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REPRESENTATIONS OF TOPOLOGICAL ALGEBRAS BY PROJECTIVE LIMITS

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ABSTRACT. It is shown that a) it is possible to define the topology of any topological algebra by a collection of F-seminorms, b) every complete locally uniformly absorbent (complete locally A-pseudoconvex) Hausdorff algebra is topologically isomorphic to a projective limit of metrizable locally uniformly absorbent algebras (respectively, A-(k-normed) algebras, where $k \in (0,1]$ varies, c) every complete locally idempotent (complete locally m-pseudoconvex) Hausdorff algebra is topologically isomorphic to a projective limit of locally idempotent Fréchet algebras (respectively, k-Banach algebras, where $k \in (0,1]$ varies) and every m-algebra is locally m-pseudoconvex. Condition for submultiplicativity of F-seminorm is given.

1. Introduction

1. Let $\mathbb{K}$ be the field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers and X a topological linear space over $\mathbb{K}$. A neighbourhood $O \subset X$ of zero is *absolutely k-convex*, if $\lambda u + \mu v \in O$ for all $u, v \in O$ and $\lambda, \mu \in \mathbb{K}$ with $|\lambda|^k + |\mu|^k \leqslant 1$ and is *absolutely pseudoconvex*, if O is absolutely k-convex for some $k \in (0,1]$, which depends on O. Then every such neighbourhood O of zero is *balanced* (that is, $\mu O \subset O$ for $|\mu| \leqslant 1$) and *pseudoconvex* (that is, O defines a number $k_O \in (0,1]$ such that

$$O + O \subset 2^{\frac{1}{k_O}} O).$$

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A topological algebra A over $\mathbb{K}$ with separately continuous multiplication (in short, a topological algebra) is *locally pseudoconvex* if it has a base $\mathcal{L}_A$ of neighbourhoods of zero, consisting of *absolutely pseudoconvex subsets*. Herewith, when

$$\inf\{k_O : O \in \mathcal{L}_A\} = k > 0,$$

then A is is a *locally k-convex* algebra and when $k = 1$, then a *locally convex algebra*. A locally pseudoconvex algebra A is *locally absorbingly pseudoconvex* (in short *locally A-pseudoconvex*), if A has a base $\mathcal{L}_A$ of absorbent (that is, for each $a \in A$ and each $O \in \mathcal{L}_A$ there exists a number $N(a, O) > 0$ such that $aO \cup Oa \subset N(a, O)O$) and pseudoconvex neighbourhoods of zero, and is a *locally multiplicatively pseudoconvex* (in short *locally m-pseudoconvex*) algebra, if every $O \in \mathcal{L}_A$ is idempotent (that is, $OO \subset O$). *Locally A-(k-convex)* and *locally m-(k-convex) algebras* are defined similarly. In case $k = 1$ these algebras are *locally A-convex* and *locally m-convex algebras*.

It is well-known (see, for example, [18, pp. 3–6] or [8, pp. 189 and 195]) that it is possible to define the topology of every locally pseudoconvex algebra A by a collection $\mathcal{P}_A = \{p_\lambda : \lambda \in \Lambda\}$ of k_λ-homogeneous seminorms, where $k_\lambda \in (0, 1]$ for each $\lambda \in \Lambda$. Recall that a seminorm p on A is *k-homogeneous* if $p(\mu a) = |\mu|^k p(a)$ for each $a \in A$. In case when for any $a \in A$ and every $p_\lambda \in \mathcal{P}_A$ there exist positive numbers $M = M(a, \lambda)$ and $N = N(a, \lambda)$ such that $p_\lambda(ab) \leqslant Mp_\lambda(b)$ and $p_\lambda(ba) \leqslant Np_\lambda(b)$ for each $b \in A$, then A is a locally A-pseudoconvex algebra, and when $M(a, \lambda) = N(a, \lambda) = p_\lambda(a)$ for each $a \in A$ and $\lambda \in \Lambda$, then a locally m-pseudoconvex algebra. Moreover, A is a *A-(k-normed) algebra*, when the topology of A is defined by a k-homogeneous norm $\|\cdot\|$, $k \in (0, 1]$, such that for any $a \in A$ there exists positive numbers $M(a)$ and $N(a)$ such that $\|ab\| \leqslant M(a)\|b\|$ and $\|ba\| \leqslant N(a)\|b\|$ for each $b \in A$, and an *m-(k-normed) algebra*, if $N(a) = M(a) = \|a\|$ for each $a \in A$.

2. A topological algebra A is a *locally idempotent algebra* if it has a base of *idempotent* neighbourhoods of zero. This class of topological algebras has been introduced in [21, p. 31]. Locally m-convex algebras (see, for example, [9, 10, 14, 15, 21, 22]) and locally m-pseudoconvex algebras (see, for example, [3, 5, 8]) have been well studied, locally idempotent algebras (without any additional requirements) have been studied only in [16, 21, 4].

We shall say that a topological algebra A is

a) a *locally absorbent algebra* if A has a base of absorbent neighbourhoods of zero.

b) a *locally uniformly absorbent algebra* if A has a base of uniformly absorbent neighbourhoods of zero (that is, for each fixed $a \in A$ and each neighbourhood O of zero in A there exists a positive number $\lambda(a)$ (which does not depend on O) such that $aO \cup Oa \subset \lambda(a)O$);

3. It is well-known (it was first published in 1952 in [15, p. 17]) that every complete locally m-convex Hausdorff algebra is topologically isomorphic to the projective limit of Banach algebras. This result has been generalized to the case of complete locally m-(k-convex) Hausdorff algebras in [2, Theorem 5], to the case of complete locally A-convex Hausdorff algebras in [7, Theorem 2.2], and

to the case of complete locally m-pseudoconvex Hausdorff algebras in [8, pp. 202–204]. Moreover, it is known (see [17, Theorem 1]) that every complete topological Hausdorff algebra with jointly continuous multiplication is topologically isomorphic to a projective limit of Fréchet algebras and every complete locally convex Hausdorff algebra with jointly continuous multiplication is topologically isomorphic to a projective limit of locally convex Fréchet algebras.

Similar representations of topological algebras (not necessarily with jointly continuous multiplication) by projective limits are considered in the present paper.

2. Topology defined by a collection of F-seminorms

1. Let X be a linear space over $\mathbb{K}$. By *F-seminorm* on X we mean a map $q : X \to \mathbb{R}^+$ which has the following properties:

(q_1) $q(\lambda x) \leqslant q(x)$ for each $x \in X$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leqslant 1$;
(q_2) $\lim_{n\to\infty} q\big(\frac{1}{n}x\big) = 0$ for each $x \in X$;
(q_3) $q(x+y) \leqslant q(x) + q(y)$ for each $x, y \in X$.

If from $q(x) = 0$ follows that $x = \theta_X$, then q is called an *F-norm* on X. In this case $d(x,y) = q(x-y)$ for each $x, y \in X$ defines a metric d on A which has the property $d(x+z, y+z) = d(x,y)$ for each $x, y, z \in X$.

2. It is well-known (see, for example, [12, p. 39, Theorem 3]) that the topology of any topological linear space X coincides with the initial topology defined by a collection of F-seminorms on X. To show that the same situation takes place in case of topological algebras, we prove first the following result.

Proposition 2.1. *Let A be an algebra over $\mathbb{K}$, $\mathcal{Q} = \{q_\lambda : \lambda \in \Lambda\}$ a non-empty collection of F-seminorms on A and $\tau_{\mathcal{Q}}$ the initial topology on A, defined by the collection $\mathcal{Q}$. Then $(A, \tau_{\mathcal{Q}})$ is a topological algebra if $\mathcal{Q}$ satisfies the condition*

(q_4) *for each fixed $a \in A$ and for any $\varepsilon > 0$ and any $\lambda \in \Lambda$ there exist $\delta_a > 0$ and $\lambda_a \in \Lambda$ such that $q_\lambda(ab) < \varepsilon$ and $q_\lambda(ba) < \varepsilon$, whenever $q_{\lambda_a}(b) < \delta_a$.*

Moreover, $(A, \tau_{\mathcal{Q}})$ is a topological algebra with jointly continuous multiplication if $\mathcal{Q}$ satisfies the condition

(q_5) *for any $\varepsilon > 0$ and any $\lambda \in \Lambda$ there exist $\delta > 0$ and $\lambda' \in \Lambda$ such that $q_\lambda(ab) < \varepsilon$, whenever $q_{\lambda'}(a) < \delta$ and $q_{\lambda'}(b) < \delta$,*

and $(A, \tau_{\mathcal{Q}})$ is a locally idempotent algebra if $\mathcal{Q}$ satisfies the condition

(q_6) *for any $\varepsilon > 0$ and any $\lambda \in \Lambda$ holds $q_\lambda(ab) < \varepsilon$, whenever $q_\lambda(a) < \varepsilon$ and $q_\lambda(b) < \varepsilon$.*

Proof. Since $\tau_{\mathcal{Q}}$ is the initial topology on A defined by the collection $\mathcal{Q}$, then $\{O_{\lambda\varepsilon} : \lambda \in \Lambda, \varepsilon > 0\}$ is a subbase of neighbourhoods of zero in $(A, \tau_{\mathcal{Q}})$, where $O_{\lambda\varepsilon} = \{a \in A : q_\lambda(a) < \varepsilon\}$ is a balanced and absorbent set by (q_1) and (q_2) for each $\varepsilon > 0$ and each $\lambda \in \Lambda$. It is easy to see that the addition $(a,b) \to a+b$ in $(A, \tau_{\mathcal{Q}})$ is continuous by (q_3) and the multiplication over $\mathbb{K}$ is continuous in $(A, \tau_{\mathcal{Q}})$ by (q_1). Therefore, $(A, \tau_{\mathcal{Q}})$ is a topological linear space.

If now, in addition, $\mathcal{Q}$ satisfies the condition (q_4), then the multiplication $(a, b) \to ab$ in $(A, \tau_{\mathcal{Q}})$ is separately continuous. To show this, let O be an arbitrary neighbourhood of zero in the topology $\tau_{\mathcal{Q}}$ on A. Then there exist $\varepsilon > 0$, $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in \Lambda$ such that

$$\bigcap_{k=1}^{n} O_{\lambda_k \varepsilon} \subset O. \tag{2.1}$$

For each fixed $a \in A$ and each $k \in \mathbb{N}_n$ there are, by the condition (q_4), a number $\delta_a(k) > 0$ and an index $\lambda_a(k) \in \Lambda$ such that $q_{\lambda_k}(ab) < \varepsilon$ and $q_{\lambda_k}(ba) < \varepsilon$ whenever $q_{\lambda_a(k)}(b) < \delta_a(k)$. Let now $\delta_a = \min\{\delta_a(1), \dots, \delta_a(n)\}$ and

$$V_a = \bigcap_{k=1} O_{\lambda_a(k)\delta_a}.$$

Then V_a is a neighbourhood of zero in A in the topology $\tau_{\mathcal{Q}}$ and

$$aV_a \cup V_a a \subset \bigcap_{k=1}^{n} [aO_{\lambda_a(k)\delta_a(k)} \cup O_{\lambda_a(k)\delta_a(k)} a] \subset \bigcap_{k=1}^{n} O_{\lambda_k \varepsilon} \subset O.$$

Hence, the multiplication in $(A, \tau_{\mathcal{Q}})$ is separately continuous. Consequently, $(A, \tau_{\mathcal{Q}})$ is a topological algebra.

If, next, $\mathcal{Q}$ satisfies the condition (q_5), then the multiplication $(a, b) \to ab$ is jointly continuous in $(A, \tau_{\mathcal{Q}})$. To show this, let again O be an arbitrary neighbourhood of zero in the topology $\tau_{\mathcal{Q}}$ on A. Then there are $\varepsilon > 0$, $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in \Lambda$ such that holds (2.1). Now, for each $k \in \mathbb{N}_n = \{1, 2, \dots, n\}$ there are, by the condition (q_5), a number $\delta_k > 0$ and an index $\lambda'_k \in \Lambda$ such that $q_{\lambda_k}(ab) < \varepsilon$, whenever $q_{\lambda'_k}(a) < \delta_k$ and $q_{\lambda'_k}(b) < \delta_k$. Let now $\delta = \min\{\delta_1, \dots, \delta_n\}$ and

$$V = \bigcap_{k=1} O_{\lambda'_k \delta}.$$

Then V is again a neighbourhood of zero in A in the topology $\tau_{\mathcal{Q}}$ and

$$VV \subset \bigcap_{k=1}^{n} O_{\lambda'_k \delta_k} O_{\lambda'_k \delta_k} \subset \bigcap_{k=1}^{n} O_{\lambda_k \varepsilon} \subset O.$$

It means that the multiplication in $(A, \tau_{\mathcal{Q}})$ is jointly continuous. Consequently, in this case $(A, \tau_{\mathcal{Q}})$ is a topological algebra with jointly continuous multiplication.

Let, in the end, $\mathcal{Q}$ satisfies the condition (q_6). Then $O_{\lambda\varepsilon} O_{\lambda\varepsilon} \subset O_{\lambda\varepsilon}$ for each $\varepsilon > 0$ and $\lambda \in \Lambda$. Therefore, $(A, \tau_{\mathcal{Q}})$ has a base of idempotent neighbourhoods of zero. Consequently, $(A, \tau_{\mathcal{Q}})$ is a locally idempotent algebra. □

Theorem 2.2. *Every topological algebra (A, τ) defines a collection $\mathcal{Q}$ of F-seminorms on A such that $(A, \tau_{\mathcal{Q}})$ is a topological algebra (in particular, when (A, τ) is a topological algebra with jointly continuous multiplication, then $(A, \tau_{\mathcal{Q}})$ is a topological algebra with jointly continuous multiplication) and $\tau = \tau_{\mathcal{Q}}$.*

Proof. Let M be the dense subset of $\mathbb{R}^+$ which consists of all non-negative rational numbers, having a finite dyadic expansions, i.e, we may write every such number

ρ on the form

$$\rho = \sum_{n=0}^{\infty} \delta_n(\rho) \cdot 2^{-n},$$

where $\delta_0(\rho) \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\delta_n(\rho) \in \{0, 1\}$ for each $n \in \mathbb{N}$ and $\delta_n(\rho) = 0$ for n sufficiently large.

Let $\mathcal{L}_{(A,\tau)}$ be a base of neighbourhoods of zero in (A, τ), consisting of closed balanced sets and $\mathcal{S} = \{S_\lambda : \lambda \in \Lambda\}$ the set of all strings $S_\lambda = (U_n(\lambda))$ in $\mathcal{L}_{(A,\tau)}$, that is, $U_n(\lambda) \in \mathcal{L}_{(A,\tau)}$ and $U_{n+1}(\lambda) + U_{n+1}(\lambda) \subset U_n(\lambda)$ for each $n \in \mathbb{N}_0$ (see [6, p. 5]). For each $\lambda \in \Lambda$, $S_\lambda = (U_n(\lambda)) \in \mathcal{S}$ and $\rho \in M$ let

$$V_\lambda(\rho) = \underbrace{U_0(\lambda) + \cdots + U_0(\lambda)}_{\delta_0(\rho) \text{ summands}} + \sum_{n=1}^{\infty} \delta_n(\rho) \cdot U_n(\lambda) \tag{2.2}$$

and

$$q_\lambda(a) = \inf\{\rho \in M : a \in V_\lambda(\rho)\}$$

for each $a \in A$ and $\lambda \in \Lambda$. Then every q_λ is a F-seminorm on A (see [12, pp. 39–40]) and

$$\ker q_\lambda = \bigcap_{n=0}^{\infty} U_n(\lambda).$$

Indeed, if $a \in \ker q_\lambda$, then $q_\lambda(a) < 2^{-n}$ for each $n \in \mathbb{N}_0$. Therefore, $a \in V_\lambda(2^{-n}) = U_n(\lambda)$ for each $n \in \mathbb{N}_0$. On the other hand, if $a \in U_n(\lambda)$ for each $n \in \mathbb{N}_0$, then $q_\lambda(a) \leqslant 2^{-n}$ for each $n \in \mathbb{N}_0$. Hence $q_\lambda(a) = 0$ or $a \in \ker q_\lambda$.

To show that $\mathcal{Q} = \{q_\lambda : \lambda \in \Lambda\}$ satisfies the condition (q_4), let $a \in A$, $\lambda \in \Lambda$ (by this we fix a string $S_\lambda = (U_n(\lambda))$ in $\mathcal{L}_{(A,\tau)}$) and $\varepsilon > 0$. Then there exists a number $n_\varepsilon \in \mathbb{N}$ such that $\frac{1}{2^{n_\varepsilon}} < \varepsilon$. Since the multiplication $(a, b) \to ab$ in (A, τ) is separately continuous, then there exists a neighbourhood $V_a \in \mathcal{L}_{(A,\tau)}$ such that $aV_a \cup V_a a \subset U_{n_\varepsilon}(\lambda)$. Let (U_n) be the string in $\mathcal{L}_{(A,\tau)}$, which is generated by V_a, that is, $U_0 = V_a$ and other members U_n of this string are defined by V_a. Hence, there exists an index $\lambda_a \in \Lambda$ such that $S_{\lambda_a} = (U_n(\lambda_a))$, where $U_0(\lambda_a) = V_a$ and $U_n(\lambda_a) = U_n$, if $n \geqslant 1$. Now, $V_a = V_{\lambda_a}(1)$ and

$$aV_a \cup V_a a \subset U_{n_\varepsilon}(\lambda) = V_\lambda\Big(\frac{1}{2^{n_\varepsilon}}\Big).$$

Therefore, for every fixed $a \in A$ and for any $\lambda \in \Lambda$ and any $\varepsilon > 0$ there exist an index $\lambda_a \in \Lambda$ and a number $\varepsilon_a > 0$ such that $q_\lambda(ab) \leqslant 2^{-n_\varepsilon} < \varepsilon$ and $q_\lambda(ab) \leqslant 2^{-n_\varepsilon} < \varepsilon$ whenever $q_{\lambda_a}(b) < 1$ (in the present case, $\varepsilon_a = 1$). Hence, the collection $\mathcal{Q}$ satisfies the condition (q_4). Consequently, $(A, \tau_\mathcal{Q})$ is a topological algebra by Proposition 2.1.

In particular, when (A, τ) is a topological algebra with jointly continuous multiplication, then the multiplication in $(A, \tau_\mathcal{Q})$ is also jointly continuous. Indeed, let $\lambda \in \Lambda$ (by this we fix again a string $S_\lambda = (U_n(\lambda))$ in $\mathcal{L}_{(A,\tau)}$) and $\varepsilon > 0$. Then there is again a number $n_\varepsilon \in \mathbb{N}$ such that $\frac{1}{2^{n_\varepsilon}} < \varepsilon$. Since the multiplication $(a, b) \to ab$ in (A, τ) is jointly continuous, then there exists an element $V \in \mathcal{L}_A$ such that $VV \subset U_{n_\varepsilon}(\lambda)$. Let now (U_n) be the string in $\mathcal{L}_{(A,\tau)}$ for which $U_0 = V$. Then there exists an index $\lambda' \in \Lambda$ such that $S_{\lambda'} = (U_n(\lambda'))$, where $U_0(\lambda') = V$. If $a, b \in V = V_{\lambda'}(1)$, then $q_{\lambda'}(a) < 1$, $q_{\lambda'}(b) < 1$ and from $ab \in VV \subset U_{n_\varepsilon}(\lambda) = V_\lambda(\frac{1}{2^{n_\varepsilon}})$ follows that $q_\lambda(ab) \leqslant \frac{1}{2^{n_\varepsilon}} < \varepsilon$. Hence, for each $\lambda \in \Lambda$ and $\varepsilon > 0$ there exist $\lambda' \in \Lambda$ and $\varepsilon' > 0$ such that $q_\lambda(ab) < \varepsilon$, whenever $q_{\lambda'}(a) < 1$ and $q_{\lambda'}(b) < 1$ (in the present case, $\varepsilon' = 1$). This shows that $\mathcal{Q}$ satisfies the condition (q_5). Consequently, $(A, \tau_{\mathcal{Q}})$ is a topological algebra with jointly continuous multiplication by Proposition 2.1.

Next we show that $\tau = \tau_{\mathcal{Q}}$. For it let $U \in \mathcal{L}_{(A,\tau)}$, $S_{\lambda_0} = (U_n(\lambda_0))$ be the string in $\mathcal{L}_{(A,\tau)}$ for which $U_0(\lambda_0) = U$ and let $u \in O_{\lambda_0 1}$. Then $q_{\lambda_0}(u) < 1$. Hence $u \in V_{\lambda_0}(1) = U_0(\lambda_0) = U$. It means that $O_{\lambda_0 1} \subset U$. Since $O_{\lambda_0 1}$ belongs to the base of neighbourhoods of zero in $(A, \tau_{\mathcal{Q}})$, then $\tau \subset \tau_{\mathcal{Q}}$. Let now $O \in \mathcal{L}_{(A,\tau_{\mathcal{Q}})}$. Then there are $\varepsilon > 0$, $m \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_m \in \Lambda$ (with this we fix m strings $S_{\lambda_1} = (U_n(\lambda_1)), \dots, S_{\lambda_m} = (U_n(\lambda_m))$ in $\mathcal{L}_{(A,\tau)}$) such that

$$\bigcap_{k=1}^{m} O_{\lambda_k \varepsilon} \subset O.$$

Again, there is a number $n_\varepsilon \in \mathbb{N}$ such that $\frac{1}{2^{n_\varepsilon}} < \varepsilon$. Now,

$$U = \bigcap_{k=1}^{m} U_{n_\varepsilon}(\lambda_k)$$

is a neighbourhood of zero of A in the topology $\tau_{\mathcal{Q}}$. Since

$$U_{n_\varepsilon}(\lambda_k) = V_{\lambda_k}\left(\frac{1}{2^{n_\varepsilon}}\right)$$

for each $k \in \mathbb{N}_m$, then from $u \in U$ follows that $q_{\lambda_k}(u) \leqslant \frac{1}{2^{n_\varepsilon}} < \varepsilon$ for each $k \in \mathbb{N}_m$. Hence, $U \subset O$. It means that $\tau_{\mathcal{Q}} \subset \tau$. Consequently, $\tau = \tau_{\mathcal{Q}}$. □

Corollary 2.3. *Let (A, τ) be a locally pseudoconvex algebra; $\mathcal{P} = \{p_\alpha : \alpha \in \mathcal{A}\}$ the collection of nonhomogeneous seminorms on A, which defines the topology τ; $\tau_{\mathcal{P}}$ the topology on A, defined by the collection $\mathcal{P}$; $\mathcal{L}_A$ the base of neighbourhoods of zero in A, which are closed and balanced sets; $\mathcal{Q} = \{q_S : S$ is a string in $\mathcal{L}_A\}$ and $\tau_{\mathcal{Q}}$ the topology on A, defined by the collection $\mathcal{Q}$ of F-seminorms on A. Then $\tau = \tau_{\mathcal{P}} = \tau_{\mathcal{Q}}$.*

Proof. It is well-known that $\tau = \tau_{\mathcal{P}}$ and $\tau = \tau_{\mathcal{Q}}$ by Theorem 2.2. Hence, all these three topologies coincide. □

3. In point of view of algebra it is important to know, when every F-seminorm, defined by a string from $\mathcal{L}_A$, is submultiplicative.

Proposition 2.4. *Let A be a topological algebra, $\mathcal{L}_A$ the base of all closed and balanced neighbourhoods of zero in A, and $S = (U_n)$ a string in $\mathcal{L}_A$. Then the F-seminorm q_S, defined by S, is submultiplicative if and only if the knots U_n of S satisfy the condition*

$$U_n U_m \subset U_{n+m} \tag{2.3}$$

for all $n, m \in \mathbb{N}$.

Proof. Let $S = (U_n)$ be a string in $\mathcal{L}_A$ such that the F-seminorm q_S, defined by S, is submultiplicative (that is $q_S(ab) \leqslant q_S(a)q_S(b)$ for all $a, b \in A$). Let $n, m \in \mathbb{N}$ be fixed, $a \in U_n = V_S(\frac{1}{2^n})$ and $b \in U_m = V_S(\frac{1}{2^m})$. Then $q_S(a) \leqslant \frac{1}{2^n}$ and $q_S(b) \leqslant \frac{1}{2^m}$. Therefore,

$$q_S(ab) \leqslant \frac{1}{2^n}\frac{1}{2^m} = \frac{1}{2^{n+m}}$$

or

$$ab \in V_S\Big(\frac{1}{2^{n+m}}\Big) = U_{n+m}.$$

Hence, $U_nU_m \subset U_{n+m}$.

Let now $S = (U_n)$ be a string in $\mathcal{L}_A$ such that the knots U_n of S satisfy the condition (2.3) for all $n, m \in \mathbb{N}$. Moreover, let ρ and σ be dyadic numbers such that $a \in V_S(\rho)$, $\rho \leqslant q_S(a)+\varepsilon$, $b \in V_S(\sigma)$ and $\sigma \leqslant q_S(b)+\delta$. Then $ab \in V_S(\rho)V_S(\sigma)$. For every $s \geqslant 1$ and $l \geqslant 0$ let

$$U_l(s) = \underbrace{U_l + \cdots + U_l}_{s \text{ summands}}.$$

Since

$$V_S(\rho)V_S(\sigma) = [U_0(\delta_0(\rho)) + \sum_{n=1}^{\infty}\delta_n(\rho)\,U_n\,][U_0(\delta_0(\sigma)) + \sum_{n=1}^{\infty}\delta_n(\sigma)\,U_n] \subset$$
$$U_0(\delta_0(\rho)\delta_0(\sigma)) + T_1 + T_2 + T_3$$

by (2.2), where

$$T_1 = \sum_{n=1}^{\infty}\delta_n(\rho)\,U_n[U_0(\delta_0(\sigma))] \subset \sum_{n=1}^{\infty}\delta_n(\rho)[U_n(\delta_0(\sigma))] = T_1'$$

$$T_2 = [U_0(\delta_0(\rho))]\sum_{n=1}^{\infty}\delta_n(\sigma)\,U_n \subset \sum_{n=1}^{\infty}\delta_n(\sigma)\,[U_n(\delta_0(\rho))] = T_2'$$

and

$$T_3 = \sum_{n=1}^{\infty}\Big[\sum_{k=1}^{n}\delta_k(\rho)\delta_{n-k+1}(\sigma)\,U_kU_{n-k+1}\Big] \subset \sum_{n=1}^{\infty}\Big[\sum_{k=1}^{n}\delta_k(\rho)\delta_{n-k+1}(\sigma)\,U_{n+1}\Big] = T_3'$$

by the condition (2.3). Hence,

$$ab \in U_0(\delta_0(\rho)\delta_0(\sigma)) + T_1' + T_2' + T_3' \subset$$
$$V_S(\delta_0(\rho)\delta_0(\sigma)) + \underbrace{V_S(K_\rho) + \cdots + V_S(K_\rho)}_{\delta_0(\sigma)\text{ summands}} + \underbrace{V_S(K_\sigma) + \cdots + V_S(K_\sigma)}_{\delta_0(\rho)\text{ summands}} + V_S(K_{\rho\sigma}) \subset$$
$$V_S(\delta_0(\rho)\delta_0(\sigma)) + \delta_0(\sigma)V_S(K_\rho) + \delta_0(\rho)V_S(K_\sigma) + V_S(K_{\rho\sigma})$$
$$= V_S([\delta_0(\rho) + K_\rho][\delta_0(\sigma) + K_\sigma]) = V_S(\rho\sigma)$$

because $V_S(\alpha) + V_S(\beta) \subset V_S(\alpha+\beta)$ for each dyadic numbers α and β (see [12, p. 39], where

$$K_l = \sum_{n=1}^{\infty}\delta_n(l)\frac{1}{2^n} \quad \text{and} \quad K_{lm} = \sum_{n=1}^{\infty}\Big[\sum_{k=1}^{n}\delta_k(l)\delta_{n-k+1}(m)\Big]\frac{1}{2^{n+1}}.$$

Therefore,

$$q_S(ab) \leqslant \rho\sigma \leqslant (q_S(a)+\varepsilon)(q_S(b)+\varepsilon),$$

from which follows that q_S is submultiplicative (because ε is an arbitrary positive number). □

Corollary 2.5. *Let A be a topological algebra, $\mathcal{L}_A$ the base of closed and balanced neighbourhoods of zero in A and $\mathcal{Q}_A = \{q_S : S$ is a string in $\mathcal{L}_A\}$ be the collection of submultiplicative F-seminorms on A, which defines the topology of A. Then A is a locally idempotent algebra.*

Proof. Let A be a topological algebra such that every F-seminorm q_S in $\mathcal{Q}_A$ is submultiplicative, $O \in \mathcal{L}_A$ an arbitrary element and $S = (U_n)$ the string in $\mathcal{L}_A$ with $U_0 = O$. Since q_S is submultiplicative, then all knots U_n of the string S satisfy the condition (2.3), by Proposition 2.4. Hence O is an idempotent set. Therefore, A is a locally idempotent algebra. □

Remark 2.6. Recall that a topological algebra, for which every F-seminorm in $\mathcal{Q}_A$ is submultiplicative, is an *m-algebra* in [16, p. 767], and an *m-convex topological algebra* in [20, p. 335]. W. Żelazko in [21, p. 39] and V. Murali in [16, p. 766] asked whether every locally idempotent algebra is an m-algebra? By Proposition 2.4, the answer is no, because idempotent knots of a string do not necessarily satisfy the condition (2.3).

3. Main results

To represent topological algebras by projective limits, we need

Lemma 3.1. *Let A be a locally uniformly absorbent (locally idempotent) Hausdorff algebra over $\mathbb{K}$, $\mathcal{L}_A$ the base of all closed, balanced and uniformly absorbent (respectively, closed, balanced and idempotent) neighbourhoods of zero in A and $S_A = (U_n)$ a string in $\mathcal{L}_A$. Then the kernel*

$$N(S_A) = \bigcap_{n=1}^{\infty} U_n$$

of S_A is a closed two-sided ideal in A.

Proof. When $N(S_A) = \{\theta_A\}$, then $N(S_A)$ is a closed two-sided ideal in A. Suppose now that $N(S_A) \neq \{\theta_A\}$. Then there are elements $a, b \in N(S_A) \setminus \{\theta_A\}$. Let $n \in \mathbb{N}$ be an arbitrary fixed number. Since $N(S_A) \subset U_{n+1}$ and $U_{n+1} + U_{n+1} \subset U_n$, then $a + b \in U_n$ for each $n \in \mathbb{N}$. Hence, $a + b \in N(S_A)$.

Let next $\lambda \in \mathbb{K}$ and $a \in N(S_A)$. Then $a \in U_n$ for each $n \in \mathbb{N}$. If $|\lambda| \leqslant 1$, then $\lambda a \in U_n$ for each $n \in \mathbb{N}$, because U_n is balanced. If $|\lambda| > 1$, let $n_0 \in \mathbb{N}$ be a natural number such that $[|\lambda|] + 1 \leqslant 2^{n_0}$ and n an arbitrary fixed natural number (Here $[r]$ denotes the entire part of a real number r). Since

$$\lambda a = [|\lambda|] \frac{\lambda}{|\lambda|} a + (|\lambda| - [|\lambda|]) \frac{\lambda}{|\lambda|} a \in \underbrace{U_{n+n_0} + \cdots + U_{n+n_0}}_{[|\lambda|]+1 \text{ summands}} \subset U_n,$$

because $\left|\frac{\lambda}{|\lambda|}\right| = 1$, $\left|(|\lambda| - [|\lambda|]) \frac{\lambda}{|\lambda|}\right| < 1$ and every U_n is balanced. Hence, $\lambda a \in U_n$ for each $n \in \mathbb{N}$. Thus, $\lambda a \in N(S_A)$.

First, we assume that A is a locally uniformly absorbent algebra, $a \in A$ and $b \in N(S_A)$. Since A is a locally uniformly absorbent algebra, then there exists a positive number $\lambda(a)$ such that $aU_n \in \lambda(a)U_n$. Therefore, $\frac{ab}{\lambda(a)} \in U_n$ for each $n \in \mathbb{N}$. Hence, $ab \in \lambda(a)N(S_A) \subset N(S_A)$. Similarly, we can show that $ba \in N(S_A)$. Consequently, $N(S_A)$ is a two-sided ideal in A.

Let now A be a locally idempotent algebra, $a \in A$, $b \in N(S_A)$ and $n \in \mathbb{N}$. Then there exists a positive number ε_n such that $a \in \varepsilon_n U_n$ (because every neighbourhood of zero absorbs points). If $|\varepsilon_n| \leqslant 1$, then $\varepsilon_n U_n \subset U_n$ because U_n is balanced, and if $|\varepsilon_n| > 1$, then, from $\varepsilon_n b \subset \varepsilon_n N(S_A) \subset N(S_A) \subset U_n$ follows that

$$ab \in (\varepsilon_n U_n)(\varepsilon_n^{-1} U_n) \subset U_n U_n \subset U_n.$$

Hence, $ab \in N(S_A)$. Similarly, we can show that $ba \in N(S_A)$. Consequently, $N(S_A)$ is again a two-sided ideal in A. □

Theorem 3.2. *For any (real or complex) locally uniformly absorbent Hausdorff algebra A there exists a projective system $\{A_\lambda; h_{\lambda\mu}, \Lambda\}$ of metrizable locally uniformly absorbent algebras and continuous homorphisms $h_{\lambda\mu}$ from A_λ to A_μ (whenever $\lambda \prec \mu$) such that A is topologically isomorphic to a dense subalgebra of the projective limit $\varprojlim A_\lambda$ of this system. In particular case, when, in addition, A is complete, then A and $\varprojlim A_\lambda$ are topologically isomorphic.*

Moreover, if A is a locally A-pseudoconvex (locally A-convex) Hausdorff algebra, then A is topologically isomorphic to a dense subalgebra of the projective limit $\varprojlim A_\lambda$ of A-(k_λ-normed) algebras (respectively, A-normed algebras). In particular, when, in addition, A is complete, then A and $\varprojlim A_\lambda$ are topologically isomorphic.

Proof. 1) Let A be a locally uniformly absorbent Hausdorff algebra, $\mathcal{L}_A$ the base of closed, balanced and uniformly absorbent neighbourhoods of zero in A and $\mathcal{S}_A = \{S_\lambda : \lambda \in \Lambda\}$ the collection of all strings in $\mathcal{L}_A$. That is, every $S_\lambda \in \mathcal{S}_A$ is a sequence (O_n^λ) in $\mathcal{L}_A$, members O_n^λ of which satisfy the condition

$$O_{n+1}^\lambda + O_{n+1}^\lambda \subset O_n^\lambda$$

for each $n \in \mathbb{N}$ (see [6, p. 5]). We define the ordering $\prec$ in Λ in the following way: we say that $\lambda \prec \mu$ in Λ if and only if $S_\mu \subset S_\lambda$, that is, if $S_\lambda = (O_n^\lambda)$ and $S_\mu = (O_n^\mu)$, then $O_n^\mu \subset O_n^\lambda$ for each $n \in \mathbb{N}$. It is easy to see that $(\Lambda, \prec)$ is a partially ordered set. To show that $(\Lambda, \prec)$ is a directed set let $S_{\lambda_1} = (O_n^{\lambda_1})$ and $S_{\lambda_2} = (O_n^{\lambda_2})$ be arbitrary fixed strings in $\mathcal{S}_A$ and let $S_\mu = (O_n^\mu)$ be the string in $\mathcal{L}_A$, which we define in the following way: let $O_1^\mu \in \mathcal{L}_A$ be such that $O_1^\mu \subset O_1^{\lambda_1} \cap O_1^{\lambda_2}$. Further, for each $n \geqslant 1$, let U_{n+1} be a neighbourhood of zero in $\mathcal{L}_A$ such that $U_{n+1} + U_{n+1} \subset O_n^\mu$ and O_{n+1}^μ be a neighbourhood in $\mathcal{L}_A$ such that

$$O_{n+1}^\mu \subset U_{n+1} \cap O_{n+1}^{\lambda_1} \cap O_{n+1}^{\lambda_2}.$$

Then

$$O_{n+1}^\mu + O_{n+1}^\mu \subset U_{n+1} \cap O_{n+1}^{\lambda_1} \cap O_{n+1}^{\lambda_2} + U_{n+1} \cap O_{n+1}^{\lambda_1} \cap O_{n+1}^{\lambda_2} \subset U_{n+1} + U_{n+1} \subset O_n^\mu$$

for each $n \in \mathbb{N}$. Since $S_\mu \in \mathcal{S}_A$ and $O_n^\mu \subset O_n^{\lambda_1} \cap O_n^{\lambda_2}$ for each $n \in \mathbb{N}$, then $\lambda_1 \prec \mu$ and $\lambda_2 \prec \mu$. It means that $(\Lambda, \prec)$ is a directed set.

For each $\lambda \in \Lambda$, let q_λ be the F-seminorm on A, defined by the string $S_\lambda = (O_n^\lambda)$,

$$N_\lambda = \bigcap_{n=1}^{\infty} O_n^\lambda$$

(N_λ is a closed two-sided ideal in A by Lemma 3.1), $A_\lambda = A/\ker q_\lambda$ and π_λ the canonical homomorphism of A onto A_λ. Moreover, let $\overline{q}_\lambda(\pi_\lambda(a)) = q_\lambda(a)$ for each $a \in A$. Since $\ker q_\lambda = N_\lambda$ (see the proof of Theorem 2.2), then $\overline{q}_\lambda$ is a F-norm on A_λ. Let τ_{A_λ} be the topology on A_λ, defined by $\overline{q}_\lambda$. Thus, $(A_\lambda, \tau_{A_\lambda})$ is a metrizable locally uniformly absorbent algebra for all $\lambda \in \Lambda$.

For any $\lambda, \mu \in \Lambda$ with $\lambda \prec \mu$ let $h_{\lambda\mu}$ be the map defined by $h_{\lambda\mu}(\pi_\mu(a)) = \pi_\lambda(a)$ for each $a \in A$. Then $h_{\lambda\mu}$ is a continuous homomorphism from A_μ onto A_λ, $h_{\lambda\lambda}$ is the identity mapping on A_λ for each $\lambda \in \Lambda$ and $h_{\lambda\mu} \circ h_{\mu\gamma} = h_{\lambda\gamma}$ for each $\lambda, \mu, \gamma \in \Lambda$ with $\lambda \prec \mu \prec \gamma$. Hence, $\{A_\lambda; h_{\lambda\mu}, \Lambda\}$ is a projective system of metrizable locally uniformly absorbent algebras A_λ with continuous homomorphisms $h_{\lambda\mu}$ and

$$\varprojlim A_\lambda = \{(\pi_\lambda(a))_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} A_\lambda : \; h_{\lambda\mu}(\pi_\mu(a)) = \pi_\lambda(a), \text{ whenever } \lambda \prec \mu\}$$

is the projective limit of this system.

Let e be the mapping defined by $e(a) = (\pi_\lambda(a))_{\lambda\in\Lambda}$ for each $a \in A$ and pr_λ the projection of $\prod_{\mu\in\Lambda} A_\mu$ onto A_λ for each $\lambda \in \Lambda$. Since $\mathrm{pr}_\lambda(e(a)) = \pi_\lambda(a)$ for each $a \in A$ and $\lambda \in \Lambda$ and π_λ is continuous for each $\lambda \in \Lambda$, then e is a continuous map from A into $\prod_{\mu\in\Lambda} A_\mu$ (see, for example, [19, Theorem 8.8]). Moreover, if $a, b \in A$ are such that $e(a) = e(b)$, then $\pi_\lambda(a) = \pi_\lambda(b)$ for each $\lambda \in \Lambda$. Therefore,

$$a - b \in \bigcap_{\lambda\in\Lambda} N_\lambda = \bigcap_{O\in\mathcal{L}_A} O = \theta_A,$$

because A is a Hausdorff space. It means that $a = b$. Hence, e is a one-to-one map.

Let now O be any neighbourhood of zero in A, α an arbitrary fixed index in Λ and

$$U = \Big[\prod_{\lambda\in\Lambda} U_\lambda\Big] \cap e(A),$$

where $U_\alpha = \pi_\alpha(O)$ and $U_\lambda = A_\lambda$, if $\lambda \neq \alpha$. Then U is a neighbourhood of zero in $e(A)$. Since

$$\mathrm{pr}_\alpha(U) \subset \pi_\alpha(O) = \mathrm{pr}_\alpha(e(O))$$

and α is arbitrary, then $U \subset e(O)$. Hence, e is an open map. Taking this into account, e is a topological isomorphism from A into $\prod_{\lambda\in\Lambda} A_\lambda$.

To show that $e(A)$ is dense in $\varprojlim A_\lambda$, let $(a_\lambda)_{\lambda\in\Lambda} \in \varprojlim A_\lambda$ be an arbitrary element and O an arbitrary neighbourhood of $(a_\lambda)_{\lambda\in\Lambda}$ in $\varprojlim A_\lambda$. Then there is a neighbourhood U of $(a_\lambda)_{\lambda\in\Lambda}$ in $\prod_{\lambda\in\Lambda} A_\lambda$ such that $O = U \cap \varprojlim A_\lambda$. Now, there is a finite subset $H \subset \Lambda$ such that $\prod_{\lambda\in\Lambda} U_\lambda \subset U$, where U_λ is a neighbourhood of a_λ in A_λ, if $\lambda \in H$, and $U_\lambda = A_\lambda$, if $\lambda \in \Lambda \setminus H$. Let $\mu \in \Lambda$ be such that $\lambda \prec \mu$ for every $\lambda \in H$ and

$$V = \bigcap_{\lambda\in H} h_{\lambda\mu}^{-1}(U_\lambda).$$

Then V is a neighbourhood of a_μ in A_μ. Take an element $a \in \pi_\mu^{-1}(V)$. Then $\pi_\mu(a) \in V$. Therefore, $\pi_\lambda(a) = h_{\lambda\mu}(\pi_\mu(a)) \in U_\lambda$ for each $\lambda \in H$. It means that $e(a) \in U \cap e(A) = O$. Consequently, $e(A)$ is dense in $\varprojlim A_\lambda$.

2) Let next A be a complete locally uniformly absorbent Hausdorff algebra, $(e(a_\alpha))_{\alpha\in\mathcal{A}}$ a Cauchy net in $e(A)$ and O any neighbourhood of zero in (A,τ). Since e is an open map from A onto $e(A)$, then $e(O)$ is a neighbourhood of zero in $e(A)$. Thus, there exists an index $\alpha_0 \in \mathcal{A}$ such that $e(a_\beta) - e(a_\gamma) \in e(O)$ or $a_\beta - a_\gamma \in O$, whenever $\beta, \gamma \in \mathcal{A}$, $\alpha_0 \prec \beta$ and $\alpha_0 \prec \gamma$. It means that $(a_\alpha)_{\alpha\in\mathcal{A}}$ is a Cauchy net in A. Since A is complete, then there is an element $a_0 \in A$ such that $(a_\alpha)_{\alpha\in\mathcal{A}}$ converges to a_0 in A . Thus $(e(a_\alpha))_{\alpha\in\mathcal{A}}$ converges to $e(a_0)$ in $e(A)$ because e is continuous. Consequently, $e(A)$ is complete and, therefore, is closed in $\varprojlim A_\lambda$.

3) Let now A be a locally A-pseudoconvex (locally A-convex) Hausdorff algebra and $\mathcal{P} = \{p_\lambda : \lambda \in \Lambda\}$ a saturated collection of k_λ-homogeneous seminorms on A with $k_\lambda \in (0,1]$ for each $\lambda \in \Lambda$ (respectively, a collection of homogeneous seminorms on A) which defines the topology of A. We put $A_\lambda = A/\ker p_\lambda$ and norms $\overline{p}_\lambda$ on A_λ we define by $\overline{p}_\lambda(\pi_\lambda(a)) = p_\lambda(a)$ for each $a \in A$ and $\lambda \in \Lambda$, where π_λ is the canonical homomorphism from A onto A_λ. Then $\ker p_\lambda$ is a two-sided ideal in A and A_λ is an A-(k_λ-normed) algebra (if A is a locally A-convex algebra, then A_λ is an A-normed algebra) for each fixed $\lambda \in \Lambda$. The ordering $\prec$ in Λ we define as follows: $\lambda \prec \mu$ if and only if $p_\lambda(a) \leqslant p_\mu(a)$ for each $a \in A$. Then $(\Lambda, \prec)$ is a directed set. Similarly as above, for each $\lambda, \mu \in \Lambda$ with $\lambda \prec \mu$, we define homomorphisms $h_{\lambda\mu}$ from A_μ into A_λ by $h_{\lambda\mu}(\pi_\mu(a)) = \pi_\lambda(a)$ for each $a \in A$. Then $h_{\lambda\mu}$ with $\lambda \prec \mu$ is a continuous map, because

$$\overline{p}_\lambda(h_{\lambda\mu}(\pi_\mu(a))) = \overline{p}_\lambda(\pi_\lambda(a)) = p_\lambda(a) \leqslant p_\mu(a) = \overline{p}_\mu(\pi_\mu(a))$$

for each $a \in A$. Again, similary as above, $\{A_\lambda; h_{\lambda\mu}, \Lambda\}$ is a projective system of A-(k_λ-normed) algebras (respectively, A-normed algebras) A_λ and A is topologically isomorphic to a dense subalgebra of the projective limit $\varprojlim A_\lambda$ of this system and in the complete case A and $\varprojlim A_\lambda$ are topologically isomorphic. □

Theorem 3.3. *For any (real or complex) locally idempotent Hausdorff algebra A there exists a projective system $\{\tilde{A}_\lambda; \tilde{h}_{\lambda\mu}, \Lambda\}$ of locally idempotent Fréchet algebras and continuous homomorphisms $\tilde{h}_{\lambda\mu}$ from $\tilde{A}_\mu$ to $\tilde{A}_\alpha$ (whenever $\lambda \prec \mu$) such that A is topologically isomorphic to a dense subalgebra of the projective limit $\varprojlim \tilde{A}_\lambda$ of this system. In particular case, when, in addition, A is complete, then A and $\varprojlim \tilde{A}_\lambda$ are topologically isomorphic.*

Moreover, if A is a locally m-pseudoconvex (locally m-convex) Hausdorff algebra, then A is topologically isomorphic to a dense subalgebra of the projective limit $\varprojlim \tilde{A}_\lambda$ of k_λ-Banach (respectively, Banach) algebras. In particular, when, in addition, A is complete, then A and $\varprojlim \tilde{A}_\lambda$ are topologically isomorphic.

Proof. a) Let A be a locally idempotent Hausdorff algebra, $\mathcal{L}_A$ the base of closed and balanced neighbourhoods of zero in A and $\mathcal{S}_A = \{S_\lambda : \lambda \in \Lambda\}$ the collection of all strings in $\mathcal{L}_A$. Similarly as in the proof of Theorem 3.2, we define the ordering $\prec$ in Λ in the following way: we say that $\lambda \prec \mu$ in Λ if and only if $S_\mu \subset S_\lambda$. Then $(\Lambda, \prec)$ is a directed set.

For each $\lambda \in \Lambda$, let q_λ be the F-seminorm on A, defined by the string $S_\lambda = (O_n^\lambda)$,

$$N_\lambda = \bigcap_{n=1}^{\infty} O_n^\lambda$$

(N_λ is a closed two-sided ideal in A by Lemma 3.1), $A_\lambda = A/N_\lambda$ and let π_λ be the canonical homomorphism of A onto A_λ. Moreover, let $\overline{q}_\lambda(\pi_\lambda(a)) = q_\lambda(a)$ for each $a \in A$, $\tilde{A}_\lambda$ be the completion of A_λ, ν_λ the topological isomorphism from A_λ onto a dense subalgebra of $\tilde{A}_\lambda$ (defined by the completion of A_λ), $\tilde{q}_\lambda$ the extension of $\overline{q}_\lambda \circ \nu_\lambda^{-1}$ to $\tilde{A}_\lambda$ and $\tilde{\tau}_\lambda$ the topology on $\tilde{A}_\lambda$, defined by $\tilde{q}_\lambda$. Then

$$\tilde{q}_\lambda[(\nu_\lambda \circ \pi_\lambda)(a)] = \overline{q}_\lambda(\pi_\lambda(a)) = q_\lambda(a)$$

for each $a \in A$ and $\ker q_\lambda = N_\lambda$. Therefore, $\tilde{q}_\lambda$ is an F-norm on $\tilde{A}_\lambda$. Since A_λ is a metrizable locally idempotent algebra, then the multiplication in A_λ is jointly continuous, because of which $\tilde{A}_\lambda$ is an algebra. Hence, $(\tilde{A}_\lambda, \tilde{\tau}_\lambda)$ is a locally idempotent Fréchet algebra for each $\lambda \in \Lambda$.

Similarly as in the proof of Theorem 3.2, for each $\lambda, \mu \in \Lambda$ with $\lambda \prec \mu$ we define the map $h_{\lambda\mu}$ by $h_{\lambda\mu}(\pi_\mu(a)) = \pi_\lambda(a)$ for each $a \in A$. Then $h_{\lambda\mu}$ is a continuous homomorphism from A_μ into A_λ, $h_{\lambda\lambda}$ is the identity mapping on A_λ for each $\lambda \in \Lambda$ and $h_{\lambda\mu} \circ h_{\mu\gamma} = h_{\lambda\gamma}$ for each $\lambda, \mu, \gamma \in \Lambda$ with $\lambda \prec \mu \prec \gamma$. Since $\nu_\lambda \circ h_{\lambda\mu} \circ \nu_\mu^{-1}$ is a continuous homomorphism from $\nu_\mu(A_\mu)$ into $\tilde{A}_\lambda$, then (by [11, Proposition 5], $\tilde{h}_{\lambda\mu}$ is continuous and linear and by the continuity of multiplication in $\tilde{A}_\mu$, $\tilde{h}_{\lambda\mu}$ is submultiplicative similarly as in the proof of [13, Proposition 1, pp. 4–5] or in the proof of [1, Proposition 3]) there exists a continuous extension $\tilde{h}_{\lambda\mu}$ from $\tilde{A}_\mu$ into $\tilde{A}_\lambda$ such that $\tilde{h}_{\lambda\mu}$ is a homomorphism and

$$\tilde{h}_{\lambda\mu}[\nu_\mu(\pi_\mu(a))] = \nu_\lambda[h_{\lambda\mu}(\pi_\mu(a))] = \nu_\lambda[\pi_\lambda(a)]$$

for each $a \in A$ and $\lambda, \mu \in \Lambda$ with $\lambda \prec \mu$. Since $\tilde{h}_{\lambda\lambda}$ is the identity map on $\tilde{A}_\lambda$ for each $\lambda \in \Lambda$ and $\tilde{h}_{\lambda\mu} \circ \tilde{h}_{\mu\gamma} = \tilde{h}_{\lambda\gamma}$, whenever $\lambda, \mu, \gamma \in \Lambda$ and $\lambda \prec \mu \prec \gamma$, then $\{\tilde{A}_\lambda; \tilde{h}_{\lambda\mu}, \Lambda\}$ is a projective system of locally idempotent Fréchet algebras A_λ with continuous homomorphisms $\tilde{h}_{\lambda\mu}$ from $\tilde{A}_\mu$ into $\tilde{A}_\lambda$ and

$$\varprojlim \tilde{A}_\lambda = \{(\nu_\lambda[\pi_\lambda(a)])_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} \tilde{A}_\lambda : \ \tilde{h}_{\lambda\mu}[\nu_\mu(\pi_\mu(a))] = \nu_\lambda(\pi_\lambda(a)), \text{ whenever } \lambda \prec \mu\}$$

is the projective limit of this system.

Let $\tilde{e}$ be the mapping which is defined by $\tilde{e}(a) = (\nu_\lambda[\pi_\lambda(a)])_{\lambda\in\Lambda}$ from A into $\prod_{\mu\in\Lambda} \tilde{A}_\mu$ for each $\lambda \in \Lambda$. Similarly as in the proof of Theorem 3.2, we can show that $\tilde{e}$ is a topological isomorphism from A onto a dense subset of $\varprojlim \tilde{A}_\lambda$. Moreover, A and $\varprojlim \tilde{A}_\lambda$ are topologically isomorphic, if A is complete.

2) Let now A be a locally m-pseudoconvex (locally m-convex) Hausdorff algebra. Then every algebra A_λ in the first part of the proof is a k_λ-normed (respectively, normed) algebra, because of which the completion $\tilde{A}_\lambda$ of A_λ is a k_λ-Banach (respectively, Banach) algebra. Similarly as in the first part of the proof, we can

show that A is topologically isomorphic to a dense subalgebra of the projective limit $\varprojlim \tilde{A}_\lambda$ of k_λ-Banach (respectively, Banach) algebras. Moreover, A and $\varprojlim \tilde{A}_\lambda$ are topologicaly isomorphic if A is complete. □

Remark 3.4. Theorem 3.3 in the case where A is complete is well known. For the sake of completeness, this case has been added.

Corollary 3.5. *Let A be a unital Hausdorff algebra, $\mathcal{L}_A$ the base of closed and balanced neighbourhoods of zero in A and $\mathcal{Q}_A = \{q_S : S \text{ is a string in } \mathcal{L}_A\}$ the collection of F-seminorms, which defines the topology of A. If every q_S is submultiplicative, then A is locally m-pseudoconvex.*

Proof. Let A be a Hausdorff algebra with unit element e_A such that every F-seminorm q_S in $\mathcal{Q}_A$ is submultiplicative. Then A is a locally idempotent algebra, by Corollary 2.5. For any S in $\mathcal{L}_A$, let $A_S = A/\ker q_S$, π_S the canonical homomorphism from A onto A_S and $\overline{q}_S$ the map defined by $\overline{q}_S(\pi_S(a)) = q_S(a)$ for each $a \in A$. Then $\overline{q}_S$ is a submultiplicative F-norm on A_S. Hence, the extension $\tilde{q}_S$ (see the proof of Theorem 3.3) is a submultiplicative F-norm on the completion $\tilde{A}_S$. To show that $\tilde{A}_S$ is a locally bounded algebra (that is, $\tilde{A}_S$ contains a bounded neighbourhood of zero), let

$$O_S = \{x \in \tilde{A}_S : \tilde{q}_S(x) \leqslant 1\},$$

x_0 an arbitrary element in O_S and (α_n) an arbitrary sequence in $\mathbb{K}$ which converges to zero. We can assume that $|\alpha_n n| \leqslant 1$ for each $n \in \mathbb{N}$ (otherwise, we can use instead of (α_n) the subsequence (α_{k_n}), for which $|\alpha_{k_n} n| \leqslant 1$, because (α_{k_n}) converges to zero as well). Since

$$0 \leqslant \tilde{q}_S(\alpha_n x_0) = \tilde{q}_S((\alpha_n e_A)x_0) \leqslant \tilde{q}_S((\alpha_n n)\frac{1}{n}e_A)\tilde{q}_S(x_0) \leqslant \tilde{q}_S(\frac{1}{n}e_A),$$

then from

$$0 \leqslant \lim_{n\to\infty} \tilde{q}_S(\lambda_n x_0) \leqslant \lim_{n\to\infty} \tilde{q}_S(\frac{1}{n}e_A) = 0$$

follows that $(\alpha_n x_0)$ converges to the zero element of A. It means (see, for example, [12, Proposition 1, p. 34]) that O_S is a bounded neighbourhood of zero in $\tilde{A}_S$. Hence, $\tilde{A}_S$ is a locally bounded algebra, and therefore, locally m-pseudoconvex. Consequently, by Theorem 3.3, A is topologically isomorphic to a dense subalgebra W of the projective limit $\varprojlim \tilde{A}_S$ of complete locally m-pseudoconvex algebras $\tilde{A}_S$. Hence, W is a subalgebra of the product $\prod_{S\subset\mathcal{L}_A} \tilde{A}_S$. Since any product of locally m-pseudoconvex algebras is locally m-convex in the product topology and any subalgebra of a locally m-pseudoconvex algebra is locally m-pseudoconvex in subset topology, then W is locally m-pseudoconvex. Consequently, A is also locally m-pseudoconvex. □

Corollary 3.6. *Every unital m-algebra is locally m-pseudoconvex.*

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