pi \) be \( \mathcal{C}_i \) and let \( w \) denote the line \( \pi_q \cap \pi \). Then it follows that in the planar representation of \( \mathcal{K}_2 \) that \( F \) is the set \( \{\mathcal{C}_1, \ldots, \mathcal{C}_{q-1}, w\} \). Thus \( \mathcal{C}_1, \ldots, \mathcal{C}_{q-1}, w \) partition the points of \( \pi \backslash n \), and it also follows that the nuclei of the \( \mathcal{C}_i \) are distinct points of \( n \backslash \{Y\} \) and that the line \( w \) intersects \( n \) in the remaining point of \( n \backslash \{Y\} \). Conversely, any such set \( \{\mathcal{C}_1, \ldots, \mathcal{C}_{q-1}, w\} \) partitioning the points of \( \pi \backslash \{n\} \) corresponds to a flock of \( \mathcal{K}_2 \).

### 3 Proof of Theorem 1.4

Suppose \( \mathcal{F}_2 \) is a flock of \( \mathcal{K}_2 \) and \( a \) an axis of the base oval of \( \mathcal{K}_2 \). If \( V \) is the vertex of \( \mathcal{K}_2 \), then \( \langle V, a \rangle \) contains the axial line \( \ell \) of \( \mathcal{K}_2 \). Then, as in Section 2, if we project the elements of \( \mathcal{F}_2 \) from a non-vertex point \( P \) of \( \ell \) onto a plane \( \pi \), not containing \( P \), we obtain a planar representation \( \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{q-1}, w\} \) of \( \mathcal{F}_2 \). Let the common point of the ovals \( \mathcal{C}_1, \ldots, \mathcal{C}_{q-1} \) be \( Y \), the common axis of the ovals be \( n \) and \( n \cap w = X' \).

Now consider two other planes \( \text{PG}(2, q) \) and \( \zeta \), such that \( \text{PG}(2, q) \cap \pi = n, \pi \cap \text{PG}(2, q) \cap \zeta = \{Y\}, \text{PG}(2, q) \cap \zeta = m \) and \( \pi \cap \zeta = u \). In \( \zeta \) we consider an oval \( \mathcal{C}'_1 \) equivalent to \( D(\alpha) \) such that \( \mathcal{C}'_1 \) has axis \( m \), contains the point \( Y \) on \( m \), and has nucleus \( N \). Let \( \{\mathcal{C}'_1, \mathcal{C}'_2, \ldots, \mathcal{C}'_q\} \) be the axial linear pencil containing \( \mathcal{C}'_1 \) with axis \( m \). The ovals \( \mathcal{C}'_1, \mathcal{C}'_2, \ldots, \mathcal{C}'_q \) partition \( \zeta \backslash m \), and in particular the points of \( u \backslash \{Y\} \). Consequently we may choose indices such that \( \mathcal{C}_i \cap \mathcal{C}'_1 = \{Y, W_i\} \), with \( W_i \in u \) and \( i = 1, 2, \ldots, q - 1 \).

We now show that for each \( i = 1, 2, \ldots, q - 1 \) there is a unique cone containing \( \mathcal{C}_i \) and \( \mathcal{C}'_1 \). Since \( n \) and \( m \) are tangents to \( \mathcal{C}_i \) and \( \mathcal{C}'_1 \) at \( Y \), respectively, it follows that the vertex of any cone containing the two ovals must be in the plane \( \langle n, m \rangle = \text{PG}(2, q) \).

Now there are \( q(q - 1) \) cones containing \( \mathcal{C}_1 \) and with vertex in \( \text{PG}(2, q) \backslash (n \cup m) \), and also \( q(q - 1) \) ovals of \( \pi \) equivalent to \( D(\alpha) \) with axis \( n \) and containing the points \( Y \) and \( W_1 \). Thus, if we can find a group fixing \( \mathcal{C}'_1, Y, W_1 \) as well as acting regularly on both the set of points of \( \text{PG}(2, q) \backslash (n \cup m) \) and the set of ovals of \( \pi \) equivalent to \( D(\alpha) \) with axis \( n \) and containing the points \( Y \) and \( W_1 \), then it follows there must be exactly one cone containing \( \mathcal{C}'_1 \) and such an oval. To show the existence of such a group we (briefly) apply coordinates. Let \( \text{PG}(2, q) : x_2 = 0, \pi : x_3 = 0, \xi : x_1 = 0 \).

We may assume that \( \mathcal{C}'_1 \) has the form \( \{(t^2, 0, 1, t) : t \in \text{GF}(q)\} \cup \{(1, 0, 0, 0)\} \). The required group has elements of the form

\[
\begin{pmatrix}
\lambda^2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & \lambda
\end{pmatrix}
\text{ for } \rho \in \text{GF}(q) \text{ and } \lambda \in \text{GF}(q) \backslash \{0\}.
\]

By the above there is a unique cone \( \mathcal{K}_1 \) containing \( \mathcal{C}_1 \) and \( \mathcal{C}'_1 \) which has vertex \( X_1 \),
say, in $\PG(2,q)$. Similarly we have cones $\mathcal{K}_i, i = 1, 2, \ldots, q - 1$, where $\mathcal{K}_i$ contains $O_i$ and $O'_i$ and has vertex $X_i$. We also define $\mathcal{K}_q$ to be the cone containing $O_q$ and with vertex $X'$. For convenience we will relabel the point $X'$ as $X_q$.

We now show that $X_i, X_j, Y, i \neq j$, are not collinear and that the two cones $\mathcal{K}_i$ and $\mathcal{K}_j, i \neq j$, intersect in exactly $Y$. Without loss of generality we will consider $\mathcal{K}_1$ and $\mathcal{K}_2$. First suppose that $Y, X_1, X_2$ are collinear on a line $o$ (which is necessarily a generator of both $\mathcal{K}_1$ and $\mathcal{K}_2$). Of the $q + 1$ planes of $\PG(3,q)$ on $o$, $\PG(2,q)$ is a tangent plane to both $\mathcal{K}_1$ and $\mathcal{K}_2$ while each of the other $q$ planes contains a second generator of both $\mathcal{K}_1$ and $\mathcal{K}_2$ and so a second point of $\mathcal{K}_1 \cap \mathcal{K}_2$. Hence $|\mathcal{K}_1 \cap \mathcal{K}_2| = q + 1$. Now we consider the planes on the line $m$. The plane $\PG(2,q)$ is tangent to both $\mathcal{K}_1$ and $\mathcal{K}_2$ while each of the other $q$ planes intersects both $\mathcal{K}_1$ and $\mathcal{K}_2$ in an oval equivalent to $D(x)$ with axis $m$, containing the point $Y$ and with nucleus $N$. Two such ovals may intersect in either $0, 1, 2$ or $q + 1$ points. Suppose that there exists a plane $\eta$ distinct from $\PG(2,q)$ on $m$ for which $\eta \cap \mathcal{K}_1 = \eta \cap \mathcal{K}_2 = \emptyset$, $\emptyset$ an oval. Now since $\emptyset$ is the set of common points of $\mathcal{K}_1$ and $\mathcal{K}_2$ it follows that $\eta \cap \emptyset = \emptyset_1 \cap \emptyset_2 = \{Y\}$. Hence the line $\eta \cap \emptyset$ is tangent to both $\mathcal{K}_1$ and $\mathcal{K}_2$ at $Y$ and so must be $n$. This implies that $m, n \subset \eta$ and so $\eta = \PG(2,q)$, a contradiction. It follows that each plane on $m$ distinct from $\PG(2,q)$ contains exactly two points of $\mathcal{K}_1 \cap \mathcal{K}_2, Y$ and one other. However this must also hold for $\xi$, a contradiction. Therefore $Y, X_1, X_2$ are not collinear.

If $N \in \langle X_1, X_2 \rangle$, then $\emptyset_1$ and $\emptyset_2$ have a common nucleus and so it follows that $N, X_1, X_2$ are not collinear. So the line $\langle X_1, X_2 \rangle$ contains a point $P$ of $m \backslash \{Y, N\}$. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are to meet in exactly $Y$, then no line of $\xi$ distinct from $m$ and incident with $P$ can contain a point of both $\emptyset_1'$ and $\emptyset_2'$. Hence $\emptyset_1'$ and $\emptyset_2'$ are compatible at $P$. From this we see that the number of cones containing $\emptyset_2'$, with vertex in $\PG(2,q)$, that meet $\mathcal{K}_1$ in exactly $Y$ is the number of points on $m$ at which $\emptyset_1'$ and $\emptyset_2'$ are compatible, multiplied by $q - 2$ for the possible vertices in $\langle X_1, P \rangle \backslash \{X_1, P\}$ not on $n$, for each such point of compatibility $P$. By Section 2 this is $q(q - 2)/2$. In the planar representation of $\mathcal{K}_2$, this is the same as the number of ovals equivalent to $D(x)$ meeting $\emptyset_1$ in exactly $Y$, containing $W_2$, and with nucleus distinct from that of $\emptyset_1$. It follows that the cones $\mathcal{K}_1$ and $\mathcal{K}_2$ meet in exactly $Y$.

We now show that the cone $\mathcal{K}_q$ and any cone $\mathcal{K}_i, i \in \{1, 2, \ldots, q - 1\}$ intersect in exactly $Y$. If we consider a plane $\pi'$ such that $u \subset \pi'$, but $m, n \not\subset \pi'$, then we have the same situation as above except that $\mathcal{K}_q \cap \pi'$ is an oval and not a line. By choosing $\pi'$ appropriately we see that $\mathcal{K}_q \cap \mathcal{K}_i = \{Y\}$ for $i = 1, 2, \ldots, q - 1$.

Since the cones $\mathcal{K}_i$ intersect pairwise in exactly $Y$ it follows that they partition the points of $\PG(3,q) \backslash \PG(2,q)$.

We now show that the set $\mathcal{O} = \{Y, X_1, X_2, \ldots, X_q\}$ is an oval with nucleus $N$. Consider the three points $X_i, X_j, X_k$ for distinct $i, j, k$ in $\{1, 2, \ldots, q - 1\}$. Suppose that $X_i, X_j, X_k$ are collinear on the line $\ell_{ijk}$. There are $q$ planes on $\ell_{ijk}$ distinct from $\PG(2,q)$, and the $q$ lines of $\mathcal{K}_i \langle Y, X_i \rangle$ lie on these planes with at most two per plane; and similarly for $X_j$ and $X_k$. It follows that there is a plane on $\ell_{ijk}$ which contains a line from at least two of the cones $\mathcal{K}_i, \mathcal{K}_j, \mathcal{K}_k$, which implies two cones intersecting in a point other than $Y$, a contradiction. Hence $X_i, X_j, X_k$ cannot be collinear. Similarly, $X_q, X_j, X_j$ are not collinear for distinct $i, j$ in $\{1, 2, \ldots, q - 1\}$ and $\mathcal{O} = \{Y, X_1, X_2, \ldots, X_q\}$ is an oval. Since the lines $\langle N, X_i \rangle, i = 1, 2, \ldots, q - 1$ and $\langle N, X_q \rangle$ are
the hyperoval completion of \( \mathcal{H}_i, i = 1, 2, \ldots, q \), respectively, and these lines intersect \( n \) in distinct points it follows that \( N \) is the nucleus of the oval \{ \( Y, X_1, X_2, \ldots, X_q \) \). 

If we construct the GQ \( T_2(\mathcal{C}) \) in \( \text{PG}(3, q) \), then the set \( \mathcal{S} = \{ Y \} \cup \{ \mathcal{H}_i \setminus \langle Y, X_i \rangle : i = 1, 2, \ldots, q \} \) is a spread of \( T_2(\mathcal{C}) \), and the cone \( \mathcal{H}_i \) has base oval equivalent to \( \mathcal{S}(x) \) and axial line \( \langle Y, X_i \rangle \).

Conversely, suppose that we have such a spread \( \mathcal{S} \) of \( T_2(\mathcal{C}) \). If we take any plane \( \pi \) on \( Y \), distinct from the plane \( \text{PG}(2, q) \) of \( \mathcal{C} \), that intersects \( \mathcal{C} \) in a secant, then the intersection of the cones of \( \mathcal{S} \) with \( \pi \) yields an \( \alpha \)-flock in the planar representation; if we take any plane \( \xi \) on \( Y \) and \( N \), distinct from \( \text{PG}(2, q) \), then the intersection of the cones of \( \mathcal{S} \) with \( \xi \) yields ovals \( \mathcal{C}_1', \mathcal{C}_2', \ldots, \mathcal{C}_q' \). It is clear that the above construction gives us the oval \( \mathcal{C} \).

Note that this result characterises the ovals \( \mathcal{C} \) that may be constructed from an \( \alpha \)-flock by the existence of the corresponding spread of \( T_2(\mathcal{C}) \). (This result was first proved algebraically in [1].)

4 Algebraic description of \( \mathcal{C} \) and \( \mathcal{S} \)

In this section we add coordinates to the construction of Theorem 1.4 to show that the hyperoval completion of \( \mathcal{C} \) is the same as the hyperoval constructed from an \( \alpha \)-flock by Cherowitzo and that the spread \( \mathcal{S} \) of \( T_2(\mathcal{C}) \) is the same as that constructed by Brown, O’Keefe, Payne, Penttila and Royle. Note that in [8] Thas showed that in the case of a flock of a quadratic cone that the \( q + 1 \) (flock, axis to base oval of cone) pairs gave rise to the \( q + 1 \) herd hyperovals constructed from a flock as formalised in Theorem 1.1.

Adding coordinates as in the proof of Theorem 1.4, let \( \text{PG}(2, q) : x_2 = 0, \pi : x_3 = 0, \bar{\xi} : x_1 = 0 \). Thus \( n : x_2 = x_3 = 0, m : x_1 = x_2 = 0 \) and \( u : x_3 = x_1 = 0 \) with \( Y(1, 0, 0, 0) \).

Let \( \mathcal{H}_2 : x_1^2 = x_0 x_3^{-1} \) and let \( \mathcal{F}_2 \) be a flock of \( \mathcal{H}_2 \). From Section 1 we may assume that \( \mathcal{F}_2 \) has elements \( \pi_2 : f(t)x_0 + t^{1/2}x_1 + ag(t)x_2 + x_3 = 0, t \in \text{GF}(q) \), where \( f \) and \( g \) are permutations such that \( f(0) = g(0) = 0 \) and \( f(1) = g(1) = 1 \) and trace \( (a) = 1 \). Let \( \mathcal{C}_i'' \) denote the oval \( \mathcal{H}_2 \cap \pi_1 \), and so

\[
\mathcal{C}_i'' = \{(s^x, s, 1, f(t) s^x + t^{1/2} s + g(t)) : s \in \text{GF}(q)\} \cup \{(1, 0, 0, f(t))\}
\]

with nucleus \((0, 1, 0, t^{1/2})\). We now choose to project these \( \mathcal{C}_i'' \) from the point \( U = (1, 0, 0, 1) \) on the axial line \( x_1 = x_2 = 0 \) of \( \mathcal{H}_2 \), onto the plane \( \pi_1 \). As \( f(1) = 1 \) the point \( U \) is contained in \( \pi_1 \) and so

\[
\mathcal{C}_1'' \mapsto w : x_3 = x_0 + x_1 + x_2 = 0.
\]

For \( t \neq 1 \)

\[
\mathcal{C}_i'' \mapsto \mathcal{C}_i = \{(1 + f(t)) s^x + t^{1/2} s + ag(t), s, 1, 0) : s \in \text{GF}(q)\} \cup \{Y\}
\]

with nucleus \((t^{1/2}, 1, 0, 0)\). Thus the planar representation of the \( \alpha \)-flock is \( \{\mathcal{C}_t : t \in \text{GF}(q) \setminus \{1\}\} \cup \{w\} \). For \( t \neq 1 \) define \( W_t \) to be the second point (other than \( Y \)) of
on $u$, that is, $W_t = (ag(t), 0, 1, 0)$ and define $W_1$ to be the intersection of $w$ and $u$, that is $W_1 = (1, 0, 1, 0)$.

Next, in the plane $\xi$ we choose the axial linear pencil of ovals equivalent to $\mathcal{D}(x)$ to be

$$\mathcal{C}_B' = \{(rx + aB, 0, 1, r) : r \in GF(q)\} \cup \{Y\}, \quad B \in GF(q), \quad \text{with nucleus } (0, 0, 0, 1).$$

The second point (other than $Y$) of the oval $O_B'$ on $u$ is $(1, 0, 1, 0)$.

For $t \neq 1$ the unique cone on $\mathcal{C}_t$ and $\mathcal{C}_{g(t)}'$ has vertex $t^{1/2}, 1, 0, (1 + f(t))^{1/2})$. Thus by Theorem 1.4 we have that

$$\{(t^{1/2}, 1, 0, (1 + f(t))^{1/2}) : t \in GF(q)\} \cup \{(1, 0, 0, 0)\}$$

is an oval of $\text{PG}(2, q)$ with nucleus $(0, 0, 0, 1)$.

Applying the collineation $x_3' = x_3 + a$ and then the automorphic collineation induced by $x$, the oval is equivalent to

$$\mathcal{C} = \{(t, 1, 0, f(t)) : t \in GF(q)\} \cup \{(1, 0, 0, 0)\} \quad \text{with nucleus } (0, 0, 0, 1).$$

This implies that the hyperoval completion of the oval is indeed the same hyperoval as that in Theorem 1.2.

Now considering the corresponding spread of $T_2(\mathcal{C})$, we see that the cone with vertex $(t, 1, 0, f(t))$ intersects the plane $\xi$ in the oval

$$\{(rx + a^2g(t)x^r, 0, 1, r) : r \in GF(q)\} \cup \{(1, 0, 0, 0)\} \quad \text{with nucleus } (0, 0, 0, 1).$$

This is the same as the spread given in Theorem 1.3.

References

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