

Classification of flocks of the quadratic cone over fields of order at most 29

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Dedicated to Adriano Barlotti on the occasion of his 80th birthday

Abstract. We complete the classification of flocks of the quadratic cone in $\text{PG}(3, q)$ for $q \leq 29$, by showing by computer that there are exactly 8 flocks of the quadratic cone in $\text{PG}(3, 19)$, 18 flocks of the quadratic cone in $\text{PG}(3, 23)$, 12 flocks of the quadratic cone in $\text{PG}(3, 25)$, 14 flocks of the quadratic cone in $\text{PG}(3, 27)$, and 28 flocks of the quadratic cone in $\text{PG}(3, 29)$, up to equivalence.

1 Introduction

The study of flocks of finite circle planes can be traced back to an error in Dembowski (1968) [13], where he mistakenly asserted that it was easy to see that a flock of an egglike finite inversive plane is linear. This mistake inspired Thas (1973) [39] who provided a proof for characteristic 2, and then W. F. Orr (1973) [28] provided a proof for odd characteristic in his thesis, published in [29]. Thas went on to consider flocks of Miquelian Minkowski planes, classifying them in Thas (1975) [40] for characteristic 2 and Thas (1990) [42] for fields of order $1 \pmod{4}$, with Bader–Lunardon (1989) [1] completing the proof for fields of order $3 \pmod{4}$. Bonisoli–Korchmaros (1992) [6] provided a different proof and their introduction was understandably optimistic about the remaining Miquelian Laguerre plane case. (Recently Durante–Siciliano [14] gave a beautiful and short new proof.) That the Laguerre case was considerably more complicated was evident by then from constructions in [15], [43], [20], [30], [22], [41], [17], [16], [21], [3], [19], [31], [18]. However, in compensation, the link with generalised quadrangles [41], [23] makes these flocks more interesting. (The links with translation planes are many, with new links via hyperbolic fibrations recently discovered [5], [4].) Since then, it has become clear that complete classification in the Miquelian Laguerre case is extremely difficult, with further constructions appearing in [10], [34], [25], [32], [9], and so attention has turned to small field orders. The previously known classification results are Thas (1987) [41] for fields of orders 2, 3, 4 and De Clerck–Gevaert–Thas (1988) [11] for fields of orders 5, 7, 8 (these are computer-free results); Mylle (1991) [27] for the field of order 9, De Clerck–Herssens (1992) [12] for the fields of orders 11, 16, Penttila–Royle (1998) [35] for the fields of orders 13,

17, Brown–O’Keefe–Payne–Penttilä–Royle [7] for the field of order 32 (these are computer-based results), see Theorem 2.6. Here we add the fields of orders 19, 23, 25, 27 and 29 to the list, finding that there are exactly 8 flocks of the quadratic cone in $\text{PG}(3, 19)$ (Corollary 4.7), 18 flocks of the quadratic cone in $\text{PG}(3, 23)$ (Corollary 5.8), 12 flocks of the quadratic cone in $\text{PG}(3, 25)$ (Corollary 6.3), 14 flocks of the quadratic cone in $\text{PG}(3, 27)$ (Corollary 7.3), and 28 flocks of the quadratic cone in $\text{PG}(3, 29)$ (Corollary 8.3), up to equivalence, with these being computer-based results.

Our methods involve the approach of Penttilä–Royle (1998) [35] via BLT-sets, made possible by the results of Bader–Lunardon–Thas (1990) [3]. In this setting our results are that there are 5 BLT-sets of $Q(4, 19)$ (Theorem 4.6), 9 BLT-sets of $Q(4, 23)$ (Theorem 5.7), 6 BLT-sets of $Q(4, 25)$ (Theorem 6.2), 6 BLT-sets of $Q(4, 27)$ (Theorem 7.2), and 9 BLT-sets of $Q(4, 29)$ (Theorem 8.2), up to equivalence. They were all constructed previously, see [35] for orders 19, 23 and 25, and [24] for orders 27 and 29 (or [25] for order 27). These papers also explain how many flocks arise from each BLT-set.

In [35], the time-consuming aspect is isomorph rejection. A hybrid algorithm, involving orderly generation of partial BLT-sets is used up to some chosen cut-off point, and then this list is used to generate a list, with redundancies, of all BLT-sets, after which isomorph rejection must take place. (For orderly algorithms, see [38], [26], [37].) While general improvements to isomorph rejection lie beyond our abilities, we are able to use some happy circumstances to improve the efficiency of isomorph rejection for BLT-sets. These happy circumstances include the availability of fast algorithms for dealing with permutation groups in MAGMA [8], including the availability of the group $\text{P}\Gamma\text{O}(5, q)$ as a permutation group on the points of $Q(4, q)$, for the relevant field orders q , and the fact that none of the presently known BLT-sets are rigid. Taking a group-theoretic approach, equivalence implies conjugacy of stabilisers; and two BLT-sets with the same stabiliser if equivalent are equivalent by an element normalising that stabiliser. These observations lead to a reduction for isomorph rejection of non-rigid BLT-sets to stabiliser, conjugacy and normaliser calculations. Now isomorph rejection is iterated, so its replacement by a cheaper equivalent improves efficiency even if the proof of equivalence is expensive, as the proof need not be iterated. Thus, rather than perform conjugacy calculations many times, we characterise the known BLT-sets of $Q(4, q)$, for each relevant value of the field order q , by an ad hoc property $P(q)$ of their stabilisers. It should be emphasised that the availability of efficient algorithms for dealing with permutation groups is the reason for our choice of this approach.

The total computing time used was about 5 weeks on a Pentium clone. A lot of memory was needed—about 450 Meg of RAM. To proceed further, even only to the field of order 31, would require considerable computing time (our best estimate is 8 years), and may well run out of memory, as the most we have access to is 512 Meg of RAM. It is also the belief of the authors that rigid BLT-sets exist in $Q(4, q)$ for some q in $\{37, 41, 43, 47, 49\}$, which would cause our methods to fail. (As yet, we have failed to find any rigid BLT-sets.) In contrast, we believe that the list of 8 BLT-sets of $Q(4, 31)$ in [24] is probably complete.

2 Preliminaries

A *flock* of a quadratic cone K in $\text{PG}(3, q)$ is a set of q planes meeting K in sections which partition K , minus its vertex. Two flocks of K are *equivalent* if there is a collineation fixing K and taking the first flock to the second. A flock is *linear* if the planes all share a line. A *partial BLT-set* of $\mathcal{Q}(4, q)$ is a set B of points, such that for any point P of $\mathcal{Q}(4, q)$ not in B , the number of points of B collinear with P is at most 2. Two partial BLT-sets are *equivalent* if they are in the same orbit of the automorphism group $\text{P}\Gamma\text{O}(5, q)$. A partial BLT-set of $\mathcal{Q}(4, q)$ has size at most $q + 1$; if equality occurs it is a *BLT-set*.

Theorem 2.1 ([3]). *Every flock of the quadratic cone in $\text{PG}(3, q)$, q odd, determines a BLT-set of $\mathcal{Q}(4, q)$. Conversely, given a BLT-set B of $\mathcal{Q}(4, q)$, q odd, and a point P of B , there arises a flock of the quadratic cone of $\text{PG}(3, q)$. Moreover, equivalent flocks give equivalent BLT-sets, and conversely, two flocks arising from the points P and Q of a BLT-set B are equivalent if and only if P and Q lie in the same orbit of the stabiliser of B in $\text{P}\Gamma\text{O}(5, q)$.*

Lemma 2.2. *For q odd, $\mathcal{Q}(4, q)$ contains a unique partial BLT-set of size 3, up to equivalence.*

Proof. By Witt's theorem, up to equivalence there is a unique plane whose polar is an external line. Again, by Witt's theorem, the stabiliser of this plane is 3-transitive on the points of $\mathcal{Q}(4, q)$ in this plane. Hence $\text{P}\Gamma\text{O}(5, q)$ is transitive on partial BLT-sets of size 3.

The known infinite families of BLT-sets are the classical BLT-sets associated with the linear flocks, the Fisher–Thas–Walker BLT-sets [15], [43] for fields of order congruent to 2 modulo 3, the Fisher BLT-sets [15] (see also [34]), the Kantor semifield BLT-sets [22] for field of non-prime order, the Kantor monomial BLT-sets [22] for fields of order congruent to 2 or 3 modulo 5, the Ganley BLT-sets [17], [16] for fields of characteristic 3, the Kantor likeable BLT-sets [17], [21] for fields of characteristic 5, the Mondello BLT-sets [34] for fields of order congruent to 1 or 4 modulo 5, and the Law–Penttila BLT-sets [25] for fields of characteristic 3. When there is a unique flock arising from the BLT-set, we give it the same name, except for the linear flocks arising from the classical BLT-sets (this covers the Fisher–Thas–Walker, Fisher, Kantor semifield and Mondello cases [3], [33], [34]). The presently known sporadic BLT-sets are surveyed in [24]. Notation for those we need to refer to will be fixed in each appropriate section. The known infinite families of flocks of the quadratic cone in characteristic 2 are the linear flocks, the Fisher–Thas–Walker flocks [15], [43] for fields of non-square order, the two classes of Payne flocks [30], [31] for fields of non-square order, the Subiaco flocks [10], and the Adelaide flocks [9] for fields of square order. No sporadic flocks of the quadratic cone in characteristic 2 are presently known.

The following theorem is a restatement of a theorem of [41].

Theorem 2.3. *A BLT-set of $Q(4, q)$ contained in a hyperplane is classical or Kantor semifield.*

The following theorem is a restatement of a theorem of [33].

Theorem 2.4. *A BLT-set meeting a classical BLT-set in at least half of its points is classical or Fisher.*

Lemma 2.5. *A BLT-set of $Q(4, q)$ with a stabiliser in $\text{PGO}(5, q)$ with Sylow r -subgroup R of order $> q + 1$, where r is an odd prime divisor of $q + 1$, is classical or Fisher.*

Proof. The only orbits of R of length $< r^2$ on points of $Q(4, q)$ are those of points on 2 planes fixed by R , since a Sylow r -subgroup of $\text{PGO}(5, q)$ fixes two planes (which each meets $Q(4, q)$ in the points of a classical BLT-set) and acts semiregularly on the points of $Q(4, q)$ outside those planes. Hence a BLT-set B admitting R , being made up of orbits of length $< r^2$, is contained in the union of these two planes. Since these planes are classical BLT-sets, it follows that B must contain at least half of the points of a classical BLT-set. So, by Theorem 2.4, it follows that B is classical or Fisher.

Theorem 2.6. (i) [41] *All flocks of the quadratic cone of $\text{PG}(3, 2)$ are linear.*

(ii) [41] *All flocks of the quadratic cone of $\text{PG}(3, 3)$ are linear.*

(iii) [41] *All flocks of the quadratic cone of $\text{PG}(3, 4)$ are linear.*

(iv) [11] *All flocks of the quadratic cone of $\text{PG}(3, 5)$ are linear or Fisher, up to equivalence.*

(v) [11] *All flocks of the quadratic cone of $\text{PG}(3, 7)$ are linear or Fisher, up to equivalence.*

(vi) [11] *All flocks of the quadratic cone of $\text{PG}(3, 8)$ are linear or Fisher–Thas–Walker, up to equivalence.*

(vii) [27] *All flocks of the quadratic cone of $\text{PG}(3, 9)$ are linear or Fisher or Kantor semifield, up to equivalence.*

(viii) [12] *All flocks of the quadratic cone of $\text{PG}(3, 11)$ are linear or Fisher or Fisher–Thas–Walker or Mondello, up to equivalence.*

(ix) [35] *All flocks of the quadratic cone of $\text{PG}(3, 13)$ are linear or Fisher or one of two flocks arising from the Kantor monomial BLT-set, up to equivalence.*

(x) [12] *All flocks of the quadratic cone of $\text{PG}(3, 16)$ are linear or Subiaco, up to equivalence.*

(xi) [35] *All flocks of the quadratic cone of $\text{PG}(3, 17)$ are linear or Fisher or Fisher–Thas–Walker or one of two flocks arising from the Kantor monomial BLT-set or one of two flocks arising from the sporadic BLT-set of De Clerck–Herssens (1992) [12], or one of two flocks arising from the sporadic BLT-set of Penttilä–Royle (1998) [35], up to equivalence.*

- (xii) [7] *All flocks of the quadratic cone of $\text{PG}(3, 32)$ are linear, Fisher–Thas–Walker, or one of the two classes of Payne or Subiaco, up to equivalence.*

3 The algorithm

For $q = 19, 23, 25, 27$ and 29 , all partial BLT-sets of size 5 of $Q(4, q)$ are generated, using an orderly algorithm. Then, for each partial BLT-set B of size 5 a graph Γ is formed, with vertices the points P of $Q(4, q)$ such that $X \cup \{P\}$ is a partial BLT-set, and edges $\{P, Q\}$ such that $X \cup \{P, Q\}$ is a partial BLT-set. The BLT-sets of $Q(4, q)$ containing B are precisely the sets $B \cup C$, where C is a clique of Γ of size $q - 4$, and this is the size of the largest clique in Γ . These cliques are calculated (using the inbuilt AllCliques command in MAGMA), and each is tested to see if property $P(q)$ holds.

Independently, algorithms are run to show if a BLT-set of $Q(4, q)$ satisfies $P(q)$ then it is known, for $q = 19, 23, 25, 27$ and 29 .

4 Classification of BLT-sets in $Q(4, 19)$

Here the definitive work on construction is [35], where 5 BLT-sets of $Q(4, 19)$ are listed, namely, the classical and Fisher BLT-sets ([15]), and three new ones with groups of orders 40, 20 and 16. The one with the group of order 40 is a member of the Mondello family of [34]; the other two we shall refer to as PR20 and PR16.

The ad hoc hypothesis $P(19)$ used to characterise these BLT-sets B of $Q(4, 19)$ is on the stabiliser of B in $\text{PGO}(5, 19)$: that it has order divisible by 5 or is isomorphic to $D_8 \times C_2$ and has 2 elements with no fixed points, 2 elements with 2 fixed points, 5 elements with 20 fixed points, 3 elements with 22 fixed points and 3 elements with 362 fixed points (in the action on the points of $Q(4, 19)$). We call the latter alternative for the stabiliser being *of PR16-type*.

Lemma 4.1. *A BLT-set of $Q(4, 19)$ with group of order divisible by 25 is classical or Fisher, up to equivalence.*

Proof. Apply Lemma 2.5, with $r = 5$.

Lemma 4.2. *A BLT-set of $Q(4, 19)$ with a group of order divisible by 5 is classical, Fisher, Mondello or PR20, up to equivalence.*

Proof. We run software in MAGMA to verify the assertions below.

(i) There are 3 conjugacy classes of subgroups of $\text{PGO}(5, 19)$ of order 5: one H_1 with 20 fixed points, two with 0 fixed points (one H_2 with centraliser in $\text{PGO}(5, 19)$ of order 400, the other H_3 with centraliser in $\text{PGO}(5, 19)$ of order 136 800).

(ii) H_1 leads only to BLT-sets with group of order divisible by 25.

(iii) H_2 leads only to BLT-sets with group of order divisible by 25 and BLT-sets with a group of order 20. All the BLT-sets with a group of order 20 arising are equivalent under the normaliser of H_2 in $\text{PGO}(5, 19)$.

(iv) H_3 leads to both BLT-sets with group of order divisible by 25 and BLT-sets

with group of order 40. All the BLT-sets with a group of order 40 arising are equivalent under the normaliser of H_3 in $\text{PGO}(5, 19)$. In conclusion, either the BLT-set has a group of order divisible by 25, in which case, by Lemma 4.1, it is classical or Fisher, or it has a group of order 40 or a group of order 20. If it has a group of order 40, then it is uniquely determined, so it is Mondello. If it has a group of order 20, then it is uniquely determined, so it is PR20.

Lemma 4.3. *There is a unique conjugacy class of subgroups of $\text{PGO}(5, 19)$ of PR16-type.*

Proof. We apply the Subgroups command in MAGMA to a Sylow 2-subgroup of $\text{PGO}(5, 19)$, and check the order distributions and fixed point distributions for the resulting subgroups of order 16.

Lemma 4.4. *There is a unique BLT-set of $Q(4, 19)$, up to equivalence, whose group is of PR16-type. This unique BLT-set is PR16.*

Proof. Taking the orbits of a representative H of this unique conjugacy class (Lemma 4.3), calculate the orbits that form partial BLT-sets and the unions of orbits that form BLT-sets by computer. There arise two classical BLT-sets and 2 with groups of order 16. The latter fall into a single orbit under the normaliser of H in $\text{PGO}(5, 19)$, so up to equivalence, there is a unique result. It is easy to verify that PR16 has these properties.

Theorem 4.5. *A BLT-set of $Q(4, 19)$ is equivalent to a classical, Fisher, Mondello, PR20 or PR16 BLT-set if and only if $P(19)$ holds.*

Proof. Combine Lemmas 4.2 and 4.4.

Theorem 4.6. *The only BLT-sets of $Q(4, 19)$ are the classical, Fisher, Mondello, PR20 and PR16 BLT-sets, up to equivalence.*

Corollary 4.7. *There are exactly 8 flocks of the quadratic cone in $\text{PG}(3, 19)$, up to equivalence.*

Proof. By [35], each of the classical, Fisher, Mondello and PR20 BLT-sets of $Q(4, 19)$ leads to a single flock of the quadratic cone in $\text{PG}(3, 19)$, while the BLT-set PR16 of $Q(4, 19)$ leads to 4 inequivalent flocks of the quadratic cone in $\text{PG}(3, 19)$. The result follows by Theorem 2.1.

5 Classification of BLT-sets in $Q(4, 23)$

Here the definitive work on construction is [35], where 9 BLT-sets of $Q(4, 23)$ are listed, namely, the classical, Fisher [15], Fisher–Thas–Walker [15], [43], Kantor monomial [22], De Clerck–Herssens [12] with a group of order 72, which we shall refer to

as DCH72, and four new ones with groups of orders 1152, 24, 16 and 6, which we shall refer to as PR1152, PR24, PR16 and PR6.

The ad hoc hypothesis P(23) used to characterise these BLT-sets B of $Q(4, 23)$ is on the stabiliser of B in $\text{PGO}(5, 23)$: that it has order divisible by 5 or is isomorphic to $D_8 \times C_2$ and has 2 elements with no fixed points, 2 elements with 2 fixed points, 5 elements with 22 fixed points, 3 elements with 26 fixed points and 3 elements with 530 fixed points (in the action on the points of $Q(4, 23)$). We call the latter alternative for the stabiliser *being of PR16-type*.

Lemma 5.1. *A BLT-set of $Q(4, 23)$ with group of order divisible by 9 is classical, Fisher, PR1152, or DCH72.*

Proof. We run software in MAGMA to verify the assertions that follow.

(i) A Sylow 3-subgroup S of $\text{PGO}(5, 23)$ has 320 orbits that are partial BLT-sets of size 9 and 16 that are partial BLT-sets of size 3.

(ii) The normaliser N of S in $\text{PGO}(5, 23)$ has 3 orbits on orbits of S of size 9 that are partial BLT-sets, with representatives O_1, O_2, O_3 . Only O_1 has the property that there are other orbits X of S of length 9 such that $O_1 \cup X$ is a partial BLT-set. There are 2 such orbits X_1 and X_2 and each has the property that $O \cup X_i$ can be completed to a unique BLT-set B_i by the addition of orbits of S of length 3. B_1 has a group of order 1152 and is not Fisher. B_2 has a group of order 72.

(iii) N has only one orbit on orbits of S that are partial BLT-sets of size 3, with representative Y . There are no BLT-sets comprised of 5 orbits of S of length 3 and 1 of length 9. There are 3 BLT-sets containing Y comprised of 8 orbits of S of length 3, two of which are Fisher and one classical.

In conclusion, since only one BLT-set with a group of order 1152 arose which is not Fisher, it must be PR1152, and since only one BLT-set with a group of order 72 arose, it must be DCH72.

Lemma 5.2. *A BLT-set of $Q(4, 23)$ with a group of order divisible by 3 is classical, Fisher, Fisher–Thas–Walker, PR1152, DCH72, PR24 or PR6.*

Proof. We run software in MAGMA to verify the assertions that follow.

(i) There are 2 conjugacy classes of subgroups of $\text{PGO}(5, 23)$ of order 3: one H_1 with 0 fixed points, the other H_2 with 24 fixed points.

(ii) H_1 has 4048 orbits of length 3 that are partial BLT-sets, which fall into one orbit under the normaliser N of H_1 in $\text{PGO}(5, 23)$, with representative Y . We calculate all BLT-sets on Y that are a union of H_1 -orbits, obtaining many with a group of order divisible by 9, 24 with a group of order 12 144 with representative B_1 , 192 with a group of order 24 with representative B_2 and 192 with a group of order 6 with representative B_3 . N has order 582 912, and meets the stabiliser of B_1 in a group of order 48, the stabiliser of B_2 in a group of order 6 and contains the stabiliser of B_3 . Hence the orbit of B_1 under N consists of 12 144 BLT-sets, the orbit of B_2 under N consists of 97 152 BLT-sets and the orbit of B_3 under N consists of 97 152 BLT-sets. Since N is transitive on H_1 -orbits of length 3 that are partial BLT-sets, each of them lies on a constant number of BLT-sets in each of these orbits. That number is

$12\,144.8/4\,048 = 24$ for the N -orbit of B_1 , $97\,152.8/4\,048 = 192$ for both the N -orbit of B_2 and the N -orbit of B_3 . Thus all the BLT-sets arising with a group of order 12 144 are equivalent under N , all the BLT-sets arising with a group of order 24 are equivalent under N , and all the BLT-sets arising with a group of order 6 are equivalent under N .

(iii) H_2 leads only to BLT-sets with a group of order divisible by 9.

In conclusion, either the BLT-set has a group of order divisible by 9, in which case, by Lemma 5.1, it is classical, Fisher, PR1152 or DCH72 or it has a group of order 12 144 or a group of order 24 or a group of order 6. If it has a group of order 12 144, then it is uniquely determined, so it is Fisher–Thas–Walker. If it has a group of order 24, then it is uniquely determined, so it is PR24. If it has a group of order 6, then it is uniquely determined, so it is PR6.

Lemma 5.3. *A BLT-set of $Q(4, 23)$ with a group of order divisible by 11 is classical, Fisher–Thas–Walker, or Kantor monomial.*

Proof. We run software in MAGMA to verify the assertions that follow.

(i) There are four conjugacy classes of subgroups of $\text{PGO}(5, 23)$ of order 11: one H_1 with 48 fixed points, another H_2 with 26 fixed points, two H_3 and H_4 with 4 fixed points.

(ii) H_1 does not stabilise any BLT-sets.

(iii) H_2 has 506 orbits of length 11 that are partial BLT-sets, which fall into one orbit under the normaliser N of H_2 in $\text{PGO}(5, 23)$, with representative Y . We calculate all BLT-sets on Y that are a union of H_2 -orbits, finding just one, which is classical.

(iv) H_3 has 44 orbits of length 11 that are partial BLT-sets, which fall into one orbit under the normaliser N of H_3 in $\text{PGO}(5, 23)$, with representative Y . We calculate all BLT-sets on Y that are a union of H_3 -orbits, finding just one, with a group of order 44.

(v) H_4 has 44 orbits of length 11 that are partial BLT-sets, which fall into one orbit under the normaliser N of H_4 in $\text{PGO}(5, 23)$, with representative Y . We calculate all BLT-sets on Y that are a union of H_4 -orbits, finding just one, with a group of order 12 144.

Since the Fisher–Thas–Walker and Kantor monomial BLT-sets have to arise here as they have groups of order 12 144 and 44, respectively, the BLT-set in (iv) is Kantor monomial and that in (v) is Fisher–Thas–Walker.

Lemma 5.4. *There is a unique conjugacy class of subgroups of $\text{PGO}(5, 23)$ of PR16-type.*

Proof. We apply the Subgroups command in MAGMA to a Sylow 2-subgroup of $\text{PGO}(5, 23)$, and check the order distributions and fixed point distributions for the resulting subgroups of order 16.

Lemma 5.5. *There is a unique BLT-set of $Q(4, 23)$, up to equivalence, whose group is of PR16-type. This unique BLT-set is PR16.*

Proof. Taking the orbits of a representative H of this unique conjugacy class (Lemma 5.4), calculate the orbits that form partial BLT-sets and the unions of orbits that form BLT-sets by computer. There arise two classical BLT-sets and 2 with groups of order 16. The latter fall into a single orbit under the normaliser of H in $\text{PGO}(5, 23)$, so up to equivalence, there is a unique result. It is easy to verify that PR16 has these properties.

Theorem 5.6. *A BLT-set of $Q(4, 23)$ is equivalent to a classical, Fisher, Mondello, PR20 or PR16 BLT-set if and only if $P(23)$ holds.*

Proof. Combine Lemmas 4.2 and 4.4.

Theorem 5.7. *The only BLT-sets of $Q(4, 23)$ are the classical, Fisher, Fisher–Thas–Walker, Kantor monomial, DCH72, PR1152, PR24, PR16 and PR6 BLT-sets, up to equivalence.*

Corollary 5.8. *There are exactly 18 flocks of the quadratic cone in $\text{PG}(3, 23)$, up to equivalence.*

Proof. By [35], each of the classical, Fisher, Fisher–Thas–Walker, PR1152 and PR24 BLT-sets of $Q(4, 23)$ leads to a single flock of the quadratic cone in $\text{PG}(3, 23)$, while the BLT-sets DCH72 and Kantor monomial of $Q(4, 23)$ each lead to 2 inequivalent flocks of the quadratic cone in $\text{PG}(3, 23)$, the BLT-set PR16 leads to 4 inequivalent flocks of the quadratic cone in $\text{PG}(3, 23)$, and the BLT-set PR6 leads to 5 inequivalent flocks of the quadratic cone in $\text{PG}(3, 23)$. The result follows by Theorem 2.1.

6 Classification of BLT-sets in $Q(4, 25)$

Here the definitive work on construction is [35], where 6 BLT-sets of $Q(4, 25)$ are listed, namely, the classical and Fisher BLT-sets ([15]), Kantor semifield BLT-sets ([22]), Kantor likeable BLT-sets ([21], [17]), and two new ones with groups of orders 16 and 8, which we shall refer to as PR16 and PR8. In order to avoid being repetitive, we shall abbreviate the treatment of each case from now on. The ad hoc hypothesis $P(25)$ used to characterise these BLT-sets B of $Q(4, 25)$ is on the stabiliser of B in $\text{PGO}(5, 25)$: that it has order divisible by 13 or contains an element of order 4 that either is in $\text{PGO}(5, 25)$ and has 4 fixed points or is not in $\text{PGO}(5, 25)$ and has 8 fixed points.

Theorem 6.1. *A BLT-set of $Q(4, 25)$ is equivalent to a classical, Fisher, Kantor semifield, Kantor likeable, PR16 or PR8 BLT-set if and only if $P(25)$ holds.*

Proof. By computer, broken down into the three cases. Conjugacy results are necessary for the appropriate subgroups of order 4.

Theorem 6.2. *Any BLT-set of $Q(4, 25)$ is equivalent to a classical, Fisher, Kantor semifield, Kantor likeable, PR16 or PR8 BLT-set.*

Corollary 6.3. *There are exactly 12 flocks of the quadratic cone in $\text{PG}(3, 25)$, up to equivalence.*

Proof. By [35], each of the classical, Fisher and Kantor semifield BLT-sets of $Q(4, 25)$ leads to a single flock of the quadratic cone in $\text{PG}(3, 25)$, while the Kantor likeable BLT-set of $Q(4, 25)$ leads to 2 inequivalent flocks of the quadratic cone in $\text{PG}(3, 25)$, the BLT-set PR16 leads to 3 inequivalent flocks of the quadratic cone in $\text{PG}(3, 25)$, and the BLT-set PR8 leads to 4 inequivalent flocks of the quadratic cone in $\text{PG}(3, 23)$. The result follows by Theorem 2.1.

7 Classification of BLT-sets in $Q(4, 27)$

Here the definitive work on construction is [24], where 6 BLT-sets of $Q(4, 27)$ are listed, namely, the classical and Fisher BLT-sets [15], Kantor semifield BLT-sets [22], Ganley BLT-sets [16], [17], Kantor monomial BLT-sets [22] and one new one with a group of order 6, now a member of the Law–Penttila family [25].

The ad hoc hypothesis P(27) used to characterise these BLT-sets B of $Q(4, 27)$ is on the stabiliser of B in $\text{PGO}(5, 27)$: that it has an element of order 3 that has 40 fixed points.

Theorem 7.1. *A BLT-set of $Q(4, 27)$ is equivalent to a classical, Fisher, Kantor semifield, Ganley, Kantor monomial or Law–Penttila BLT-set if and only if P(27) holds.*

Theorem 7.2. *Any BLT-set of $Q(4, 27)$ is equivalent to a classical, Fisher, Kantor semifield, Ganley, Kantor monomial or Law–Penttila BLT-set.*

Corollary 7.3. *There are exactly 14 flocks of the quadratic cone in $\text{PG}(3, 27)$, up to equivalence.*

Proof. By [24], each of the classical, Fisher and Kantor semifield BLT-sets of $Q(4, 27)$ leads to a single flock of the quadratic cone in $\text{PG}(3, 27)$, while the Kantor monomial and Ganley BLT-sets of $Q(4, 27)$ each lead to 2 inequivalent flocks of the quadratic cone in $\text{PG}(3, 27)$, and the Law–Penttila BLT-set leads to 7 inequivalent flocks of the quadratic cone in $\text{PG}(3, 27)$. The result follows by Theorem 2.1.

8 Classification of BLT-sets in $Q(4, 29)$

Here the definitive work on construction is [24], where 9 BLT-sets of $Q(4, 29)$ are listed, namely, the classical and Fisher BLT-sets [15], the Fisher–Thas–Walker BLT-sets [15], [43], the Mondello BLT-sets [34], and five new ones with groups of orders 720, 48, 8, 6 and 3, which we will refer to as LP720, LP48, LP8, LP6 and LP3.

The ad hoc hypothesis P(29) used to characterise these BLT-sets B of $Q(4, 29)$ is on the stabiliser of B in $\text{PGO}(5, 29)$: that it has an element of order 3 that has no fixed points or an element of order 5 that has no fixed points and a centraliser of order 730 800 or has a dihedral group of order 8 with 2 elements with 2 fixed points, 4 elements with 30 fixed points and 1 element with 32 fixed points.

Theorem 8.1. *A BLT-set of $Q(4, 29)$ is equivalent to a classical, Fisher, Fisher–Thas–Walker, Mondello, LP720, LP48, LP8, LP6 or LP3 BLT-set if and only if P(29) holds.*

Theorem 8.2. *Any BLT-set of $Q(4, 29)$ is equivalent to a classical, Fisher, Fisher–Thas–Walker, Mondello, LP720, LP48, LP8, LP6 or LP3 BLT-set.*

Corollary 8.3. *There are exactly 28 flocks of the quadratic cone in $PG(3, 29)$, up to equivalence.*

Proof. By [24], each of the classical, Fisher, Fisher–Thas–Walker, Mondello and LP720 BLT-sets of $Q(4, 29)$ leads to a single flock of the quadratic cone in $PG(3, 29)$, while the BLT-set LP48 of $Q(4, 29)$ leads to 2 inequivalent flocks of the quadratic cone in $PG(3, 29)$, the BLT-set LP8 of $Q(4, 29)$ leads to 5 inequivalent flocks of the quadratic cone in $PG(3, 29)$, the BLT-set LP6 of $Q(4, 29)$ leads to 6 inequivalent flocks of the quadratic cone in $PG(3, 29)$, and the BLT-set LP3 of $Q(4, 29)$ leads to 10 inequivalent flocks of the quadratic cone in $PG(3, 29)$. The result follows by Theorem 2.1.

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