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Hyperbolic structure on a complement of tori in the 4-sphere

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Abstract. It is well known that many noncompact hyperbolic 3-manifolds are topologically complements of links in the 3-sphere. We extend this phenomenon to dimension 4 by exhibiting an example of a noncompact hyperbolic 4-manifold that is topologically the complement of 5 tori in the 4-sphere. We also exhibit examples of hyperbolic manifolds that are complements of $5n$ tori in a simply-connected 4-manifold with Euler characteristic $2n$. All the examples are based on a construction of Ratcliffe and Tschantz, who produced 1171 noncompact hyperbolic manifolds with Euler characteristic 1. Our examples are finite covers of the Ratcliffe–Tschantz manifold with the biggest symmetry group.

1 Introduction

Let $\mathbb{H}^n$ be the n-dimensional hyperbolic space and let G be a discrete subgroup of $\operatorname{Isom}\mathbb{H}^n$, the isometries of $\mathbb{H}^n$. If G is torsion-free, then $M = \mathbb{H}^n/G$ is a hyperbolic manifold of dimension n. In this paper, the term "hyperbolic manifold" will always be used for a manifold that is also complete, noncompact and has finite volume. Such a manifold M is the interior of a compact manifold with boundary $\overline{M}$ (see, for example, [1]). Every boundary component of $\overline{M}$ is a compact flat (Euclidean) manifold, i.e. a manifold of the form $\mathbb{R}^{n-1}/K$, where K is a discrete subgroup of $\operatorname{Isom}\mathbb{R}^{n-1}$, the isometries of $\mathbb{R}^{n-1}$.

We say that M is a (codimension-2) complement in a closed n-manifold N if $M = N - A$, where A is a closed $(n-2)$-submanifold of N that has a tubular neighborhood and has as many components as $\partial\overline{M}$. Typically one would like N to be a familiar manifold, such as $\mathbb{S}^n$.

It is a well-known fact (see [9] and [11]) that many hyperbolic 3-manifolds are complements of links in the 3-sphere. The main purpose of this paper is to generalize this phenomenon to dimension 4, that is, to provide an example of a hyperbolic 4-manifold that is a complement inside the 4-sphere and to provide some examples where the hyperbolic manifolds are complements in other simply-connected 4-manifolds.

In [5] we considered the general problem of when M may be thought of as a complement. Let $M = N - A$ be a complement. Then every component of $\partial\overline{M}$ must be