Simplicial maps from the 3-sphere to the 2-sphere

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Abstract. We give minimal simplicial maps \( h : S^3 \rightarrow S^2 \) and \( \xi : S^3_{12} \rightarrow S^2_4 \) of Hopf invariant one and two respectively. In general we give, for each \( n \geq 3 \), a simplicial map \( \xi : S^3_{2n} \rightarrow S^2_4 \) (not necessarily minimal) of Hopf invariant \( n \). We use the notation \( S^k_v \) to denote a \( v \)-vertex triangulation of the \( k \)-sphere.

Key words. Hopf invariant, Seifert fibrations, simplicial maps.

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Introduction. Let \( f : S^3 \rightarrow S^2 \) be any map; we may assume it simplicial relative to some triangulations of \( S^3 \) and \( S^2 \). Note that the pre-image, \( f^{-1}(x) \), of an interior point \( x \) of any 2-simplex of \( S^2 \) is a simple closed curve in \( S^3 \) and by using orientations of \( S^2 \) and \( S^3 \) a natural orientation can be assigned to it. As we know any simple closed curve bounds in \( S^3 \); so by subdividing \( S^3 \) suitably, we can choose a 2-chain \( \Sigma^2 \) (say) bounded by \( f^{-1}(x) \) and orient all its 2-simplices such that the induced orientation on \( f^{-1}(x) \) coincides with its natural orientation. In this way we get a homomorphism \( f_* : H_2(\Sigma^2, f^{-1}(x)) \rightarrow H_2(S^2, x) \), defined as \( f_*([x]) = d \cdot [\beta] \) for some integer \( d \). The number \( d \) is independent of the choice of the interior point \( x \), it depends only on the homotopy class of the map \( f \) and is called Hopf invariant of the map \( f \).

We know that any two maps from the 3-sphere to the 2-sphere are homotopic if and only if they have the same Hopf invariant. In this article we give, for each \( n \in \mathbb{N} \), a simplicial map (from the 3-sphere to the 2-sphere) of Hopf invariant \( n \) and we observe that for \( n = 1, 2 \) these are minimal simplicial maps.

We say that a simplicial map \( \xi : S^3_{v(n)} \rightarrow S^2_4 \) is a minimal simplicial map of Hopf invariant \( n \) if there is no simplicial map \( S^3_{v(n)-1} \rightarrow S^2_4 \) of Hopf invariant \( n \). It seems to us that it is a hard problem to find such minimal simplicial maps. However if we restrict ourselves to the category of some special maps, e.g. if we consider only those maps \( S^3 \rightarrow S^2 \) in which the inverse image of each point of \( S^2 \) is homeomorphic to \( S^1 \) then we hope to get minimal simplicial maps of any given Hopf invariant. Note that in each homotopy class of maps \( S^3 \rightarrow S^2 \), such a special map exists. So we shall give simplicial maps \( S^3_{v(n)} \rightarrow S^2_4 \) in this category.
Indeed we are going to triangulate a particular type of the Seifert fibrations \([3]\) of the 3-sphere. We know that any Seifert fibration of \(S^3\) gives a quotient map \(S^3 \to S^2\) (see \([3]\)) under which a circle maps to a point and at most two points of \(S^2\) can be exceptional and all other points are ordinary. But in our constructions at most one point of \(S^2\) will be exceptional. We first recall definitions of ordinary and exceptional points of \(S^2\) for a given quotient map \(S^3 \to S^2\) having only one exceptional point.

**Ordinary points.** Let \(f : S^3 \to S^2\) be a map in which \(f^{-1}(x)\) is homeomorphic to \(S^1\) for all points \(x\) of \(S^2\) and let \(D^2\) denote the unit 2-disk in \(\mathbb{R}^2\) with centre at the origin. We say \(x \in S^2\) is an ordinary point or a regular point if there exists a neighbourhood \(U_x \subseteq S^2\) of \(x\), and homeomorphisms

\[
\begin{align*}
  f^{-1}(U_x) &\xrightarrow{h} D^2 \times S^1 \\
  f &\downarrow \\
  U_x &\xrightarrow{h} D^2
\end{align*}
\]

\(h : f^{-1}(U_x) \to D^2 \times S^1\) and \(\tilde{h} : U_x \to D^2\) with \(\tilde{h}(x) = 0\) for which the above diagram commutes.

**Exceptional points.** Let \(\rho : D^2 \to D^2\) be a rotation map defined as \(\rho(r \cdot e^{i\theta}) = r \cdot e^{i(\theta + 2\pi/n)}\) for some \(n \geq 2\). Consider the quotient space, \(\Gamma_n\), obtained from \(D^2 \times I\) by identifying the points \((x, 0)\) with \((\rho(x), 1)\) for all \(x \in D^2\). This identification gives a quotient map \(q_n : D^2 \times I \to \Gamma_n\) under which exactly \(n\) fibers, i.e. \(x \times I, \rho(x) \times I, \rho^2(x) \times I, \ldots, \rho^{n-1}(x) \times I\) (here \(x \neq 0\)), of \(D^2 \times I\), together form a single circle of \(\Gamma_n\) and the fiber \(0 \times I\) of \(D^2 \times I\) maps to the middle circle of \(\Gamma_n\).

Let a map \(\eta : D^2 \to D^2\) be defined as \(\eta(z) = z^n\) for some \(n \geq 2\). Then there is a unique map \(g : \Gamma_n \to D^2\) which makes the following diagram commutative.

\[
\begin{align*}
  D^2 \times I &\xrightarrow{q} \Gamma_n \\
  \eta &\downarrow \\
  D^2 &\xrightarrow{g} D^2
\end{align*}
\]

We call a point \(x\) of \(S^2\) an **exceptional** or a **singular point** of multiplicity \(n\), for a given map \(f : S^3 \to S^2\), if for each neighbourhood \(U_x\) of \(x\) there are homeomorphisms \(h : f^{-1}(U_x) \to \Gamma_n\) and \(\tilde{h} : U_x \to D^2\) with \(\tilde{h}(x) = 0\) which make the following diagram commutative.

\[
\begin{align*}
  f^{-1}(U_x) &\xrightarrow{h} \Gamma_n \\
  f &\downarrow \\
  U_x &\xrightarrow{\tilde{h}} D^2
\end{align*}
\]

A fiber corresponding to an exceptional point will be called an **exceptional fiber**.
Theorem. There exist minimal simplicial maps \( \eta, \xi : S^3_{12} \to S^2_4 \) of Hopf invariants one and two respectively, and in general for each \( n > 2 \) there is a simplicial map \( \xi : S^3_{6n} \to S^2_4 \) of Hopf invariant \( n \).

Proof. A simplicial map \( \eta : S^3_{12} \to S^2_4 \) has been shown and defined by the vertex labelling in Figures 1 and 2.

![Figure 1](image)

We have proved in [2] that the map \( \eta \) is the minimal triangulation of the well-known Hopf fibration \( h : S^3 \to S^2 \), so the map has Hopf invariant one. Now we shall give a minimal simplicial map \( \xi : S^3_{12} \to S^2_4 \) of Hopf invariant two.

Construction of a minimal simplicial map of Hopf invariant two. As we wish to make simplicial maps in which the pre-image of each point of \( S^2_4 \) (it is the minimal triangulation of the 2-sphere) is homeomorphic to \( S^1 \), so corresponding to four vertices of \( S^2_4 \) there are four simplicial circles in \( S^3 \) and each will have at least three vertices. So at least 12 vertices are needed in \( S^3 \) to make a simplicial map of any non-zero Hopf invariant. Moreover it is interesting to know that any simplicial map \( S^3_v \to S^2_4 \), with \( v \leq 11 \), is a homotopically trivial map (see Theorem IIa of [1]).

The simplicial map \( \xi : S^3_{12} \to S^2_4 = \partial[ABCD] \). Our simplicial complex \( S^3_{12} \) is the union of two solid tori. One of them, \( M^3_9 \), is the pre-image of a 2-simplex \( ABC \). Its triangulated boundary has been shown in Figure 3 below. This solid torus consists of nine
Figure 2. $T_o^2 = \partial N_{12}^3$ and $N_{12}^3$

Figure 3. $T_o^2 = \partial M_{0}^3$ and $M_{0}^3$
3-simplices, three of them are $A_0 B_0 C_0 B_1$, $A_0 A_1 B_1 C_0$, $A_1 B_1 C_1 C_0$ and the remaining six can be obtained from these by using the permutation $\sigma = (A_0 A_1 A_2)(B_0 B_1 B_2) \cdot (C_0 C_1 C_2)(D_0 D_1 D_2)$.

The second solid torus, $N^3_{12}$, is the pre-image of $S^2_4 \setminus \text{Int.} \ ABC$. Its triangulated boundary (shown in Figure 4 above) is isomorphic to, and will be identified with, the boundary of $M^3_2$. It has thirty-six 3-simplices, twelve of them are $A_0 C_0 C_2 D_0$, $A_1 A_2 C_1 D_0$, $A_1 A_2 B_2 D_0$, $A_2 B_0 B_2 D_0$, $B_0 B_2 C_2 D_0$, $B_0 C_0 C_2 D_0$, $A_0 B_1 D_0 D_1$, $B_1 C_1 D_0 D_1$, $A_1 C_1 D_0 D_1$, $A_1 B_2 D_0 D_1$, $B_2 C_2 D_0 D_1$, $A_0 C_2 D_0 D_1$ and the remaining twenty-four can be obtained from these by using the permutation $\sigma$. Note that the solid torus $N^3_{12}$ is homeomorphic to $\Gamma_2$ under a fiber preserving homeomorphism, its middle fiber $\xi^{-1}(D)$ is exceptional of multiplicity 2.

The simplicial map $\xi : S^3_{12} \to S^2_4$ given by $X_i \mapsto X$ for all $X \in \{A, B, C, D\}$ is well defined, as under this map simplices of $S^3_{12}$ get mapped onto the simplices of $S^2_4$.

**Remarks.** 1. The solid torus $N^3_{12}$ contains pre-images of $CAD$, $ABD$ and $BCD$. We
have given simplices of $\xi^{-1}(CAD)$, in Figure 5, explicitly and simplices of $\xi^{-1}(ABD)$ and $\xi^{-1}(BCD)$ can be obtained similarly. The simplicial complex $\xi^{-1}(CAD)$ consists of twelve 3-simplices, four of them are $A_0 C_0 D_0 D_2$, $A_0 A_1 C_0 D_2$, $A_1 C_0 D_1 D_2$, $A_1 C_0 C_1 D_1$ and the remaining eight can be obtained from these by using the permutation $\alpha$.

2. From here it is very clear that pre-images of $AD$, $BD$ and $CD$ are Möbius strips bounded by $\xi^{-1}(A)$, $\xi^{-1}(B)$ and $\xi^{-1}(C)$ respectively. In each case the middle circle of the Möbius strip is $\xi^{-1}(D)$. Further note that pre-images of $AB$, $BC$ and $CA$ are cylinders.

3. Here vertices $A$, $B$, $C$ are ordinary vertices while the vertex $D$ is an exceptional vertex of multiplicity 2. So in order to verify the Hopf invariant of the map $\xi$, we choose a 2-chain, in $S^3$ bounded by one of the pre-images of $A$, $B$ or $C$ and see the restriction of the map $\xi$ to this 2-chain. In particular let us see in Figure 6 the restriction of $\xi$ to a 2-chain bounded by the pre-image of the vertex $A$ of $S^3$.

It is clear from Figure 6 that the restricted map $\bar{\xi} : (\Sigma^2, \xi^{-1}(x)) \to (S^2, x)$ has degree $\pm 2$, so the Hopf invariant of the map is $\pm 2$ depending upon the choice of the orientations of $S^3$ and $S^2$.

Now we shall give for each $n \geq 3$, a simplicial map $\xi : S^3_{6n} \to S^2_5$ of Hopf invariant $n$ but their minimality is yet to be verified.
In order to make a simplicial map $\xi : S^3_{6n} \to S^2_4$ of Hopf invariant $n \geq 3$, we take a sphere $S^3_{6n}$ which is the union of two solid tori ($M$ and $N$ say) whose common boundary is shown in Figure 7 above.
The solid torus $M$ has been triangulated with $3(2n-1)$ 3-simplices. Three of them are $A_0B_0C_0B_1$, $A_1B_1C_0A_0A_1B_1C_0$ and rest of the 3-simplices can be obtained from these by using the permutation $\pi = \prod_{X \in \{A, B, C, D\}} (X_0X_1 \ldots X_{2n-2})$.

The second solid torus has been triangulated with $3(2n-1)(3n-2)$ 3-simplices; $3(3n-2)$ of these are $C_iA_iA_{i+1}D_0$, $A_iA_{i+1}B_{i+1}D_0$, $A_{i+1}B_{i+1}B_{i+2}D_0$, $B_{i+1}B_{i+2}C_{i+1}D_0$, $B_{i+2}C_{i+1}C_{i+2}D_0$, $C_{i+1}C_{i+2}A_{i+2}D_0$, $A_0B_1D_0D_1$, $B_1C_1D_0D_1$, $A_1C_1D_0D_1$, $A_2B_{i+1}D_0D_1$, $B_{i+1}C_{i+1}D_0D_1$, $C_{i+1}A_{i+2}D_0D_1$ for each odd $i \in \mathbb{Z}/(2n-1)$ i.e. $i$ lies in the set $\{1, 3, 5, \ldots, 2n-3\}$ and the rest of the 3-simplices can be obtained from these by using the permutation $\pi$. It is easy to verify that the simplicial map $\xi : S^3_{2n} \to S^2_4$ defined as $X_i \to X$ for all $X \in \{A, B, C, D\}$ has Hopf invariant $n$.

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**References**


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