

## On the enumerative geometry of real algebraic curves having many real branches

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**Abstract.** Let  $C$  be a smooth real plane curve. Let  $c$  be its degree and  $g$  its genus. We assume that  $C$  has at least  $g$  real branches. Let  $d$  be a nonzero natural integer strictly less than  $c$ . Let  $\mathbf{e}$  be a partition of  $cd$  of length  $g$ . Let  $\nu$  be the number of all real plane curves of degree  $d$  that are tangent to  $g$  real branches of  $C$  with orders of tangency  $\mathbf{e}_1, \dots, \mathbf{e}_g$ . We show that  $\nu$  is finite and we determine  $\nu$  explicitly.

**Key words.** Enumerative geometry, real algebraic curve, real branch,  $M$ -curve,  $(M - 1)$ -curve, Picard group, real space curve, real plane curve, conic, cubic, quartic, quintic.

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### 1 Introduction

Enumerative real algebraic geometry has known a growing attention throughout the last decade or so (see [10] for a survey). In his work on intersection theory, Fulton attracted attention to the number of real solutions of enumerative problems in algebraic geometry [1, p 55]. As an example, he explicitly posed the question of how many of the 3264 conics tangent to five given real conics can be real. He proved that all of them can be real (unpublished). Independently, Ronga, Tognoli and Vust also proved this fact [7]. Sottile proved analogous results [9]. Fulton's question naturally leads to the more general question of how many real curves of degree  $d$  are tangent to a certain number of real branches of a given real plane curve of degree  $c$ .

In this paper we answer the above question for any smooth real plane curve of any degree  $c$  that has many real branches. Let us explain what we mean by a real plane curve having many real branches. By Harnack's Inequality [2], a smooth real plane curve  $C$  of degree  $c$  has at most  $g + 1$  real branches, where  $g$  is the genus of  $C$ , i.e.,  $g = \frac{1}{2}(c - 1)(c - 2)$ . Harnack also showed that, for any natural number  $c$ , there is a smooth real plane curve of degree  $c$  having  $g + 1$  real branches. Such real plane curves are also known as  $M$ -curves. They are subject to intensive study ever since Hilbert included them in his 16th problem. We say that a smooth real plane curve has *many real branches* if its genus  $g$  is at least 1 and if it has at least  $g$  real branches. To

put it otherwise, a smooth real plane curve has many real branches if it is either a nonrational  $M$ -curve or a nonrational  $(M - 1)$ -curve.

Before we can state one of our main results, we need to fix some terminology. Let  $n$  be a natural number. A *partition* of  $n$  is an element  $\mathbf{e}$  of  $\mathbf{N}^\ell$ , for some natural integer  $\ell$ , such that

$$n = \mathbf{e}_1 + \cdots + \mathbf{e}_\ell \quad \text{and} \quad \mathbf{e}_1 \geq \cdots \geq \mathbf{e}_\ell \geq 1.$$

The integer  $\ell$  is called the *length* of the partition. Let  $\mathbf{e}$  be a partition of  $n$  of length  $\ell$ . Let  $k$  be a natural number. We say that  $k$  is a *member* of  $\mathbf{e}$  if there is an  $i \in \{1, \dots, \ell\}$  with  $\mathbf{e}_i = k$ . The partition  $\mathbf{e}$  of  $n$  is *even* if all its members are even. If  $k$  is a member of  $\mathbf{e}$ , then the *multiplicity* of  $k$  in  $\mathbf{e}$  is the number of indices  $i \in \{1, \dots, \ell\}$  such that  $\mathbf{e}_i = k$ . A *multiplicity* of  $\mathbf{e}$  is a multiplicity of a member of  $\mathbf{e}$ .

Let  $C$  be a smooth real algebraic curve in  $\mathbb{P}^2$  having many real branches. Let  $c$  be its degree and let  $g$  be its genus. Let  $d$  be a nonzero natural integer. Let  $\mathbf{e}$  be a partition of  $cd$  of length  $g$ . A real algebraic curve  $D$  in  $\mathbb{P}^2$  of degree  $d$  is said to *have tangency*  $\mathbf{e}$  to  $g$  real branches of  $C$  if  $D$  is tangent to  $g$  real branches of  $C$  with orders of tangency  $\mathbf{e}_1, \dots, \mathbf{e}_g$ .

**Theorem 1.1.** *Let  $C$  be a smooth real algebraic curve in  $\mathbb{P}^2$  having many real branches. Let  $c$  be its degree and let  $g$  be its genus. Let  $\mathbf{e}$  be a partition of  $c(c - 1)$  of length  $g$ . Let  $v$  be the number of real plane curves of degree  $c - 1$  having tangency  $\mathbf{e}$  to  $g$  real branches of  $C$ . Then,  $v$  is finite. Moreover,  $v \neq 0$  if and only if  $\mathbf{e}$  is an even partition. In that case,*

$$v = \begin{cases} \frac{g!}{m_1! \cdots m_r!} \cdot \prod_{i=1}^g \mathbf{e}_i & \text{if } C \text{ is an } (M - 1)\text{-curve, and} \\ \frac{(g + 1)!}{m_1! \cdots m_r!} \cdot \prod_{i=1}^g \mathbf{e}_i & \text{if } C \text{ is an } M\text{-curve,} \end{cases}$$

where  $m_1, \dots, m_r$  are the multiplicities of  $\mathbf{e}$ .

In Section 4 we show a similar statement for the number of real curves of degree  $c - 2$  that are tangent to  $g$  real branches of a given real plane curve  $C$  of degree  $c$  (cf. Theorem 4.3). By Bezout, there are no real curves of degree strictly less than  $c - 2$  that are tangent to  $g$  real branches of a given real plane curve  $C$  of degree  $c$ .

**Example 1.2.** Let  $C$  be a smooth real cubic curve in  $\mathbb{P}^2$ . Since the degree of  $C$  is equal to 3, the genus of  $C$  is equal to 1. Moreover,  $C(\mathbb{R}) \neq \emptyset$ . Hence,  $C$  necessarily has many real branches. Let  $v$  be the number of real conics tangent to one real branch of  $C$  with order of tangency equal to 6. Then, according to Theorem 1.1,  $v = 6$  if  $C$  has only one real branch, and  $v = 12$  if  $C$  has exactly 2 real branches. This statement is the well-known fact that a real elliptic curve has either 6 or 12 real points whose order is a divisor of 6 [8].

**Example 1.3.** Let  $C$  be a smooth real quartic curve in  $\mathbb{P}^2$ . Since the degree of  $C$  is equal to 4, the genus of  $C$  is equal to 3. Therefore, in order to apply Theorem 1.1, we assume that  $C$  has at least 3 real branches. The partitions of  $4 \cdot 3 = 12$  of length 3 and in even numbers are  $(8, 2, 2)$ ,  $(6, 4, 2)$  or  $(4, 4, 4)$ . Let  $\mathbf{e}$  be one of these partitions. Let  $\nu$  be the number of real cubics tangent to 3 real branches of  $C$  with orders of tangency  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ . When one applies Theorem 1.1 to the current situation one obtains the following values for  $\nu$ . If  $C$  has exactly 3 real branches then

$$\nu = \begin{cases} 96 & \text{if } \mathbf{e} = (8, 2, 2), \\ 288 & \text{if } \mathbf{e} = (6, 4, 2), \\ 64 & \text{if } \mathbf{e} = (4, 4, 4). \end{cases}$$

If  $C$  has exactly 4 real branches then

$$\nu = \begin{cases} 384 & \text{if } \mathbf{e} = (8, 2, 2), \\ 1152 & \text{if } \mathbf{e} = (6, 4, 2), \\ 256 & \text{if } \mathbf{e} = (4, 4, 4). \end{cases}$$

The cases where  $\mathbf{e} = (4, 4, 4)$  have already been shown in [4]. According to my knowledge, the other cases are new.

We refer to Section 4 for examples of higher degree (see Remark 4.1 and Example 4.2).

Theorem 1.1 is an application of Theorem 2.1 below that may be of independent interest. Section 2 is devoted to the proof of Theorem 2.1. In Section 3, we give applications to enumerative problems for real curves in any projective space. In Section 4, we specialize to plane curves; we prove Theorem 1.1 and formulate and prove Theorem 4.3. We also give several examples.

## 2 Divisor classes on real algebraic curves

Let  $C$  be a smooth geometrically integral proper real algebraic curve. A *real branch* of  $C$  is a connected component of the set  $C(\mathbb{R})$  of real points of  $C$ . By Harnack's Inequality [2],  $C$  has at most  $g + 1$  real branches, where  $g$  is the genus of  $C$ . We will say that  $C$  has *many real branches* if  $g \geq 1$  and the number of real branches of  $C$  is at least  $g$ .

**Theorem 2.1.** *Let  $C$  be a smooth geometrically integral proper real algebraic curve having many real branches. Let  $g$  be the genus of  $C$ . Let  $B_1, \dots, B_g$  be mutually distinct real branches of  $C$  and put*

$$B = \prod_{i=1}^g B_i.$$

Let  $e_1, \dots, e_g$  be nonzero natural integers, and let

$$\varphi : B \rightarrow \text{Pic}(C)$$

be the map defined by  $\varphi(P) = \text{cl}(\sum_{i=1}^g e_i P_i)$ , where  $\text{cl}$  denotes the divisor class. Then,  $\varphi$  is a topological covering of its image of degree  $\prod_{i=1}^g e_i$ .

*Proof.* Since  $B$  is connected, there is a connected component  $X$  of  $\text{Pic}(C)$  such that  $\varphi(B) \subseteq X$ . Since  $B$  and  $X$  are of the same dimension, it suffices to show that the map  $\varphi$  is unramified, in order to show that  $\varphi$  is a topological covering map.

Let  $P \in B$  and let  $v$  be a tangent vector to  $B$  at  $P$ . Suppose that the tangent map  $T\varphi$  of  $\varphi$  maps  $v$  to 0. We have to show that  $v$  is equal to 0, in order to show that  $\varphi$  is unramified.

Since  $B = \prod B_i$ ,  $P = (P_1, \dots, P_g)$  and  $v = (v_1, \dots, v_g)$ , where  $P_i \in B_i$  and  $v_i$  is a tangent vector to  $B_i$  at  $P_i$ . Let  $T = \text{Spec}(\mathbb{R}[\varepsilon])$ , where  $\mathbb{R}[\varepsilon]$  is the  $\mathbb{R}$ -algebra of dual numbers [3]. Each pair  $(P_i, v_i)$  determines a morphism

$$f_i : T \rightarrow C' = C \times_{\text{Spec}(\mathbb{R})} T,$$

the image of each  $f_i$  is a relative Cartier divisor  $D_i$  of  $C'/T$  [6]. If  $x_i$  is a local equation for  $P_i$  on  $C$ , then  $x_i - \lambda_i \varepsilon$  is a local equation for  $D_i$  on  $C'$ , for some  $\lambda_i \in \mathbb{R}$ . We have to show that  $\lambda_i = 0$  for  $i = 1, \dots, g$ , in order to show that  $v = 0$ .

Recall [3] that one has a short exact sequence

$$0 \rightarrow H^1(C, \mathcal{O}_C) \rightarrow \text{Pic}(C') \rightarrow \text{Pic}(C) \rightarrow 0.$$

In fact, this short exact sequence is naturally split since  $C$  can be identified with the special fiber of  $C'/T$ . Let  $D$  be the relative Cartier divisor  $\sum e_i D_i$  on  $C'/T$ . Consider the class  $\text{cl}(D)$  of the divisor  $D$  in  $\text{Pic}(C')$ . The hypothesis that  $T\varphi$  maps  $v$  onto 0 implies that  $\text{cl}(D)$  is contained in the image of the natural section of the map  $\text{Pic}(C') \rightarrow \text{Pic}(C)$ . Hence, the natural projection from  $\text{Pic}(C')$  onto  $H^1(C, \mathcal{O}_C)$  maps  $\text{cl}(D)$  onto 0. Now, let us compute the image of  $\text{cl}(D)$  by this natural projection.

Recall [3] that  $H^1(C, \mathcal{O}_C)$  can be identified with the cokernel  $R$  of the natural map

$$K \rightarrow \bigoplus_{Q \in C} K/\mathcal{O}_Q,$$

where  $\mathcal{O}_Q$  is the local ring of  $C$  at  $Q$  and  $K$  is the function field of  $C$ . Since

$$(x_i - \lambda_i \varepsilon)^{e_i} = x_i^{e_i} - e_i \lambda_i x_i^{e_i-1} \varepsilon = x_i^{e_i} \left( 1 - e_i \lambda_i \frac{1}{x_i} \varepsilon \right),$$

the image of  $\text{cl}(D)$  in  $H^1(C, \mathcal{O}_C)$  is equal to the element  $\rho = (\rho_Q)$  of  $R$  defined by

$$\rho_Q = \begin{cases} -e_i \lambda_i \frac{1}{x_i} & \text{if } Q = P_i, \\ 0 & \text{otherwise} \end{cases}$$

Take some  $i \in \{1, \dots, g\}$  and let us show that  $\lambda_i = 0$ . By the Riemann–Roch Theorem, there is a nonzero differential form  $\omega$  on  $C$  such that  $\omega$  has a zero at the points  $P_j$ ,  $j = 1, \dots, g$ ,  $j \neq i$ . Since the divisor of  $\omega$  is of even degree on each real branch of  $C$ ,  $\omega$  has at least 2 zeros on each of the real branches  $B_j$ ,  $j = 1, \dots, g$ ,  $j \neq i$ . Since  $\omega$  has exactly  $2(g - 1)$  zeros, it follows that  $\omega$  does not vanish on  $B_i$ . In particular,  $\omega$  does not vanish at  $P_i$ . Let  $t$  be the trace map from  $H^1(C, \Omega_C)$  into  $\mathbb{R}$  [3]. Since  $\rho = 0$  in  $R$ , one has  $t(\rho\omega) = 0$ . From the definition of the trace map, it follows that the residue of  $-e_i \lambda_i \frac{1}{x_i} \omega$  vanishes at  $P_i$ . Therefore,  $\lambda_i = 0$ . This proves that  $\varphi$  is unramified.

In order to finish the proof, we show the statement concerning the topological degree of  $\varphi$ . Choose a base point  $O \in B$  and write  $O = (O_1, \dots, O_g)$ . Let

$$\psi : B \rightarrow \text{Pic}(C)$$

be the map defined by letting  $\psi(P)$  be the divisor class  $\text{cl}(\sum P_i - O_i)$ . By [5, Theorem 3.1],  $\psi$  is a homeomorphism onto the neutral component  $\text{Pic}(C)^0$  of  $\text{Pic}(C)$ . Let  $\tau$  be the translation by  $-\text{cl}(\sum e_i O_i)$  on  $\text{Pic}(C)$ . Clearly,  $\tau$  maps the image  $X$  of  $\varphi$  homeomorphically onto  $\text{Pic}(C)^0$ . In order to show that the degree of  $\varphi$  is equal to  $\prod e_i$ , we show that the self-map  $\tau \circ \varphi \circ \psi^{-1}$  of  $\text{Pic}(C)^0$  has degree  $\prod e_i$ .

Each factor  $B_i$  of  $B$  gives rise to an element  $\beta_i$  of the first homology group  $H_1(B, \mathbb{Z})$ . Clearly,  $\beta_1, \dots, \beta_g$  is a basis of  $H_1(B, \mathbb{Z})$ . Then,  $\psi_*(\beta_1), \dots, \psi_*(\beta_g)$  is a basis of  $H_1(\text{Pic}(C)^0, \mathbb{Z})$ . Since the multiplication-by- $e_i$  map on  $\text{Pic}(C)^0$  induces the multiplication-by- $e_i$  map on  $H_1(\text{Pic}(C)^0, \mathbb{Z})$ ,

$$(\tau \circ \varphi \circ \psi^{-1})_*(\psi_*(\beta_i)) = e_i \cdot \psi_*(\beta_i).$$

It follows that  $\tau \circ \varphi \circ \psi^{-1}$  is of degree  $\prod e_i$ . □

**Corollary 2.2.** *Let  $C$  be a smooth geometrically integral proper real algebraic curve having many real branches. Let  $g$  be the genus of  $C$ . Let  $B_1, \dots, B_g$  be mutually distinct real branches of  $C$ . Let  $S$  be a complete linear system on  $C$  of degree  $e$ . Let  $\varepsilon_i$  be the degree mod 2 of  $S$  on  $B_i$ , for  $i = 1, \dots, g$ . Let  $e_1, \dots, e_g$  be nonzero natural integers satisfying*

$$\sum_{i=1}^g e_i = e.$$

*Let  $v$  be the number of divisors  $D$  of the form  $\sum_{i=1}^g e_i P_i$ , for some  $P_i \in B_i$ , that belong to  $S$ . Then,  $v$  is finite. Moreover,  $v \neq 0$  if and only if*

$$e_i \equiv \varepsilon_i \pmod{2} \quad \text{for } i = 1, \dots, g.$$

*In that case,  $v = \prod_{i=1}^g e_i$ .*

*Proof.* Let  $B = \prod B_i$ , as before, and let  $\varphi : B \rightarrow \text{Pic}(C)$  be defined by  $\varphi(P) =$

$\text{cl}(\sum e_i P_i)$ . Let  $X$  be the  $\varphi$ -image of  $B$ . Suppose there is an integer  $i$ ,  $1 \leq i \leq g$ , such that  $e_i \not\equiv \varepsilon_i \pmod{2}$ . Then,  $\text{cl}(S)$  does not belong to  $X$ . Hence, there is no divisor  $D$  in  $S$  of the form  $\sum e_i P_i$  for some  $P_i \in B_i$ . In that case,  $v = 0$ . Assume, now, that  $e_i \equiv \varepsilon_i \pmod{2}$  for all  $i = 1, \dots, g$ . Then,  $\text{cl}(S)$  belongs to  $X$ . According to Theorem 2.1, the number of  $P \in B$  such that  $\varphi(P) = \text{cl}(S)$  is equal to  $\prod e_i$ . It follows that the number of divisors  $D$  in  $S$  of the form  $\sum_{i=1}^g e_i P_i$ , for some  $P_i \in B_i$ , is equal to  $\prod_{i=1}^g e_i$ . Therefore,  $v = \prod_{i=1}^g e_i$ .  $\square$

### 3 Enumerative problems for real space curves

Let  $C$  be a smooth geometrically integral real algebraic curve in  $\mathbb{P}^n$ , where  $n \geq 2$ . Let  $d$  be a nonzero natural number. We say that the linear system of all real hypersurfaces of degree  $d$  in  $\mathbb{P}^n$  cuts out a complete linear system on  $C$  if the restriction map

$$H^0(\mathbb{P}^n, \mathcal{O}(d)) \rightarrow H^0(C, \mathcal{O}(d))$$

is an isomorphism.

Let  $B$  be a real branch of  $C$ . Then,  $B$  is a compact connected smooth real analytic curve in the real projective space  $\mathbb{P}^n(\mathbb{R})$ . Since the fundamental group of  $\mathbb{P}^n(\mathbb{R})$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ , two cases can occur:  $B$  is contractible in  $\mathbb{P}^n(\mathbb{R})$  or  $B$  is not. In the latter case, we say that  $B$  is a *pseudo-line* of  $C$ . In the former case,  $B$  is an *oval* of  $C$ .

**Corollary 3.1.** *Let  $n \geq 2$  be an integer. Let  $C$  be a smooth geometrically integral real algebraic curve in  $\mathbb{P}^n$ . Let  $c$  be its degree and let  $g$  be its genus. Suppose that  $C$  has many real branches and let  $B_1, \dots, B_g$  be mutually distinct real branches of  $C$ . Let  $d$  be a nonzero natural integer such that the linear system of all real hypersurfaces of degree  $d$  in  $\mathbb{P}^n$  cuts out a complete linear system on  $C$ . Let  $\mathbf{e}$  be a partition of  $cd$  of length  $g$ . Let  $v$  be the number of real hypersurfaces  $D$  in  $\mathbb{P}^n$  of degree  $d$  such that  $D$  is tangent to the real branches  $B_1, \dots, B_g$  of  $C$  with order of tangency  $\mathbf{e}_1, \dots, \mathbf{e}_g$ , respectively. Then,  $v$  is finite. Moreover,  $v \neq 0$  if and only if one of the following conditions is satisfied:*

1.  $d$  is even and  $\mathbf{e}$  is an even partition, or
2.  $d$  is odd and, for all  $i = 1, \dots, g$ ,  $B_i$  is an oval of  $C$  if and only if  $\mathbf{e}_i$  is even.

Furthermore, if  $v \neq 0$  then  $v = \prod_{i=1}^g \mathbf{e}_i$ .

*Proof.* Let  $S$  be the linear system on  $C$  cut out by all real hypersurfaces of  $\mathbb{P}^n$  of degree  $d$ . By hypothesis,  $S$  is complete. Moreover,  $v$  is equal to the number of divisors  $D$  in  $S$  of the form  $\sum \mathbf{e}_i P_i$ , for some  $P_i \in B_i$ . We determine the latter number.

The degree of  $S$  is equal to  $e = cd$ . Let  $\varepsilon_i$  be the degree mod 2 of  $S$  on  $B_i$ . If  $d$  is even then  $\varepsilon_i \equiv 0 \pmod{2}$  for  $i = 1, \dots, g$ . If  $d$  is odd then  $\varepsilon_i \equiv 0 \pmod{2}$  if and only if  $B_i$  is an oval of  $C$ . By Corollary 2.2, the number of divisors  $D$  in  $S$  of the form  $\sum \mathbf{e}_i P_i$ , for some  $P_i \in B_i$ , is finite and is nonzero if and only if condition 1 or 2 is satisfied. Moreover, in that case, this number is equal to  $\prod \mathbf{e}_i$ .  $\square$

Corollary 3.1 is a generalization of [4, Theorem 3.1], where we have counted only hypersurfaces tangent to  $g$  real branches with one and the same order of tangency to each of these branches.

As immediate consequences of Corollary 3.1, we mention the following two statements.

**Corollary 3.2.** *Let  $n \geq 2$  be an integer. Let  $C$  be a smooth geometrically integral real algebraic curve in  $\mathbb{P}^n$  having many real branches. Let  $c$  be its degree and let  $g$  be its genus. Let  $d$  be a nonzero even natural integer such that the linear system of all real hypersurfaces of degree  $d$  in  $\mathbb{P}^n$  cuts out a complete linear system on  $C$ . Let  $\mathbf{e}$  be a partition of  $cd$  of length  $g$ . Let  $v$  be the number of real hypersurfaces in  $\mathbb{P}^n$  of degree  $d$  having tangency  $\mathbf{e}$  to  $g$  real branches of  $C$ . Then,  $v$  is finite. Moreover,  $v \neq 0$  if and only if  $\mathbf{e}$  is an even partition. In that case,*

$$v = \begin{cases} \frac{g!}{m_1! \dots m_r!} \cdot \prod_{i=1}^g \mathbf{e}_i & \text{if } C \text{ is an } (M-1)\text{-curve, and} \\ \frac{(g+1)!}{m_1! \dots m_r!} \cdot \prod_{i=1}^g \mathbf{e}_i & \text{if } C \text{ is an } M\text{-curve,} \end{cases}$$

where  $m_1, \dots, m_r$  are the multiplicities of  $\mathbf{e}$ .

**Corollary 3.3.** *Let  $n \geq 2$  be an integer. Let  $C$  be a smooth geometrically integral real algebraic curve in  $\mathbb{P}^n$  having many real branches. Let  $c$  be its degree and let  $g$  be its genus. Let  $\delta$  be the number of pseudo-lines of  $C$ . Let  $d$  be an odd natural integer such that the linear system of all real hypersurfaces of degree  $d$  in  $\mathbb{P}^n$  cuts out a complete linear system on  $C$ . Let  $\mathbf{e}$  be a partition of  $cd$  of length  $g$ . Let  $v$  be the number of real hypersurfaces in  $\mathbb{P}^n$  of degree  $d$  having tangency  $\mathbf{e}$  to  $g$  real branches of  $C$ . Then,  $v$  is finite. Moreover,  $v \neq 0$  if and only if the number of odd members of  $\mathbf{e}$  is equal to  $\delta$ . In that case,*

$$v = \begin{cases} \frac{\delta! \cdot (g-\delta)!}{m_1! \dots m_r!} \cdot \prod_{i=1}^g \mathbf{e}_i & \text{if } C \text{ is an } (M-1)\text{-curve, and} \\ \frac{\delta! \cdot (g+1-\delta)!}{m_1! \dots m_r!} \cdot \prod_{i=1}^g \mathbf{e}_i & \text{if } C \text{ is an } M\text{-curve.} \end{cases}$$

where, as before,  $m_1, \dots, m_r$  are the multiplicities of  $\mathbf{e}$ .

#### 4 Enumerative problems for real plane curves

*Proof of Theorem 1.1.* Let  $d = c - 1$ . Let us show that the linear system of all real curves of degree  $d$  in  $\mathbb{P}^2$  cuts out a complete linear system on  $C$ . The restriction map

$$H^0(\mathbb{P}^2, \mathcal{O}(d)) \rightarrow H^0(C, \mathcal{O}(d))$$

is injective since  $d < c$  and  $C$  is irreducible. The dimension of  $H^0(\mathbb{P}^2, \mathcal{O}(d))$  is equal to  $\frac{1}{2}(d+2)(d+1)$ . We need to show that  $H^0(C, \mathcal{O}(d))$  is of the same dimension. The degree of  $\mathcal{O}(d)$  on  $C$  is equal to  $cd = 2g + 2(c-1)$ . In particular, its degree is strictly greater than  $2g - 2$ . Hence,  $\mathcal{O}(d)$  is nonspecial on  $C$ . By the Riemann–Roch Theorem,

$$\dim H^0(C, \mathcal{O}(d)) = cd - g + 1 = \frac{1}{2}(d+2)(d+1) = \dim H^0(\mathbb{P}^2, \mathcal{O}(d)).$$

It follows that the linear system of all real curves of degree  $d$  in  $\mathbb{P}^2$  cuts out a complete linear system on  $C$ .

Now, there are two cases to consider: the case  $d$  is even and the case  $d$  is odd. If  $d$  is even then the statement of Theorem 1.1 follows from Corollary 3.2. If  $d$  is odd then  $c$  is even and the number  $\delta$  of pseudo-lines of  $C$  is equal to 0. Therefore, if  $d$  is odd, the statement of Theorem 1.1 follows from Corollary 3.3.  $\square$

**Remark 4.1.** Let  $C$ ,  $c$  and  $g$  be as in Theorem 1.1. Observe that there are many partitions  $\mathbf{e}$  of  $c(c-1)$  to which Theorem 1.1 applies, i.e. partitions  $\mathbf{e}$  of  $c(c-1)$  in even numbers and of length  $g$ . Indeed, there are as many as the number of partitions of the integer  $c-1$ , if  $c \geq 4$ . Let us show this fact.

Let  $\mathbf{d}$  be any partition of  $c-1$ . Let  $k$  be its length. Since  $g = \frac{1}{2}(c-1)(c-2)$ , the number  $k$  satisfies

$$k \leq c-1 = \frac{2g}{c-2} \leq g.$$

Define  $\mathbf{e} \in \mathbb{N}^g$  by  $\mathbf{e}_i = 2d_i + 2$  if  $i \leq k$ , and  $\mathbf{e}_i = 2$  if  $i > k$ . Then,

$$\mathbf{e}_1 + \cdots + \mathbf{e}_g = 2(c-1) + 2k + 2(g-k) = c(c-1).$$

It follows that  $\mathbf{e}$  is a partition of  $c(c-1)$  in even numbers and of length  $g$ .

Conversely, any partition of  $c(c-1)$  in even numbers and of length  $g$  arises in this way. Therefore, the number of partitions  $\mathbf{e}$  of  $c(c-1)$  in even numbers and of length  $g$  is equal to the number of partitions of the integer  $c-1$ , if  $c \geq 4$ .

**Example 4.2.** Let  $C$  be a smooth real quintic curve in  $\mathbb{P}^2$  having many real branches. Here,  $c = 5$  and  $g = 6$  and  $C$  has at least 6 real branches. There are 5 partitions of  $c-1 = 4$ :

$$(1, 1, 1, 1), \quad (2, 1, 1), \quad (2, 2), \quad (3, 1), \quad (4).$$

The corresponding partitions of  $c(c-1) = 20$  in even numbers and of length 6 are

$$(4, 4, 4, 4, 2, 2), \quad (6, 4, 4, 2, 2, 2), \quad (6, 6, 2, 2, 2, 2), \\ (8, 4, 2, 2, 2, 2), \quad (10, 2, 2, 2, 2, 2).$$

Then, for example, Theorem 1.1 states that there are exactly 7680 real quartics tangent to 6 real branches of  $C$  with orders of tangency 4, 4, 4, 4, 2, 2, if  $C$  is an  $(M - 1)$ -curve. There are 53760 of such real quartics of  $C$  is an  $M$ -curve.

**Theorem 4.3.** *Let  $C$  be a smooth real algebraic curve in  $\mathbb{P}^2$  having many real branches. Let  $c$  be its degree and let  $g$  be its genus. Let  $\mathbf{e}$  be a partition of  $c(c - 2)$  of length  $g$ . Let  $v$  be the number of real plane curves of degree  $c - 2$  having tangency  $\mathbf{e}$  to  $g$  real branches of  $C$ . Then,  $v$  is finite. Moreover,  $v \neq 0$  if and only if one of the following conditions is satisfied:*

1.  $c$  is even and  $\mathbf{e}$  is an even partition,
2.  $c$  is odd and exactly one of the members of  $\mathbf{e}$  is odd.

Furthermore, in Case 1,

$$v = \begin{cases} \frac{g!}{m_1! \dots m_r!} \cdot \prod_{i=1}^g \mathbf{e}_i & \text{if } C \text{ is an } (M - 1)\text{-curve, and} \\ \frac{(g + 1)!}{m_1! \dots m_r!} \cdot \prod_{i=1}^g \mathbf{e}_i & \text{if } C \text{ is an } M\text{-curve,} \end{cases}$$

where, as before,  $m_1, \dots, m_r$  are the multiplicities of  $\mathbf{e}$ . In Case 2,

$$v = \begin{cases} \frac{(g - 1)!}{m_1! \dots m_r!} \cdot \prod_{i=1}^g \mathbf{e}_i & \text{if } C \text{ is an } (M - 1)\text{-curve, and} \\ \frac{g!}{m_1! \dots m_r!} \cdot \prod_{i=1}^g \mathbf{e}_i & \text{if } C \text{ is an } M\text{-curve.} \end{cases}$$

*Proof.* Let  $d = c - 2$ . We again need to show that the restriction map

$$H^0(\mathbb{P}^2, \mathcal{O}(d)) \rightarrow H^0(C, \mathcal{O}(d))$$

is an isomorphism. For the same reasons as above, the map is injective. The degree of  $\mathcal{O}(d)$  on  $C$  is equal to  $cd = 2g + c - 2$ . In particular, its degree is strictly greater than  $2g - 2$ . Hence,  $\mathcal{O}(d)$  is nonspecial on  $C$ . By the Riemann–Roch Theorem,

$$\dim H^0(C, \mathcal{O}(d)) = cd - g + 1 = \frac{1}{2}(d + 2)(d + 1) = \dim H^0(\mathbb{P}^2, \mathcal{O}(d)).$$

It follows that the linear system of all real curves of degree  $d$  in  $\mathbb{P}^2$  cuts out a complete linear system on  $C$ .

There are again two cases to consider: the case  $d$  is even and the case  $d$  is odd. If  $d$  is even then the statement of Theorem 1.1 follows from Corollary 3.2. If  $d$  is odd then  $c$  is odd as well and the number  $\delta$  of pseudo-lines of  $C$  is equal to 1. Therefore, if  $d$  is odd, the statement of Theorem 1.1 follows from Corollary 3.3.  $\square$

**Example 4.4.** Let  $C$  be a smooth real cubic curve in  $\mathbb{P}^2$ . Let  $\nu$  be the number of real lines tangent to one real branch of  $C$  with order of tangency equal to 3. Then, according to Theorem 4.3,  $\nu = 3$ . This statement is the well-known fact that a real cubic curve has exactly 3 real inflection points [8].

**Example 4.5.** Let  $C$  be a smooth plane real quartic curve in  $\mathbb{P}^2$ . Let  $\nu$  be the number of real conics tangent to 3 real branches of  $C$  with orders of tangency 4, 2, 2. If  $C$  has exactly 3 real branches then  $\nu = 48$  by Theorem 4.3. If  $C$  has exactly 4 real branches then  $\nu = 192$  by Theorem 4.3.

**Remark 4.6.** Let  $C$ ,  $c$  and  $g$  be as in Theorem 4.3. If  $c$  is even and  $c \geq 4$ , then the number of partitions  $\mathbf{e}$  of  $c(c-2)$  satisfying condition 1 of Theorem 4.3 is equal to the number of partitions of  $\frac{1}{2}(c-2)$ . This can be shown in exactly the same manner as in Remark 4.1.

If  $c$  is odd and  $c \geq 5$ , then there is a finite-to-one correspondence between the set of partitions  $\mathbf{e}$  of  $c(c-2)$  satisfying condition 2 of Theorem 4.3 and the set of all partitions  $\mathbf{d}$  of  $\frac{1}{2}(c-1)$ . However, the correspondence is not bijective. Indeed, let  $\mathbf{d}$  be a partition of  $\frac{1}{2}(c-1)$ . Let  $k$  be its length. Let  $r$  be a natural integer satisfying either  $1 \leq r \leq k$  and  $d_{r+1} < d_r$ , or  $r = g$ . Define  $\mathbf{e} \in \mathbb{N}^g$  by  $\mathbf{e}_i = 2d_i + 2$  if  $i \leq k$  and  $i \neq r$ ,  $\mathbf{e}_i = 2d_i + 1$  if  $i \leq k$  and  $i = r$ ,  $\mathbf{e}_i = 2$  if  $i > k$  and  $i \neq r$ , and  $\mathbf{e}_i = 1$  if  $i > k$  and  $i = r$ . Then,  $\mathbf{e}$  is a partition of  $c(c-2)$  of length  $g$  and satisfying condition 2 of Theorem 4.3. Moreover, each partition of  $c(c-2)$  of length  $g$  and satisfying condition 2 of Theorem 4.3 arises in this way.

**Example 4.7.** Let  $C$  be a smooth real quintic curve in  $\mathbb{P}^2$  having many real branches. Then,  $c = 5$  and  $g = 6$  and  $C$  has at least 6 real branches. There are two partitions of  $\frac{1}{2}(c-1) = 2$ : (1, 1) and (2). The corresponding partitions of  $c(c-2) = 15$  of length 6 and satisfying condition 2 of Theorem 4.3 are

$$(4, 4, 2, 2, 2, 1), \quad (4, 3, 2, 2, 2, 2) \\ (6, 2, 2, 2, 2, 1), \quad (5, 2, 2, 2, 2, 2).$$

Then, for example, Theorem 4.3 states that there are exactly 960 real cubics tangent to 6 real branches of  $C$  with orders of tangency 4, 3, 2, 2, 2, 2, if  $C$  is an  $(M-1)$ -curve. There are 6720 of such real cubics if  $C$  is an  $M$ -curve.

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