

The classification of SPG-systems of index 2

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(Communicated by H. Van Maldeghem)

Abstract. In [4] Thas introduced the concept of an SPG-system of a polar space for every $n \in \mathbb{N}_0$ and classified these systems for $n \in \{1, 2\}$. We provide a classification for $n = 3$ in the nonsingular case and for all n in the singular case.

1 Introduction and definitions

Let $Q(2n + 2, q)$, $n \geq 1$, be a nonsingular quadric of $\text{PG}(2n + 2, q)$. An *SPG-system* of $Q(2n + 2, q)$ is a set τ of $(n - 1)$ -dimensional totally singular subspaces of $Q(2n + 2, q)$ such that the elements of τ on any nonsingular elliptic quadric $Q^-(2n + 1, q) \subset Q(2n + 2, q)$ form a spread of $Q^-(2n + 1, q)$.

Let $Q^+(2n + 1, q)$, $n \geq 1$, be a nonsingular hyperbolic quadric of $\text{PG}(2n + 1, q)$. An *SPG-system* of $Q^+(2n + 1, q)$ is a set τ of $(n - 1)$ -dimensional totally singular subspaces of $Q^+(2n + 1, q)$ such that the elements of τ on any nonsingular quadric $Q(2n, q) \subset Q^+(2n + 1, q)$ constitute a spread of $Q(2n, q)$.

Let $H(2n + 1, q^2)$, $n \geq 1$, be a nonsingular Hermitian variety of $\text{PG}(2n + 1, q^2)$. An *SPG-system* of $H(2n + 1, q^2)$ is a set τ of $(n - 1)$ -dimensional totally singular subspaces of $H(2n + 1, q^2)$ such that the elements of τ on any nonsingular Hermitian variety $H(2n, q^2) \subset H(2n + 1, q^2)$ constitute a spread of $H(2n, q^2)$.

Let P be a singular polar space with ambient space $\text{PG}(d, q)$, having as radical the point x . Assume that the projective index of P is n , with $n \geq 1$, that is, n is the dimension of the maximal totally singular subspaces on P . An *SPG-system* of P is a set τ of $(n - 1)$ -dimensional totally singular subspaces of P , not containing x , such that the elements of τ which are (maximal) totally singular for the polar subspace P' of P induced by any $\text{PG}(d - 1, q) \subset \text{PG}(d, q)$ not containing x constitute a spread of P' . (Note that $P' \cong P/\{x\}$.)

In all cases we will call the dimension $n - 1$ of the elements of τ *the index* of the SPG-system. Note that an SPG-system τ of a polar space $P \in \{Q(4, q), Q^+(3, q), H(3, q^2)\}$ (here τ is of index 0) is the set of points on P . If P is a singular polar space of projective index 1 with radical $\{x\}$, then τ is the point set of $P \setminus \{x\}$.

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In [4] Thas constructed new and old classes of semipartial geometries from these systems. However it would lead us too far to give the explicit construction here and so we will restrict ourselves to state the most important results of [4] concerning the existence of SPG-systems.

Theorem 1.1 ([4]). *If τ is an SPG-system of the nonsingular polar space $P \in \{Q(2n + 2, q), Q^+(2n + 1, q), H(2n + 1, q^2)\}$, then $|\tau| = |P|$.*

Let P be a singular polar space having as radical the point x , with projective index n , and for which the quotient $P/\{x\}$ is the nonsingular polar space P' . If τ is an SPG-system of P then $|\tau| = o(P')q^n$, with $o(P')$ the number of elements of a (hypothetical) spread of P' .

The following characterization of SPG-systems will be used frequently without being explicitly mentioned.

Theorem 1.2 ([4]). *Let τ be a set of $(n - 1)$ -dimensional totally singular subspaces of the nonsingular polar space $P \in \{Q(2n + 2, q), Q^+(2n + 1, q), H(2n + 1, q^2)\}$. Then τ is an SPG-system of P if and only if the following conditions are satisfied:*

- (i) $|\tau| = |P|$,
- (ii) if $\pi, \pi' \in \tau$, with $\pi \neq \pi'$ and $\pi \cap \pi' \neq \emptyset$, then $\langle \pi, \pi' \rangle$ contains a generator of P .

Let τ be a set of $(n - 1)$ -dimensional totally singular subspaces of the singular polar space P with projective index n and having as radical the point x . Assume also that no element of τ contains x . The nonsingular quotient $P/\{x\}$ will be denoted by P' . Then τ is an SPG-system of P if and only if the following conditions are satisfied:

- (i) $|\tau| = o(P')q^n$,
- (ii) if $\pi, \pi' \in \tau$, with $\pi \neq \pi'$ and $\pi \cap \pi' \neq \emptyset$, then $\langle \pi, \pi' \rangle$ contains a generator of P .

Using the two previous theorems one can prove (see [4]) that if \mathcal{R} is a spread of $P \in \{Q(2n + 2, q), Q^+(2n + 1, q), H(2n + 1, q^2)\}$ and τ is the set of all $(n - 1)$ -dimensional subspaces contained in the elements of \mathcal{R} then τ is an SPG-system of P . Similarly if P is a singular polar space of projective index n , with ambient space $PG(d, q)$, having as radical the point x , and if \mathcal{R} is a set of n -dimensional totally singular subspaces of P (hence containing x), with the property that \mathcal{R} induces a spread in the polar subspace P' induced by any $PG(d - 1, q)$ not containing x (i.e. the intersections of the elements of \mathcal{R} with P' constitute a spread of P'), then the set τ of all $(n - 1)$ -dimensional subspaces of the ambient space of P contained in the elements of \mathcal{R} , but not containing x , is an SPG-system of P . An SPG-system constructed in the above way will be called a *spread-SPG-system*.

Thas classified all SPG-systems of index 1.

Theorem 1.3 ([4]). *There are exactly two classes of SPG-systems of index 1 on a nonsingular polar space and both are SPG-systems of $Q(6, q)$. One of them is a spread-SPG-system, the other one consists of the lines of the classical hexagon $H(q)$ embedded in $Q(6, q)$.*

Note that $Q(6, q)$ has at least one spread for all values of q , except possibly for the case where q is odd, with $q \equiv 1 \pmod{3}$ and not prime, in which case the existence of a spread is still open, see for instance [1].

The following two propositions will be used frequently in the following sections. The first one generalizes a result for $n = 2$ that can be found in [4], the second one characterizes spread-SPG-systems.

Proposition 1.4. *Let τ be an SPG-system of a polar space P .*

(1) *If P is nonsingular of projective index n then each point of P is contained in exactly $(q^n - 1)/(q - 1)$ elements of τ .*

(2) *If P is singular of projective index n then each point of P (except the radical) is contained in exactly q^{n-1} elements of τ .*

Proof. (1) Assume τ is an SPG-system of a nonsingular polar space P . We give the proof for $P = Q(2n + 2, q)$, the other cases are similar. The polar space $Q(2n + 2, q)$ contains $q^{n+1}(q^{n+1} - 1)/2$ nonsingular polar spaces $Q^-(2n + 1, q)$ which implies that each point of $Q(2n + 2, q)$ is contained in $q^{n+1}(q^n - 1)/2$ of these $Q^-(2n + 1, q)$. Furthermore an easy counting shows that each $PG(n - 1, q) \subset Q(2n + 2, q)$ is contained in $q^{n+1}(q - 1)/2$ of these polar subspaces. A double counting now proves that each point of $Q(2n + 2, q)$ is contained in exactly $(q^n - 1)/(q - 1)$ elements of τ .

(2) Assume τ is an SPG-system of a singular polar space P with radical x and let $PG(d, q)$ be the ambient space of P . Then each element of τ is contained in $(q^{d-n+1} - 1)/(q - 1) - (q^{d-n} - 1)/(q - 1) = q^{d-n}$ hyperplanes of $PG(d, q)$ not containing x . On the other hand a point of $P \setminus \{x\}$ is contained in q^{d-1} hyperplanes not through x . Again an easy double counting proves that each point of $P \setminus \{x\}$ is contained in q^{n-1} elements of τ . □

Proposition 1.5. *Let τ be an SPG-system of index $n - 1 \geq 2$ of a polar space P . Then τ is a spread-SPG-system if and only if every two intersecting elements of τ intersect in an $(n - 2)$ -dimensional space.*

Proof. One direction of the lemma is trivial, so assume that τ is an SPG-system such that every two intersecting elements of τ intersect in an $(n - 2)$ -dimensional space. Let $\pi \in \tau$ and let $\alpha_1, \dots, \alpha_k$ be the generators of P through π . Now each element of τ intersecting π should be completely contained in some α_i . Assume that γ_1 and γ_2 are elements of τ such that $\gamma_1 \in \alpha_i$ and $\gamma_2 \in \alpha_j$ with $i \neq j$ and $\gamma_1 \neq \pi \neq \gamma_2$. Since $\gamma_1 \cap \gamma_2 \neq \emptyset$, it follows that $\gamma_1 \cap \pi = \gamma_2 \cap \pi$. This implies that $\langle \gamma_1, \gamma_2 \rangle$ cannot contain a generator of P , contradicting the fact that τ is an SPG-system. So all elements of τ intersecting π are contained in a unique generator α_i . It now follows easily that τ is a spread-SPG-system. □

2 The cases $Q^+(7, q)$ and $H(7, q^2)$

Theorem 2.1. *The only SPG-systems of $Q^+(7, q)$ are spread-SPG-systems. The polar space $H(7, q^2)$ does not admit SPG-systems.*

Proof. Let τ be an SPG-system of $Q^+(7, q)$ and assume that $\alpha, \beta_1 \in \tau$ with α and β_1 intersecting in a unique point. Now let $Q(6, q) \subset Q^+(7, q)$ be a nonsingular parabolic quadric containing β_1 and let $\beta_2, \dots, \beta_{q+1}$ be the q other elements of τ contained in $Q(6, q)$ and intersecting the line $L := \alpha \cap Q(6, q)$. Suppose π_1 and π_2 are the two generators of $Q^+(7, q)$ containing α . Using the fact that $\langle \alpha, \beta_i \rangle$ must contain a generator, it follows that either $\beta_i \cap \pi_1$ or $\beta_i \cap \pi_2$ is a line, for each $i \in \{1, \dots, q+1\}$. Since $\pi_j \cap Q(6, q)$ is a plane $\bar{\pi}_j$, with $j \in \{1, 2\}$, the above implies that we have found $q+1$ disjoint lines in the union of the two planes $\bar{\pi}_1$ and $\bar{\pi}_2$, clearly a contradiction. We conclude that any two distinct intersecting elements of τ always intersect in a line or, using Proposition 1.5, that each SPG-system of $Q^+(7, q)$ is a spread-SPG-system.

Using the same technique on $H(7, q^2)$ (here one constructs q^2+1 disjoint lines in the union of $q+1$ planes), we find that $H(7, q^2)$ can only admit spread-SPG-systems. But since $H(7, q^2)$ does not admit a spread [2], this implies that $H(7, q^2)$ does not admit an SPG-system. □

Remark. Note that $Q^+(7, q)$ contains a spread if q is even as well as in the case that q is odd, except when q is odd, with $q \equiv 1 \pmod{3}$ and not prime, in which case the existence of a spread of $Q^+(7, q)$ is still open, see for instance [1].

3 The case $Q(8, q)$

3.1 Determination of the local structure. Let p be a point of the polar space P and let τ be an SPG-system of P . Then we define *the local structure of τ at p* as the structure in the polar space P/p , induced by the elements of τ containing p . Remark that P/p is contained in $T_p(P)/p$, with $T_p(P)$ the tangent space of P at p if θ is not symplectic and with $T_p(P) = p^\theta$ in the symplectic case, where θ is the polarity defining P .

By Proposition 1.4 and Theorem 1.2 (ii) the local structure of τ at any point of $Q(8, q)$ consists of q^2+q+1 lines L_0, \dots, L_{q^2+q} contained in a nonsingular parabolic quadric $Q(6, q)$ with the property that $\langle L_i, L_j \rangle$ contains a generator of $Q(6, q)$ for every $i, j \in \{0, \dots, q^2+q\}$ with $i \neq j$.

Assume that all L_i are two by two disjoint; then for every pair $\{L_i, L_j\}$ there is a unique line M_{ij} with $M_{ij} \cap L_i \neq \emptyset \neq M_{ij} \cap L_j$ and such that $\langle L_i, M_{ij} \rangle$ and $\langle L_j, M_{ij} \rangle$ are planes of $Q(6, q)$. Furthermore it is clear that if a line M_{ij} intersects the line L_k , with $i \neq j \neq k \neq i$, there holds that $M_{ij} = M_{ik} = M_{jk}$ (the span of L_i, L_j and L_k in $Q(6, q)$ must be a cone $M_{ij}Q(2, q)$). So in fact we have that if a line L_i meets a line M_{kl} the two span a plane of $Q(6, q)$ and $M_{kl} = M_{ij}$ for precisely q different indices j . We now consider the incidence structure $\mathbb{S} = (\mathbb{P}, \mathbb{L}, \mathbb{I})$, with \mathbb{P} the set of lines L_i , \mathbb{L} the set of lines M_{ij} and with \mathbb{I} the incidence relation defined by having nonempty intersection. We will call the elements of \mathbb{P} the \mathbb{S} -points, while the elements of \mathbb{L} will be called the \mathbb{S} -lines. We will show that \mathbb{S} is a projective plane. It is clear that the number of \mathbb{S} -points is q^2+q+1 and that two different \mathbb{S} -points define a unique \mathbb{S} -line. Consider the line L_0 of $Q(6, q)$ and suppose that the point x of L_0 is contained in both M_{0i} and M_{0j} with $M_{0i} \neq M_{0j}$. This implies that $\langle L_i, L_j \rangle \subset x^\perp$, giving a $PG(3, q) = \langle L_i, M_{ij}, x \rangle \subset Q(6, q)$, a contradiction. Hence an \mathbb{S} -point L_i is incident with at most $q+1$ \mathbb{S} -lines. Since an \mathbb{S} -line contains at most $q+1$ \mathbb{S} -points and each \mathbb{S} -point

is collinear with each \mathbb{S} -point we easily see that each \mathbb{S} -point is incident with $q + 1$ \mathbb{S} -lines and each \mathbb{S} -line is incident with $q + 1$ \mathbb{S} -points. From this it follows trivially that two \mathbb{S} -lines must always intersect. Hence we find from the above that \mathbb{S} is a projective plane of order q . The flag-geometry of \mathbb{S} has as point set the set of points on the lines L_i and as line set the set of lines L_i and M_{ij} . This flag-geometry is embedded in $Q(6, q)$. By [5] all such embeddings in $PG(6, q)$ are isomorphic and moreover q must be a power of 3.

We call a point of $Q(8, q)$ of *type H* if its local structure consists of $q^2 + q + 1$ disjoint lines. Hence, we have proved the following lemma.

Lemma 3.1. *If τ is an SPG-system of $Q(8, q)$ such that there exists a point of type H, then q is a power of 3 and the incidence structure \mathbb{S} defined as above is a projective plane.*

Remark. A set of $q^2 + q + 1$ lines as above can easily be constructed as follows (see [5]). Let q be a power of 3 and consider the embedding of the classical hexagon $H(q)$ in $Q(6, q)$. Now let $Q^+(5, q)$ be a nonsingular hyperbolic quadric contained in $Q(6, q)$. It is well known that $Q^+(5, q)$ contains two disjoint planes, called *ideal planes*, each consisting of $q^2 + q + 1$ points that are two by two at distance four in $H(q)$. Since q is a power of three, $H(q)$ is self-dual, so there exist in $H(q)$ two sets of $q^2 + q + 1$ mutually disjoint lines, where any two distinct lines of such a set are at distance four in $H(q)$. In fact it is even clear that if we call the lines in one set L_i (with $i \in \{0, \dots, q^2 + q\}$) the other set consists of the lines M_{ij} .

Now we shall investigate the possible configurations for the local structure if two of its lines intersect. A first possibility is clearly that the $q^2 + q + 1$ lines are the lines of a plane of $Q(6, q)$. We will call a point with such a local structure a point of *type S*. For the rest of this section we assume that the local structure is not of type S.

Lemma 3.2. *For every two intersecting lines in the local structure at a point of an SPG-system τ on $Q(8, q)$, the $q + 1$ lines in the pencil defined by these two lines belong to the local structure.*

Proof. Let π_1 and π_2 be elements of τ such that $\pi_1 \cap \pi_2$ is a line M and let $\alpha_0 = \langle \pi_1, \pi_2 \rangle, \dots, \alpha_q$ be the $q + 1$ generators of $Q(8, q)$ through π_1 and $\omega_0 = \alpha_0, \dots, \omega_q$ the generators through π_2 . Furthermore take a third plane π_3 through M in α_0 and consider a nonsingular elliptic quadric $Q^-(7, q) \subset Q(8, q)$ containing π_3 . Assume by way of contradiction that $\pi_3 \notin \tau$. The $2q + 1$ generators $\alpha_0, \dots, \omega_q$ will intersect $Q^-(7, q)$ in $2q + 1$ planes $\bar{\alpha}_0 = \bar{\omega}_0 = \pi_3, \dots, \bar{\omega}_q$ containing M . Now let $m \in M$ and call π_m the element of τ through m in $Q^-(7, q)$. The plane π_m must then contain a line N_1 of $\bar{\alpha}_0 \cup \dots \cup \bar{\alpha}_q$ different from M (since $\langle \pi_1, \pi_m \rangle$ must contain a generator), as well as a line N_2 of $\bar{\omega}_0 \cup \dots \cup \bar{\omega}_q$ different from M . This is only possible if $N_1 = N_2 \subset \bar{\alpha}_0$ (since otherwise one could construct a $PG(3, q)$ on $Q^-(7, q)$). Analogously we find for a point $m' \in M \setminus \{m\}$ a line N' in $\bar{\omega}_0$. Since N_1 and N' must have nonempty intersection, we find a contradiction. This proves that the $q + 1$ planes through M in $\langle \pi_1, \pi_2 \rangle$ belong to τ , hence the lemma. □

Suppose the point l is not of type H and not of type S , so without loss of generality we may assume that in its local structure the lines L_0, \dots, L_q form a pencil of lines in the plane $\langle L_0, L_1 \rangle$. Using the previous lemma it is evident that no other L_i (with $i \in \{q + 1, \dots, q^2 + q\}$) can be contained in this plane. There are two cases.

(1) For all $i \in \{q + 1, \dots, q^2 + q\}$ it holds that $L_i \cap \langle L_0, L_1 \rangle = \emptyset$. If we call p the vertex of the pencil, this implies that $L_i \subset p^\perp$ (since each of the lines L_0, \dots, L_q must contain a point collinear with all points of L_i) and that L_j with $j \notin \{0, \dots, q, i\}$ intersects $\langle p, L_i \rangle$ in a unique point; this last assertion follows from the fact that $L_j \not\subset \langle p, L_i \rangle$ (since otherwise the previous lemma would imply that there is a line L_k through p not in $\langle L_0, L_1 \rangle$, leading to a $\text{PG}(3, q) \subset \mathcal{Q}(6, q)$, a contradiction) and the fact that $L_j \cap \langle p, L_i \rangle \neq \emptyset$ (since otherwise $p \notin \langle L_i, L_j \rangle \subset p^\perp$ would lead to a $\text{PG}(3, q) \subset \mathcal{Q}(6, q)$). Now if $L_i \cap L_j \neq \emptyset$ we would find a $\text{PG}(3, q) \subset \mathcal{Q}(6, q)$, a contradiction. So we see that each point of $\langle p, L_i \rangle \setminus (\{p\} \cup L_i)$ is contained in a unique L_j . Put $s := L_k \cap \langle p, L_i \rangle$ and $t := L_j \cap \langle p, L_i \rangle$ with L_k and L_j chosen in such a way that p, s and t are not collinear. It is clear that $L_k \not\subset t^\perp$ since otherwise we would have $\langle p, t \rangle \subset L_k^\perp$. So there exists a $t' \in L_j \setminus \{t\}$ and an $s' \in L_k \setminus \{s\}$ such that $L_k \subset t'^\perp$ and $L_j \subset s'^\perp$. This implies that $\langle t, t', s' \rangle \subset s^\perp$, a contradiction since $s \notin \langle t, t', s' \rangle$. Hence this case cannot occur.

(2) Assume $L_{q+1} \cap L_0$ is a point r . This means that the pencil of lines through r in $\langle L_0, L_{q+1} \rangle$ belongs to the local structure. If $L_j \cap \langle L_0, L_1 \rangle = \emptyset = L_j \cap \langle L_0, L_{q+1} \rangle$ for a certain line L_j we would have $L_j \subset (pr)^\perp$, a contradiction. So without loss of generality we may assume that there is a line L_j intersecting $\langle L_0, L_1 \rangle$ in a point r' . Now suppose $r' \notin L_0$, so without loss of generality $r' \in L_1$. Clearly $L_j \cap L_{q+1} = \emptyset$. Then there exists a point $t \in L_j$ with $L_{q+1} \subset t^\perp$. We now see that $t = r'$ (respectively $t \neq r'$) leads to the contradiction $\langle L_0, L_{q+1} \rangle \subset t^\perp$ (respectively $\langle L_0, L_1 \rangle \subset t^\perp$). Hence for any line L_j not belonging to $\langle L_0, L_1 \rangle$ and $\langle L_0, L_{q+1} \rangle$ we have $r' \in L_0 \setminus \{p, r\}$ and the only possibility that is left for the local structure is the following one. Let $\alpha_0, \dots, \alpha_q$ be the $q + 1$ planes of $\mathcal{Q}(6, q)$ through a line L and let r_0, \dots, r_q be the $q + 1$ points of L . Then the local structure consists of the pencils of lines through r_i in α_i with $i = 0, \dots, q$. A point of $\mathcal{Q}(8, q)$ with such a local structure will be said to be of type O . In fact have proved the following lemma.

Lemma 3.3. *A point of $\mathcal{Q}(8, q)$ is either of type S , type O or type H . If there exists a point of type H then q is a power of 3.*

3.2 The global structure. If a point x of $\mathcal{Q}(8, q)$ is of type O then from the previous section we know that there is a special element of τ through x that we will denote by ξ_x such that for each generator α_i containing ξ_x there is a unique line M_i ($i = 0, \dots, q$) through x in ξ_x with the property that all planes in α_i containing M_i belong to τ . Furthermore we will denote the union of all points in all generators containing ξ_x by \tilde{x} .

Lemma 3.4. *Let τ be an SPG-system of $\mathcal{Q}(8, q)$. If there is a point p of type S in $\mathcal{Q}(8, q)$, then no point of type O can exist.*

Proof. Let Σ be the $\text{PG}(3, q)$ spanned by the elements of τ through p and assume that no other point in Σ is of type S . Then a point $x \in \Sigma \setminus \{p\}$ is necessarily of type O and since the line xp is already contained in $q + 1$ elements of τ it follows that one of these planes must be ξ_x . This implies that $p \in \xi_x$ for all $x \in \Sigma \setminus \{p\}$. Consequently there are distinct points $x, y \in \Sigma \setminus \{p\}$ with the property that $\xi_x = \xi_y$. Consider a line L_1 through x in ξ_x not through y or p . Then L_1 determines a 3-dimensional space M containing ξ_x with the property that all planes through L_1 in M belong to τ . Now there must be a line L_2 through y in ξ_x such that all planes through L_2 in M belong to τ . With $z := L_1 \cap L_2$ we see that M contains $2q + 1$ elements of τ through z . Hence z is of type S , a contradiction since $z \in \Sigma$.

So we may assume that in Σ there is a second point n of type S . An arbitrary point $z \in \Sigma \setminus \langle p, n \rangle$ will then be contained in $2q + 1$ elements of τ that are completely contained in Σ . Hence z is of type S . It now easily follows that all points of Σ are of type S (i.e. that all planes of Σ belong to τ). Suppose now that $x \notin \Sigma$ is of type O . Then we know that each point of \tilde{x} is contained in an element of τ through x . Since $\tilde{x} \cap \Sigma \neq \emptyset$ we find a point in Σ that would be contained in $q^2 + q + 2$ elements of τ , a contradiction. \square

Lemma 3.5. *Let τ be an SPG-system of $Q(8, q)$. If there exists a point of type O , then all points of $Q(8, q)$ will be of type O .*

Proof. Suppose that x and y are two distinct points of type O such that $\xi_x = \xi_y$. Like in the previous lemma this leads to a point of type S , contradicting the conclusion of the previous lemma. So for all x and y of type O we have that $\xi_x \neq \xi_y$.

Using Lemma 3.4 it is evident that $p \in \xi_l$ for certain l of type O implies that p is of type O . The line $\langle p, l \rangle$ then determines a solid Σ . Since each point $x \in \langle p, l \rangle$ is contained in the $q + 1$ elements of τ through $\langle p, l \rangle$ in Σ it follows that one of these planes must be ξ_x , hence each plane through $\langle p, l \rangle$ in Σ is of the form ξ_x for a certain $x \in \langle p, l \rangle$. This implies that each point of \tilde{l} is of type O . Let z be a point outside \tilde{l} and suppose $z \in p^\perp$ with $p \in \xi_l$. Then there will be a line $\langle p, n \rangle \in \xi_p$ such that $\langle p, n \rangle \subset z^\perp$. Putting Ω the $\text{PG}(3, q)$ through ξ_p determined by $\langle p, n \rangle$ it is clear that if $z \in \Omega$ we can use the foregoing to see that z will be of type O , so assume that $z \notin \Omega$. Now there is a plane $\alpha \supset \langle p, n \rangle$ in Ω such that $\alpha \subset z^\perp$. Since we know that α is of the form ξ_y for certain $y \in \langle p, n \rangle$, we see that $z \in \tilde{y}$, proving that z is of type O . \square

Lemma 3.6. *Let τ be an SPG-system of $Q(8, q)$ with the property that each point is of type O . Then the incidence structure $\mathfrak{O} = (P, B, \text{I})$ with P the point set of $Q(8, q)$, with $B = \{L \mid L \text{ a line of } Q(8, q) \text{ with } x \in L \subset \xi_x \text{ for some point } x\}$ and with I the natural incidence is a generalized octagon.*

Proof. First of all notice that $a \in \xi_b$ implies that $b \in \xi_a$. We recall from the previous lemma that for $x \in L \subset \xi_x$ (x a point of type O) the map $y \mapsto \xi_y$ is a bijection from the points of L to the planes on L in the solid determined by x and L .

First we show that $x \notin z^\perp$ implies that $\xi_x \cap \tilde{z} = \emptyset$. Assume that $u \in \xi_x \cap \tilde{z}$. If $u \in \xi_z$ we would find $x \in \xi_u \subset \tilde{z}$, a contradiction. If $u \notin \xi_z$ it follows that $u \in \xi_y$ for certain

$y \in \xi_z$. This implies that $\xi_u = \langle u, y, x \rangle$. Since $\xi_u \subset \tilde{y}$ we have that $\xi_y \subset x^\perp$ and so $z \in x^\perp$, a contradiction.

Let z be an arbitrary point of $Q(8, q)$ and let d be the distance function in the incidence graph of Φ . It is clear that if $x \in \xi_z, x \neq z$, we have $d(x, z) = 2$ and for all points u collinear with x in Φ but not on the line $\langle x, z \rangle$ we have that $d(u, z) > 2$.

If $x \in \tilde{z} \setminus \xi_z$ we see that x belongs to a unique plane ξ_y with $y \in \xi_z$, hence $d(x, z) = 4$, the path is unique and for all u collinear with x in Φ but not on the line $\langle x, y \rangle$ we have that $d(u, z) > 4$ (the last assertion since $\xi_x \cap \tilde{z} = \langle x, y \rangle$, i.e. all points collinear with x not on $\langle x, y \rangle$ lay outside \tilde{z}).

In the case $x \notin \tilde{z}$ and $x \in z^\perp$ there is a unique line $\langle z, y \rangle \subset \xi_z$ with $\langle z, y \rangle \subset x^\perp$ and $\xi_y \subset x^\perp$, i.e. there is a unique $y \in \xi_z$ such that $x \in \tilde{y}$. Hence $d(x, z) = 6$ and there is a unique path of this length.

Finally assume that $x \notin z^\perp$. Then we know that $\xi_x \cap \tilde{z} = \emptyset$ and that each line through x contains a point collinear with z in $Q(8, q)$, in other words $d(x, z) = 8$. From this it follows that the diameter of the incidence graph of Φ is 8.

From the uniqueness of the paths of length 2, 4 and 6 it is easy to conclude that no circuits of length smaller than 14 can occur. Assume that $z \sim y_1 \sim y_2 \sim x \sim y_3 \sim y_4 \sim y_5 \sim z$ would determine a path of length 14. Using the foregoing we see that $d(x, z) = 6$. Furthermore we notice that $\xi_x = \langle x, y_2, y_3 \rangle$ and that $\{x, y_2, y_3\} \subset z^\perp$ (since $d(y, z) \leq 6$ implies that $y \in z^\perp$). This implies that $z \in \tilde{x}$ and so $d(x, z) = 4$, a contradiction. Hence circuits have minimal length 16.

So we have proved that the incidence graph of Φ has diameter 8 and girth 16, in other words that Φ is a generalized octagon (see e.g. [6]). □

Proposition 3.7. *Let τ be an SPG-system of $Q(8, q)$, then there cannot exist a point of type O .*

Proof. This follows immediately from the Lemmas 3.5, 3.6 and the fact that no generalized octagon with $s = t > 1$ can exist (with $s + 1$ being the number of points on a line and $t + 1$ being the number of lines through a point), see [6]. □

Now we will show that points of type H cannot exist either. We use the same notation as mentioned before.

Lemma 3.8. *Let \mathcal{O} be the point set of the local structure L_0, \dots, L_{q^2+q} of a point of type H . If $|\mathcal{O} \cap \alpha| > q + 1$ for a plane $\alpha \subset Q(6, q)$, then α contains a line L_i .*

Proof. Assume that α does not contain a line L_i . It then follows from Lemma 3.1 that in this case α cannot contain a line M_{ij} . Assume that there exists a line K of α containing at least three distinct points x_1, x_2 and x_3 of \mathcal{O} . Furthermore let $x_1 \in L_1, x_2 \in L_2$ and $x_3 \in L_3$. Since there is a point on L_1 collinear with all points of L_2 we find that either $\langle L_1, K \rangle$ or $\langle L_2, K \rangle$ is a plane of $Q(6, q)$. So we may assume that $\langle L_1, K \rangle$ is contained in $Q(6, q)$. This plane will intersect exactly q other lines L_i along the line M_{12} . Since K is contained in this plane and intersects L_1, L_2 as well as L_3 it follows that $K = M_{12}$, a contradiction. From this we have that a line of α contains at

most two points of \mathcal{O} , in other words, $\alpha \cap \mathcal{O}$ is a (partial) oval in α . Since q is a power of 3 (Lemma 3.1), so certainly not even, we see that $|\alpha \cap \mathcal{O}| \leq q + 1$. \square

Proposition 3.9. *The quadric $Q(8, q)$ cannot contain a point of type H with respect to an SPG-system τ .*

Proof. Let x be a point of type H and let α be an element of τ containing x . It is clear that each point of α must be of type H . Now let Σ be a $\text{PG}(3, q) \subset Q(8, q)$ containing α . By looking at the local structure, each point of α is contained in q elements of τ that intersect Σ in a line; also, α and each of these q planes of τ intersect a common plane of $Q(8, q)$ in a line. In this way each point of α determines q^2 points of $\Sigma \setminus \alpha$. Since $q^2(q^2 + q + 1)/q^3 > q + 1$, this means that there is a point n in $\Sigma \setminus \alpha$ with the property that more than $q + 1$ elements of τ through n have nonempty intersection with α . The previous lemma (notice that n must be of type H) now implies that there is an element of τ through n that intersects α in a line L . Hence a point of L can never be of type H , a contradiction. \square

Bringing together the results of this section, we see we have proved the following theorem.

Theorem 3.10. *The only SPG-systems of $Q(8, q)$ are spread-SPG-systems.*

Proof. Immediate from Propositions 3.7 and 3.9. \square

Remark. Note that $Q(8, q)$ with q odd does not admit a spread (see for example [3] and [2]) and hence this theorem implies that only $Q(8, 2^h)$ admits an SPG-system.

4 The singular case

In this section we will classify the SPG-systems of the singular polar spaces P (having as radical a unique point) for every $n \in \mathbb{N}_0$. One easily sees that the same technique as used in the cases $Q^+(7, q)$ and $H(7, q^2)$ would provide a classification in the case P has projective index 3, but since here an element of τ is contained in a unique generator of P we can do better.

Theorem 4.1. *The only SPG-systems of a singular polar space P with projective index n ($n \in \mathbb{N}_0$) are spread-SPG-systems.*

Proof. Note that the cases $n \in \{1, 2\}$ were already handled in [4]; this proof will work for general n .

Let $\text{PG}(d, q)$ be the ambient space of P , let α be an element of τ and let P' be a polar subspace of P induced by a $\text{PG}(d - 1, q)$ not containing α nor the radical of P . If Σ is the generator of P containing α then all elements γ of τ contained in P' and intersecting $\beta := \alpha \cap P'$ in a $\text{PG}(i - 1, q) =: K$ will intersect $\Omega := \Sigma \cap P'$ in a $\text{PG}(i, q) =: L$ (since $\langle \gamma, \alpha \rangle$ must contain a generator of P , see Theorem 1.2). So K

determines q^i points of $\Omega \setminus \beta$. We will then say that the average number of points of $\Omega \setminus \beta$ determined by a point of K is $\delta_i := q^i(q-1)/(q^i-1)$. If we define δ to be the average number of points of $\Omega \setminus \beta$ determined by a point of β we see that we must have $\delta \leq \delta_{n-1} = q^{n-1}(q-1)/(q^{n-1}-1)$ (since all elements of τ in P' intersecting β have to be disjoint). We want to prove that there is a unique element σ of τ in P' that contains β , so assume the contrary, i.e. for no point of β the average number of points of $\Omega \setminus \beta$ determined by that point equals δ_{n-1} . If we define a_i to be the number of points in β that determine δ_i points in $\Omega \setminus \beta$ we find

$$\delta = \frac{q-1}{q^{n-1}-1} \sum_{i=1}^{n-2} a_i \delta_i > \frac{q-1}{q^{n-1}-1} \delta_{n-1} \sum_{i=1}^{n-2} a_i = \delta_{n-1},$$

clearly a contradiction. It follows that all SPG-systems of P are as constructed in [4], i.e. are all spread-SPG-systems. \square

Acknowledgement. The author wishes to express his gratitude to J. A. Thas for his interesting and helpful remarks and to F. De Clerck for thorough proof-reading of the manuscript.

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Received 23 April, 2002

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