Hyperbolic structure on a complement of tori in the 4-sphere

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Abstract. It is well known that many noncompact hyperbolic 3-manifolds are topologically complements of links in the 3-sphere. We extend this phenomenon to dimension 4 by exhibiting an example of a noncompact hyperbolic 4-manifold that is topologically the complement of 5 tori in the 4-sphere. We also exhibit examples of hyperbolic manifolds that are complements of $5n$ tori in a simply-connected 4-manifold with Euler characteristic $2^n$. All the examples are based on a construction of Ratcliffe and Tschantz, who produced 1171 noncompact hyperbolic manifolds with Euler characteristic 1. Our examples are finite covers of the Ratcliffe–Tschantz manifold with the biggest symmetry group.

1 Introduction

Let $\mathbb{H}^n$ be the $n$-dimensional hyperbolic space and let $G$ be a discrete subgroup of $\text{Isom} \mathbb{H}^n$, the isometries of $\mathbb{H}^n$. If $G$ is torsion-free, then $M = \mathbb{H}^n/G$ is a hyperbolic manifold of dimension $n$. In this paper, the term “hyperbolic manifold” will always be used for a manifold that is also complete, noncompact and has finite volume. Such a manifold $M$ is the interior of a compact manifold with boundary $\overline{M}$ (see, for example, [1]). Every boundary component of $\overline{M}$ is a compact flat (Euclidean) manifold, i.e., a manifold of the form $\mathbb{R}^{n-1}/K$, where $K$ is a discrete subgroup of $\text{Isom} \mathbb{R}^{n-1}$, the isometries of $\mathbb{R}^{n-1}$.

We say that $M$ is a (codimension-2) complement in a closed $n$-manifold $N$ if $M = N - A$, where $A$ is a closed $(n - 2)$-submanifold of $N$ that has a tubular neighborhood and has as many components as $\partial \overline{M}$. Typically one would like $N$ to be a familiar manifold, such as $S^n$.

It is a well-known fact (see [9] and [11]) that many hyperbolic 3-manifolds are complements of links in the 3-sphere. The main purpose of this paper is to generalize this phenomenon to dimension 4, that is, to provide an example of a hyperbolic 4-manifold that is a complement inside the 4-sphere and to provide some examples where the hyperbolic manifolds are complements in other simply-connected 4-manifolds.

In [5] we considered the general problem of when $M$ may be thought of as a complement. Let $M = N - A$ be a complement. Then every component of $\partial \overline{M}$ must be