The connected components of the projective line over a ring

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Abstract. The main result of the present paper is that the projective line over a ring $R$ is connected with respect to the relation “distant” if, and only if, $R$ is a GE$_2$-ring.

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1 Introduction

One of the basic notions for the projective line $\mathbb{P}(R)$ over a ring $R$ is the relation distant ($\triangle$) on the point set. Non-distant points are also called parallel. This terminology goes back to the projective line over the real dual numbers, where parallel points represent parallel spears of the Euclidean plane [4, 2.4].

We say that $\mathbb{P}(R)$ is connected (with respect to $\triangle$) if the following holds: For any two points $p$ and $q$ there is a finite sequence of points starting at $p$ and ending at $q$ such that each point other than $p$ is distant from its predecessor. Otherwise $\mathbb{P}(R)$ is said to be disconnected. For each connected component a distance function and a diameter (with respect to $\triangle$) can be defined in a natural way.

One aim of the present paper is to characterize those rings $R$ for which $\mathbb{P}(R)$ is connected. Here we use certain subgroups of the group $\text{GL}_2(R)$ of invertible $2 \times 2$-matrices over $R$, namely its elementary subgroup $E_2(R)$ and the subgroup GE$_2(R)$ generated by $E_2(R)$ and the set of all invertible diagonal matrices. It turns out that $\mathbb{P}(R)$ is connected exactly if $R$ is a GE$_2$-ring, i.e., if $\text{GE}_2(R) = \text{GL}_2(R)$.

Next we turn to the diameter of connected components. We show that all connected components of $\mathbb{P}(R)$ share a common diameter.

It is well known that $\mathbb{P}(R)$ is connected with diameter $\leq 2$ if $R$ is a ring of stable rank 2. We give explicit examples of rings $R$ such that $\mathbb{P}(R)$ has one of the following properties: $\mathbb{P}(R)$ is connected with diameter 3, $\mathbb{P}(R)$ is connected with diameter $\infty$, and $\mathbb{P}(R)$ is disconnected with diameter $\infty$. In particular, we show that there are chain geometries over disconnected projective lines.

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2 Preliminaries

Throughout this paper we shall only consider associative rings with a unit element 1, which is inherited by subrings and acts unitally on modules. The trivial case 1 = 0 is not excluded. The group of invertible elements of a ring $R$ will be denoted by $R^\times$.

Firstly, we turn to the projective line over a ring: Consider the free left $R$-module $R^2$. Its automorphism group is the group $\text{GL}_2(R)$ of invertible $2 \times 2$-matrices with entries in $R$. A pair $(a, b) \in R^2$ is called admissible, if there exists a matrix in $\text{GL}_2(R)$ with $(a, b)$ being its first row. Following [14, p. 785], the projective line over $R$ is the orbit of the free cyclic submodule $R(1, 0)$ under the action of $\text{GL}_2(R)$. So

$$\mathbb{P}(R) := R(1, 0)^{\text{GL}_2(R)}$$

or, in other words, $\mathbb{P}(R)$ is the set of all $p \in R^2$ such that $p = R(a, b)$ for an admissible pair $(a, b) \in R^2$. As has been pointed out in [8, Proposition 2.1], in certain cases $R(x, y) \in \mathbb{P}(R)$ does not imply the admissibility of $(x, y) \in R^2$. However, throughout this paper we adopt the convention that points are represented by admissible pairs only. Two such pairs represent the same point exactly if they are left-proportional by a unit in $R$.

The point set $\mathbb{P}(R)$ is endowed with the symmetric relation $\text{distant} \ (\triangle)$ defined by

$$\triangle := (R(1, 0), R(0, 1))^{\text{GL}_2(R)}. \quad (1)$$

Letting $p = R(a, b)$ and $q = R(c, d)$ gives then

$$p \triangle q \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(R).$$

In addition, $\triangle$ is anti-reflexive exactly if $1 \neq 0$.

The vertices of the distant graph on $\mathbb{P}(R)$ are the points of $\mathbb{P}(R)$, the edges of this graph are the unordered pairs of distant points. Therefore basic graph-theoretical concepts are at hand: $\mathbb{P}(R)$ can be decomposed into connected components (maximal connected subsets), for each connected component there is a distance function (dist($p, q$) is the minimal number of edges needed to go from vertex $p$ to vertex $q$), and each connected component has a diameter (the supremum of all distances between its points).

Secondly, we recall that the set of all elementary matrices

$$B_{12}(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{21}(t) := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \quad \text{with} \ t \in R \quad (2)$$

generates the elementary subgroup $E_2(R)$ of $\text{GL}_2(R)$. The group $E_2(R)$ is also generated by the set of all matrices

$$E(t) := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} = B_{12}(1) \cdot B_{21}(-1) \cdot B_{12}(1) \cdot B_{21}(t) \quad \text{with} \ t \in R, \quad (3)$$
since \(B_{12}(t) = E(-t) \cdot E(0)^{-1}\) and \(B_{21}(t) = E(0)^{-1} \cdot E(t)\). Moreover, we have \(E(t)^{-1} = E(0) \cdot E(-t) \cdot E(0)\), which implies that all finite products of matrices \(E(t)\) already comprise the group \(E_2(R)\).

The subgroup of \(\text{GL}_2(R)\) which is generated by \(E_2(R)\) and the set of all invertible diagonal matrices is denoted by \(\text{GE}_2(R)\). By definition, a \(\text{GE}_2\)-ring is characterized by \(\text{GL}_2(R) = \text{GE}_2(R)\); see, among others, [10, p. 5] or [18, p. 114].

## 3 Connected components

We aim at a description of the connected components of the projective line \(\mathbb{P}(R)\) over a ring \(R\). The following lemma, although more or less trivial, will turn out useful:

**Lemma 3.1.** Let \(X' \in \text{GL}_2(R)\) and suppose that the \(2 \times 2\)-matrix \(X\) over \(R\) has the same first row as \(X'\). Then \(X\) is invertible if, and only if, there is a matrix

\[
M = \begin{pmatrix} 1 & 0 \\ s & u \end{pmatrix} \in \text{GE}_2(R)
\]

such that \(X = MX'\).

**Proof.** Given \(X'\) and \(X\) then \(XX'^{-1} = \begin{pmatrix} 1 & 0 \\ s & u \end{pmatrix} =: M\) for some \(s, u \in R\). Further, \(X = MX'\) is invertible exactly if \(u \in R^*\). This in turn is equivalent to (4).

Here is our main result, where we use the generating matrices of \(E_2(R)\) introduced in (3).

**Theorem 3.2.** Denote by \(C_\infty\) the connected component of the point \(R(1, 0)\) in the projective line \(\mathbb{P}(R)\) over a ring \(R\). Then the following holds:

(a) The group \(\text{GL}_2(R)\) acts transitively on the set of connected components of \(\mathbb{P}(R)\).
(b) Let \(t_1, t_2, \ldots, t_n \in R, n \geq 0,\) and put

\[
(x, y) := (1, 0) \cdot E(t_n) \cdot E(t_{n-1}) \cdots E(t_1).
\]

Then \(R(x, y) \in C_\infty\) and, conversely, each point \(r \in C_\infty\) can be written in this way.

(c) The stabilizer of \(C_\infty\) in \(\text{GL}_2(R)\) is the group \(\text{GE}_2(R)\).
(d) The projective line \(\mathbb{P}(R)\) is connected if, and only if, \(R\) is a \(\text{GE}_2\)-ring.

**Proof.** (a) This is immediate from the fact that the group \(\text{GL}_2(R)\) acts transitively on the point set \(\mathbb{P}(R)\) and preserves the relation \(\triangle\).

(b) Every matrix \(E(t_i)\) appearing in (5) maps \(C_\infty\) onto \(C_\infty\), since \(R(0, 1) \in C_\infty\) goes over to \(R(1, 0) \in C_\infty\). Therefore \(R(x, y) \in C_\infty\).
On the other hand let \( r \in C_{\infty} \). Then there exists a sequence of points \( p_i = R(a_i, b_i) \in \mathbb{P}(R), i \in \{0, 1, \ldots, n\} \), such that

\[
R(1, 0) = p_0 \triangle p_1 \triangle \cdots \triangle p_n = r.
\]

Now the arbitrarily chosen admissible pairs \((a_i, b_i)\) are “normalized” recursively as follows: First define \((x_{-1}, y_{-1}) := (0, -1)\) and \((x_0, y_0) := (1, 0)\). So \( p_0 = R(x_0, y_0) \). Next assume that we already are given admissible pairs \((x_j, y_j)\) with \( p_j = R(x_j, y_j) \) for all \( j \in \{0, 1, \ldots, i - 1\}, 1 \leq i \leq n \). From Lemma 3.1, there are \( s_i \in R \) and \( u_i \in R^* \) such that

\[
\begin{pmatrix}
  x_{i-1} & y_{i-1} \\
  a_i & b_i
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s_i & u_i \end{pmatrix} \begin{pmatrix}
  x_{i-1} & y_{i-1} \\
  -x_{i-2} & -y_{i-2}
\end{pmatrix}.
\]

By putting \( x_i := u_i^{-1} a_i, y_i := u_i^{-1} b_i, \) and \( t_i := u_i^{-1} s_i \) we get

\[
\begin{pmatrix}
  x_i & y_i \\
  -x_{i-1} & -y_{i-1}
\end{pmatrix} = E(t_{i}) \cdot \begin{pmatrix}
  x_{i-1} & y_{i-1} \\
  -x_{i-2} & -y_{i-2}
\end{pmatrix}
\]

and \( p_i = R(x_i, y_i) \). Therefore, finally, \((x_n, y_n)\) is the first row of the matrix

\[
G' := E(t_n) \cdot E(t_{n-1}) \cdots E(t_1) \in E_2(R),
\]

and \( r = R(x_n, y_n) \).

(c) As has been noticed at the end of Section 2, the set of all matrices (3) generates \( E_2(R) \). This together with (b) implies that \( E_2(R) \) stabilizes \( C_{\infty} \). Further, \( R(1, 0) \) remains fixed under each invertible diagonal matrix. Therefore \( GE_2(R) \) is contained in the stabilizer of \( C_{\infty} \).

Conversely, suppose that \( G \in GL_2(R) \) stabilizes \( C_{\infty} \). Then the first row of \( G \), say \((a, b)\), determines a point of \( C_{\infty} \). By (5) and (9), there is a matrix \( G' \in E_2(R) \) and a unit \( u \in R^* \) such that \((a, b) = (1, 0) \cdot (u G')\). Now Lemma 3.1 can be applied to \( G \) and \( u G' \in GE_2(R) \) in order to establish that \( G \in GE_2(R) \).

(d) This follows from (a) and (c).

From Theorem 3.2 and (9), the connected component of \( R(1, 0) \in \mathbb{P}(R) \) is given by all pairs of \((1, 0) \cdot E_2(R)\) or, equivalently, by all pairs of \((1, 0) \cdot GE_2(R)\). Each product (5) gives rise to a sequence

\[
(x_i, y_i) = (1, 0) \cdot E(t_i) \cdot E(t_{i-1}) \cdots E(t_1), \ i \in \{0, 1, \ldots, n\},
\]

which in turn defines a sequence \( p_i := R(x_i, y_i) \) of points with \( p_0 = R(1, 0) \). By putting \( p_n := r \) and by reversing the arguments in the proof of (b), it follows that (6) is true. So, if the diameter of \( C_{\infty} \) is finite, say \( m \geq 0 \), then in order to reach all points of \( C_{\infty} \) it is sufficient that \( n \) ranges from 0 to \( m \) in formula (5).
By the action of GL₂(ℝ), the connected component \( C_p \) of any point \( p \in \mathbb{P}(R) \) is GL₂(ℝ)-equivalent to the connected component \( C_\infty \) of \( R(1,0) \) and the stabilizer of \( C_p \) in GL₂(ℝ) is conjugate to GE₂(ℝ). Observe that in general GE₂(ℝ) is not normal in GL₂(ℝ). Cf. the example in 5.7 (c). All connected components are isomorphic subgraphs of the distant graph.

4 Generalized chain geometries

If \( K \subseteq R \) is a (not necessarily commutative) subfield, then the \( K \)-sublines of \( \mathbb{P}(R) \) give rise to a generalized chain geometry \( \Sigma(K,R) \); see [7]. In contrast to an ordinary chain geometry (cf. [14]) it is not assumed that \( K \) is in the centre of \( R \). Any three mutually distant points are on at least one \( K \)-chain. Two distinct points are distant exactly if they are on a common \( K \)-chain. Therefore each \( K \)-chain is contained in a unique connected component. Each connected component \( C \) together with the set of \( K \)-chains entirely contained in it defines an incidence structure \( \Sigma(C) \). It is straightforward to show that the automorphism group of the incidence structure \( \Sigma(K,R) \) is isomorphic to the wreath product of \( \text{Aut} \Sigma(C) \) with the symmetric group on the set of all connected components of \( \mathbb{P}(R) \).

If \( \Sigma(K,R) \) is a chain geometry then the connected components are exactly the maximal connected subspaces of \( \Sigma(K,R) \) [14, p. 793, p. 821]. Cf. also [15] and [16].

An \( R \)-semilinear bijection of \( R^2 \) induces an automorphism of \( \Sigma(K,R) \) if, and only if, the accompanying automorphism of \( R \) takes \( K \) to \( u^{-1}Ku \) for some \( u \in R^* \). On the other hand, if \( \mathbb{P}(R) \) is disconnected then we cannot expect all automorphisms of \( \Sigma(K,R) \) to be semilinearly induced. More precisely, we have the following:

**Theorem 4.1.** Let \( \Sigma(K,R) \) be a disconnected generalized chain geometry, i.e., the projective line \( \mathbb{P}(R) \) over \( R \) is disconnected. Then \( \Sigma(K,R) \) admits automorphisms that cannot be induced by any semilinear bijection of \( R^2 \).

**Proof.** (a) Suppose that two semilinearly induced bijections \( \gamma_1, \gamma_2 \) of \( \mathbb{P}(R) \) coincide on all points of one connected component \( C \) of \( \mathbb{P}(R) \). We claim that \( \gamma_1 = \gamma_2 \). For a proof choose two distant points \( R(a,b) \) and \( R(c,d) \) in \( C \). Also, write \( \alpha \) for that projectivity which is given by the matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Then \( \beta := \alpha \gamma_1 \gamma_2^{-1} \alpha^{-1} \) is a semilinearly induced bijection of \( \mathbb{P}(R) \) fixing the connected component \( C_\infty \) of \( R(1,0) \) pointwise. Hence \( R(1,0), R(0,1), \) and \( R(1,1) \) are invariant under \( \beta \), and we get

\[
R(x, y)^\beta = R(x^\zeta u, y^\zeta u) \quad \text{for all} \ (x, y) \in R^2
\]

with \( \zeta \in \text{Aut}(R) \) and \( u \in R^* \), say. For all \( x \in R \) the point \( R(x,1) \) is distant from \( R(1,0) \); so it remains fixed under \( \beta \). Therefore \( x = u^{-1}x^\zeta u \) or, equivalently, \( x^\zeta u = ux \) for all \( x \in R \). Finally, \( R(x, y)^\beta = R(ux, uy) = R(x, y) \) for all \( (x, y) \in R^2 \), whence \( \gamma_1 = \gamma_2 \).
(b) Let $\gamma$ be a non-identical projectivity of $\mathbb{P}(R)$ given by a matrix $G \in \text{GE}_2(R)$, for example, $G = B_{12}(1)$. From Theorem 3.2, the connected component $C_\infty$ of $R(1, 0)$ is invariant under $\gamma$. Then

$$
\delta : \mathbb{P}(R) \to \mathbb{P}(R) : \begin{cases} 
p \mapsto p^\gamma & \text{for all } p \in C_\infty 
p \mapsto p & \text{for all } p \in \mathbb{P}(R) \setminus C_\infty
\end{cases}
$$

(11)

is an automorphism of $\Sigma(K, R)$. The projectivity $\gamma$ and the identity on $\mathbb{P}(R)$ are different and both are linearly induced. The mapping $\delta$ coincides with $\gamma$ on $C_\infty$ and with the identity on every other connected component. There are at least two distinct connected components of $\mathbb{P}(R)$. Hence it follows from (a) that $\delta$ cannot be semi-linearly induced.

If a cross-ratio in $\mathbb{P}(R)$ is defined according to [14, 1.3.5] then four points with cross-ratio are necessarily in a common connected component. Therefore, the automorphism $\delta$ defined in (11) preserves all cross-ratios. However, cross-ratios are not invariant under $\delta$ if one adopts the definition in [4, p. 90] or [14, 7.1] which works for commutative rings only. This is due to the fact that here four points with cross-ratio can be in two distinct connected components.

We shall give examples of disconnected (generalized) chain geometries in the next section.

## 5 Examples

There is a widespread literature on (non-)GE$_2$-rings. We refer to [1], [9], [10], [11], [12], [13], and [18]. We are particularly interested in rings containing a field and the corresponding generalized chain geometries.

**Remark 5.1.** Let $R$ be a ring. Then each admissible pair $(x, y) \in R^2$ is unimodular, i.e., there exist $x', y' \in R$ with $xx' + yy' = 1$. We remark that

$$(x, y) \in R^2 \text{ unimodular} \Rightarrow (x, y) \text{ admissible} \tag{12}$$

is satisfied, in particular, for all commutative rings, since $xx' + yy' = 1$ can be interpreted as the determinant of an invertible matrix with first row $(x, y)$. Also, all rings of stable rank 2 [19, p. 293] satisfy (12); cf. [19, 2.11]. For example, local rings, matrix rings over fields, and finite-dimensional algebras over commutative fields are of stable rank 2. See [13, 4.1B], [19, §2], [20], and the references given there.

The following example shows that (12) does not hold for all rings: Let $R := K[X, Y]$ be the polynomial ring over a proper skew field $K$ in independent central indeterminates $X$ and $Y$. There are $a, b \in K$ with $c := ab - ba \neq 0$. From

$$(X + a)(Y + b)c^{-1} - (Y + b)(X + a)c^{-1} = 1,$$

the pair $(X + a, -(Y + b)) \in R^2$ is unimodular. However, this pair is not admissible: Assume to the contrary that $(X + a, -(Y + b))$ is the first row of a matrix
\( M \in \text{GL}_2(R) \) and suppose that the second column of \( M^{-1} \) is the transpose of \((v_0, w_0) \in R^2 \). Then
\[
P := \{(v, w) \in R^2 \mid (X + a)v - (Y + b)w = 0\} = (v_0, w_0)R.
\]

On the other hand, by [17, Proposition 1], the right \( R \)-module \( P \) cannot be generated by a single element, which is a contradiction.

**Examples 5.2.** (a) If \( R \) is a ring of stable rank 2 then \( \mathbb{P}(R) \) is connected and its diameter is \( \leq 2 \) [14, Proposition 1.4.2]. In particular, the diameter is 1 exactly if \( R \) is a field and it is 0 exactly if \( R = \{0\} \).

As has been pointed out in [2, (2.1)], the points of the projective line over a ring \( R \) of stable rank 2 are exactly the submodules \( R(t_2t_1 + 1, t_2) \) of \( R^2 \) with \( t_1, t_2 \in R \). Clearly, this is just a particular case of our more general result in Theorem 3.2 (b).

Conversely, if an arbitrary ring \( R \) satisfies (12) and \( \mathbb{P}(R) \) is connected with diameter \( \leq 2 \), then \( R \) is a ring of stable rank 2 [14, Proposition 1.1.3].

(b) The projective line over a (not necessarily commutative) Euclidean ring \( R \) is connected, since every Euclidean ring is a GE₂-ring [13, Theorem 1.2.10].

Our next examples are given in the following theorem:

**Theorem 5.3.** Let \( U \) be an infinite-dimensional vector space over a field \( K \) and put \( R := \text{End}_K(U) \). Then the projective line \( \mathbb{P}(R) \) over \( R \) is connected and has diameter 3.

**Proof.** We put \( V := U \times U \) and denote by \( \mathcal{G} \) those subspaces \( W \) of \( V \) that are isomorphic to \( V/W \). By [5, 2.4], the mapping
\[
\Phi : \mathbb{P}(R) \to \mathcal{G} : R(\alpha, \beta) \mapsto \{(u^\alpha, u^\beta) \mid u \in U\}
\] is bijective and two points of \( \mathbb{P}(R) \) are distant exactly if their \( \Phi \)-images are complementary. By an abuse of notation, we shall write \( \text{dist}(W_1, W_2) = n \), whenever \( W_1, W_2 \) are \( \Phi \)-images of points at distance \( n \), and \( W_1 \triangle W_2 \) to denote complementary elements of \( \mathcal{G} \). As \( V \) is infinite-dimensional, \( 2 \dim W = \dim V = \dim W \) for all \( W \in \mathcal{G} \).

We are going to verify the theorem in terms of \( \mathcal{G} \): So let \( W_1, W_2 \in \mathcal{G} \). Put \( Y_{12} := W_1 \cap W_2 \) and choose \( Y_{23} \leq W_2 \) such that \( W_2 = Y_{12} \oplus Y_{23} \). Then \( W_1 \cap Y_{23} = \{0\} \) so that there is a \( W_3 \in \mathcal{G} \) through \( Y_{23} \) with \( W_1 \triangle W_3 \). By the law of modularity,
\[
W_2 \cap W_3 = (Y_{23} + Y_{12}) \cap W_3 = Y_{23} + (Y_{12} \cap W_3) = Y_{23}.
\]

Finally, choose \( Y_{14} \leq W_1 \) with \( W_1 = Y_{12} \oplus Y_{14} \) and \( Y_{34} \leq W_3 \) with \( W_3 = Y_{23} \oplus Y_{34} \). Hence we arrive at the decomposition
\[
V = Y_{14} \oplus Y_{12} \oplus Y_{23} \oplus Y_{34}.
\] As \( W_2 \in \mathcal{G} \), so is also \( W_4 := Y_{14} \oplus Y_{34} \). Now there are two possibilities:
Case 1: There exists a linear bijection $\sigma : Y_{14} \to Y_{23}$. We define $Y := \{v + v^\sigma \mid v \in Y_{14}\}$. Then $Y_{14}$, $Y_{23}$, and $Y$ are easily seen to be mutually complementary subspaces of $Y_{14} \oplus Y_{23}$. Therefore, from (14),

$$V = Y_{14} \oplus Y_{12} \oplus Y \oplus Y_{34} = Y \oplus Y_{12} \oplus Y_{23} \oplus Y_{34},$$

(15)

i.e., $W_1 \triangle (Y \oplus Y_{34}) \triangle W_2$. So dist$(W_1, W_2) \leq 2$.

Case 2: $Y_{14}$ and $Y_{23}$ are not isomorphic. Then $\dim Y_{12} = \dim W_1$, since otherwise $\dim Y_{12} < \dim W_1 = \dim W_2$ together with well-known rules for the addition of infinite cardinal numbers would imply

$$\dim W_1 = \max\{\dim Y_{12}, \dim Y_{14}\} = \dim Y_{14},$$

$$\dim W_2 = \max\{\dim Y_{12}, \dim Y_{23}\} = \dim Y_{23},$$

a contradiction to $\dim Y_{14} \neq \dim Y_{23}$.

Likewise, it follows that $\dim Y_{34} = \dim W_3$. But this means that $Y_{12}$ and $Y_{34}$ are isomorphic, whence the proof in case 1 can be modified accordingly to obtain a $Y \leq Y_{12} \oplus Y_{34}$ such that $W_1 \triangle W_3 \triangle (Y \oplus Y_{14}) \triangle W_2$. So now dist$(W_1, W_2) \leq 3$.

It remains to establish that in $\mathscr{G}$ there are elements with distance 3: Choose any subspace $W_1 \in \mathcal{G}$ and a subspace $W_2 \leq W_1$ such that $W_1/W_2$ is 1-dimensional. With the previously introduced notations we get $Y_{12} = W_2$, $\dim Y_{14} = 1$, $Y_{23} = \{0\}$, $Y_{34} = W_3 \in \mathcal{G}$, and $W_4 = Y_{14} \oplus Y_{12}$. As before, $V = W_2 \oplus W_4$ and from $\dim W_2 = 1 + \dim W_1 = \dim W_3 = 1 + \dim W_3 = \dim W_4$ we obtain $W_2, W_4 \in \mathcal{G}$.

By construction, dist$(W_1, W_2) \neq 0, 1$. Also, this distance cannot be 2, since $W \triangle W_1$ implies $W + W_2 \neq V$ for all $W \in \mathcal{G}$.

This completes the proof.

If $K$ is a proper skew field, then $K$ can be embedded in $\text{End}_K(U)$ in several ways [6, p. 17]; each embedding gives rise to a connected generalized chain geometry. (In [6] this is just called a “chain geometry”.) If $K$ is commutative, then $\text{End}_K(U)$ is a $K$-algebra and $x \mapsto x \text{id}_U$ is a distinguished embedding of $K$ into the centre of $\text{End}_K(U)$.

In this way an ordinary connected chain geometry arises; cf. [14, 4.5. Example (4)].

Our next goal is to show the existence of chain geometries with connected components of infinite diameter.

**Remark 5.4.** If $R$ is an arbitrary ring then each matrix $A \in \text{GE}_2(R)$ can be expressed in standard form

$$A = \text{diag}(u, v) \cdot E(t_n) \cdot E(t_{n-1}) \cdots E(t_1),$$

(16)

where $u, v \in R^*$, $t_1, t_n \in R$, $t_2, t_3, \ldots, t_{n-1} \in R \setminus (R^* \cup \{0\})$, and $t_1, t_2 \neq 0$ in case $n = 2$ [10, Theorem (2.2)]. Since $E(0)^2 = \text{diag}(-1, -1)$, each matrix $A \in \text{GE}_2(R)$ can also be written in the form (16) subject to the slightly modified conditions $u, v \in R^*$, $t_1, t_n \in R$, $t_2, t_3, \ldots, t_{n-1} \in R \setminus (R^* \cup \{0\})$, and $n \geq 1$. We call this a modified standard form of $A$.  

Suppose that there is a unique standard form for GE$_2(R)$. For all non-diagonal matrices in GE$_2(R)$ the unique representation in standard form is at the same time the unique representation in modified standard form. Any diagonal matrix $A \in$ GE$_2(R)$ is already expressed in standard form, but its unique modified standard form reads $A = -A \cdot E(0)^2$. Therefore there is also a unique modified standard form for GE$_2(R)$.

By reversing these arguments it follows that the existence of a unique modified standard form for GE$_2(R)$ is equivalent to the existence of a unique standard form for GE$_2(R)$.

**Theorem 5.5.** Let $R$ be a ring with a unique standard form for GE$_2(R)$ and suppose that $R$ is not a field. Then every connected component of the projective line $\mathbb{P}(R)$ over $R$ has infinite diameter.

**Proof.** Since $R$ is not a field, there exists an element $t \in R \setminus (R^* \cup \{0\})$. We put

$$q_m := R(c_m, d_m) \text{ where } (c_m, d_m) := (1, 0) \cdot E(t)^m \text{ for all } m \in \{0, 1, \ldots \}.$$  \hspace{1cm} (17)

Next fix one $m \geq 1$, and put $n - 1 := \text{dist}(q_0, q_{m-1}) \geq 0$. Hence there exists a sequence

$$p_0 \triangle p_1 \triangle \cdots \triangle p_{n-1} \triangle p_n$$  \hspace{1cm} (18)

such that $p_0 = q_0$, $p_{n-1} = q_{m-1}$, and $p_n = q_m$. Now we proceed as in the proof of Theorem 3.2 (b): First let $p_i = R(a_i, b_i)$ and put $(x_{-1}, y_{-1}) := (0, -1), (x_0, y_0) := (1, 0)$. Then pairs $(x_i, y_i) \in R^2$ and matrices $E(t_i) \in E_2(R)$ are defined in such a way that $p_i = R(x_i, y_i)$ and that (8) holds for $i \in \{1, 2, \ldots, n\}$. It is immediate from (8) that a point $p_i$, $i \geq 2$, is distant from $p_{i-2}$ exactly if $t_i \in R^*$. Also, $p_i = p_{i-2}$ holds if, and only if, $t_i = 0$. We infer from (8) and $\text{dist}(p_i, p_j) = |i - j|$ for all $i, j \in \{0, 1, \ldots, n - 1\}$ that

$$\begin{pmatrix} x_n & y_n \\ -x_{n-1} & -y_{n-1} \end{pmatrix} = E(t_n) \cdot E(t_{n-1}) \cdots E(t_1),$$  \hspace{1cm} (19)

where $t_i \in R \setminus (R^* \cup \{0\})$ for all $i \in \{2, 3, \ldots, n - 1\}$. On the other hand, by (17) and $(c_{m-1}, d_{m-1}) = (0, -1) \cdot E(t)^m$, there are $v, v' \in R^*$ with

$$\begin{pmatrix} x_n & y_n \\ -x_{n-1} & -y_{n-1} \end{pmatrix} = \text{diag}(v, v') \cdot E(t)^m.$$  \hspace{1cm} (20)

From Remark 5.4, the modified standard forms (19) and (20) are identical. Therefore, $n = m$, $\text{dist}(q_0, q_{m-1}) = m - 1$, and the diameter of the connected component of $q_0$ is infinite.

By Theorem 3.2 (a), all connected components of $\mathbb{P}(R)$ have infinite diameter.

**Remark 5.6.** Let $R$ be a ring such that $R^* \cup \{0\}$ is a field, say $K$, and suppose that we have a degree function, i.e. a function $\deg : R \to (-\infty) \cup \{0, 1, \ldots\}$ satisfying
\[ \deg a = -\infty \quad \text{if, and only if, } a = 0, \]
\[ \deg a = 0 \quad \text{if, and only if, } a \in R^*, \]
\[ \deg(a + b) \leq \max\{\deg a, \deg b\}, \]
\[ \deg(ab) = \deg(a) + \deg(b), \]
for all \( a, b \in R \). Then, following [10, p. 21], \( R \) is called a K-ring with a degree function.

If \( R \) is a K-ring with a degree function, then there is a unique standard form for \( \text{GE}_2(R) \) [10, Theorem (7.1)].

**Examples 5.7.** (a) Let \( R \) be a K-ring with a degree-function such that \( R \neq K \). From Remark 5.6 and Theorem 5.5, all connected components of the projective line \( \mathbb{P}(R) \) have infinite diameter.

The associated generalized chain geometry \( \Sigma(K, R) \) has a lot of strange properties. For example, any two distant points are joined by a unique K-chain. However, we do not enter a detailed discussion here.

(b) Let \( K[X] \) be the polynomial ring over a field \( K \) in a central indeterminate \( X \). From (a) and Example 5.2 (b), the projective line \( \mathbb{P}(K[X]) \) is connected and its diameter is infinite. On the other hand, if \( K \) is commutative then \( K[X] \) has stable rank 3 [20, 2.9]; see also [3, Chapter V, (3.5)]. So there does not seem to be an immediate connection between stable rank and diameter.

(c) Let \( R := K[X_1, X_2, \ldots, X_m] \) be the polynomial ring over a field \( K \) in \( m > 1 \) independent central indeterminates. Then, by an easy induction and by [10, Proposition (7.3)],

\[ A_n := \begin{pmatrix} 1 + X_1 X_2 & X_1^2 \\ -X_2^2 & 1 - X_1 X_2 \end{pmatrix}^n = \begin{pmatrix} 1 + nX_1 X_2 & nX_1^2 \\ -nX_2^2 & 1 - nX_1 X_2 \end{pmatrix} \]  \hspace{1cm} (21)

is in \( \text{GL}_2(R) \backslash \text{GE}_2(R) \) for all \( n \in \mathbb{Z} \) that are not divisible by the characteristic of \( K \). Also, the inner automorphism of \( \text{GL}_2(R) \) arising from the matrix \( A_1 \) takes \( B_{12}(1) \in \text{E}_2(R) \) to a matrix that is not even in \( \text{GE}_2(R) \); see [18, p. 121–122]. So neither \( \text{E}_2(R) \) nor \( \text{GE}_2(R) \) is a normal subgroup of \( \text{GL}_2(R) \).

We infer that the projective line over \( R \) is not connected. Further, it follows from (21) that the number of right cosets of \( \text{GE}_2(R) \) in \( \text{GL}_2(R) \) is infinite, if the characteristic of \( K \) is zero, and \( \geq \text{char } K \) otherwise. From Theorem 3.2, this number of right cosets is at the same time the number of connected components in \( \mathbb{P}(R) \). Even in case of \( \text{char } K = 2 \) there are at least three connected components, since the index of \( \text{GE}_2(R) \) in \( \text{GL}_2(R) \) cannot be two. From (a), all connected components of \( \mathbb{P}(R) \) have infinite diameter.

So, for each commutative field \( K \), we obtain a disconnected chain geometry \( \Sigma(K, R) \), whereas for each skew field \( K \) a disconnected generalized chain geometry arises.

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References


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