REGULARITY AND FREE BOUNDARY REGULARITY FOR THE $p$ LAPLACIAN IN LIPSCHITZ AND $C^1$ DOMAINS

John L. Lewis and Kaj Nyström

University of Kentucky, Department of Mathematics
Lexington, KY 40506-0027, U.S.A.; john@ms.uky.edu

Umeå University, Department of Mathematics
S-90187 Umeå, Sweden; kaj.nystrom@math.umu.se

Abstract. In this paper we study regularity and free boundary regularity, below the continuous threshold, for the $p$ Laplace equation in Lipschitz and $C^1$ domains. To formulate our results we let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with constant $M$. Given $p$, $1 < p < \infty$, $w \in \partial \Omega$, $0 < r < r_0$, suppose that $u$ is a positive $p$ harmonic function in $\Omega \cap B(w, 4r)$, that $u$ is continuous in $\bar{\Omega} \cap \bar{B}(w, 4r)$ and $u = 0$ on $\Delta(w, 4r)$. We first prove, Theorem 1, that $\nabla u(y) \to \nabla u(x)$, for almost every $x \in \Delta(w, 4r)$, as $y \to x$ non tangentially in $\Omega$. Moreover, $\| \log |\nabla u| \|_{BMO(\Delta(w,r))} \leq c(p, n, M)$. If, in addition, $\Omega$ is $C^1$ regular then we prove, Theorem 2, that $\log |\nabla u| \in VMO(\Delta(w,r))$. Finally we prove, Theorem 3, that there exists $\hat{M}$, independent of $u$, such that if $M \leq \hat{M}$ and if $\log |\nabla u| \in VMO(\Delta(w,r))$ then the outer unit normal to $\partial \Omega$, $n$, is in $VMO(\Delta(w, r/2))$.

1. Introduction

In this paper, which is the last paper in a sequence of three, we complete our study of the boundary behaviour of $p$ harmonic functions in Lipschitz domains. In [LN] we established the boundary Harnack inequality for positive $p$ harmonic functions, $1 < p < \infty$, vanishing on a portion of the boundary of a Lipschitz domain $\Omega \subset \mathbb{R}^n$ and we carried out an in depth analysis of $p$ capacitary functions in starlike Lipschitz ring domains. The study in [LN] was continued in [LN1] where we established Hölder continuity for ratios of positive $p$ harmonic functions, $1 < p < \infty$, vanishing on a portion of the boundary of a Lipschitz domain $\Omega \subset \mathbb{R}^n$. In [LN1] we also studied the Martin boundary problem for $p$ harmonic functions in Lipschitz domains. In this paper we establish, in the setting of Lipschitz domains $\Omega \subset \mathbb{R}^n$, the analog for the $p$ Laplace equation, $1 < p < \infty$, of the program carried out in the papers [D], [JK], [KT], [KT1] and [KT2] on regularity and free boundary regularity, below the continuous threshold, for the Poisson kernel associated to the Laplace operator when $p = 2$. Except for the work in [LN], where parts of this program

2000 Mathematics Subject Classification: Primary 35J25, 35J70.
Key words: $p$ harmonic function, Lipschitz domain, regularity, free boundary regularity, elliptic measure, blow-up.
Lewis was partially supported by an NSF grant.
were established for \( p \) capacitary functions in starlike Lipschitz ring domains, the results of this paper are, in analogy with the results in [LN] and [LN1], completely new in case \( p \neq 2, 1 < p < \infty \). We also refer to [LN2] for a survey of the results established in [LN], [LN1] and in this paper.

To put the contributions of this paper into perspective we consider the case of harmonic functions and we recall that in [D] B. Dahlberg showed for \( p = 2 \), that if \( \Omega \) is a Lipschitz domain then the harmonic measure with respect to a fixed point, \( d\omega \), and surface measure, \( d\sigma \), are mutually absolutely continuous. In fact if \( k = d\omega / d\sigma \), then Dahlberg showed that \( k \) is in a certain \( L^2 \) reverse Hölder class from which it follows that \( \log k \in BMO(d\sigma) \), the functions of bounded mean oscillation with respect to the surface measure on \( \partial \Omega \). Jerison and Kenig [JK] showed that if, in addition, \( \Omega \) is a \( C^1 \) domain then \( \log k \in VMO(d\sigma) \), the functions in \( BMO(d\sigma) \) of vanishing mean oscillation. In [KT] this result was generalized to ‘chord arc domains with vanishing constant’. Concerning reverse conclusions, Kenig and Toro [KT2] were able to prove that if \( \Omega \subset \mathbb{R}^n \) is \( \delta \) Reifenberg flat for some small enough \( \delta > 0 \), \( \partial \Omega \) is Ahlfors regular and if \( \log k \in VMO(d\sigma) \), then \( \Omega \) is a chord arc domain with vanishing constant, i.e., the measure theoretical normal \( n \) is in \( VMO(d\sigma) \).

The purpose of this paper is to prove for \( p \) harmonic functions, \( 1 < p < \infty \), and in the setting of Lipschitz domains, \( \Omega \subset \mathbb{R}^n \), the results stated above for harmonic functions (i.e., \( p = 2 \)). We also note that we intend to establish, in a subsequent paper, the full program in the setting of Reifenberg flat chord arc domains.

To state our results we need to introduce some notation. Points in Euclidean \( n \) space \( \mathbb{R}^n \) are denoted by \( x = (x_1, \ldots, x_n) \) or \( (x', x_n) \) where \( x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \) and we let \( E, \partial E, \text{diam } E, \) be the closure, boundary, diameter, of the set \( E \subset \mathbb{R}^n \). We define \( d(y, E) \) to equal the distance from \( y \in \mathbb{R}^n \) to \( E \) and we let \( \langle \cdot, \cdot \rangle \) denote the standard inner product on \( \mathbb{R}^n \). Moreover, \( |x| = \langle x, x \rangle^{1/2} \) is the Euclidean norm of \( x \), \( B(x, r) = \{ y \in \mathbb{R}^n : |x - y| < r \} \) is defined whenever \( x \in \mathbb{R}^n, r > 0 \), and \( dx \) denotes the Lebesgue \( n \) measure on \( \mathbb{R}^n \). If \( O \subset \mathbb{R}^n \) is open and \( 1 \leq q \leq \infty \) then by \( W^{1,q}(O) \) we denote the space of equivalence classes of functions \( f \) with distributional gradient \( \nabla f = (f_{x_1}, \ldots, f_{x_n}) \), both of which are \( q \) th power integrable on \( O \). We let \( ||f||_{1,q} = ||f||_q + ||\nabla f||_q \) be the norm in \( W^{1,q}(O) \) where \( || \cdot ||_q \) denotes the usual Lebesgue \( q \) norm in \( O, C_0^\infty(O) \) denotes the class of infinitely differentiable functions with compact support in \( O \) and we let \( W^{1,q}_0(O) \) be the closure of \( C_0^\infty(O) \) in the norm of \( W^{1,q}(O) \).

Given a bounded domain \( G \), i.e., a connected open set, and \( 1 < p < \infty \) we say that \( u \) is \( p \) harmonic in \( G \) provided \( u \in W^{1,p}(G) \) and provided

\[
\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, dx = 0
\]

whenever \( \theta \in W^{1,p}_0(G) \). Observe that, if \( u \) is smooth and \( \nabla u \neq 0 \) in \( G \), then

\[
\nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0 \quad \text{in } G
\]

and \( u \) is a classical solution to the \( p \) Laplace partial differential equation in \( G \). Here, as in the sequel, \( \nabla \cdot \) is the divergence operator. We note that \( \phi : E \to \mathbb{R} \) is said to
be Lipschitz on $E$ provided there exists $b$, $0 < b < \infty$, such that
\begin{equation}
|\phi(z) - \phi(w)| \leq b|z - w|, \quad \text{whenever } z, w \in E.
\end{equation}

The infimum of all $b$ such that (1.3) holds is called the Lipschitz norm of $\phi$ on $E$ and is denoted $\|\phi\|_E$. It is well known that if $E = \mathbb{R}^{n-1}$, then $\phi$ is differentiable almost everywhere on $\mathbb{R}^{n-1}$ and $\|\phi\|_{\mathbb{R}^{n-1}} = |||\nabla \phi|||_\infty$.

In the following we let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, i.e., we assume that there exists a finite set of balls $\{B(x_i, r_i)\}$, with $x_i \in \partial \Omega$ and $r_i > 0$, such that $\{B(x_i, r_i)\}$ constitutes a covering of an open neighbourhood of $\partial \Omega$ and such that, for each $i$,
\begin{align}
\Omega \cap B(x_i, 4r_i) = \{y = (y', y_n) \in \mathbb{R}^n: y_n > \phi_i(y')\} \cap B(x_i, 4r_i), \\
\partial \Omega \cap B(x_i, 4r_i) = \{y = (y', y_n) \in \mathbb{R}^n: y_n = \phi_i(y')\} \cap B(x_i, 4r_i),
\end{align}
in an appropriate coordinate system and for a Lipschitz function $\phi_i$. The Lipschitz constant of $\Omega$ is defined to be $M = \max_i |||\nabla \phi_i|||_\infty$.

If the defining functions $\{\phi_i\}$ can be chosen to be $C^1$ regular then we say that $\Omega$ is a $C^1$ domain. If $\Omega$ is Lipschitz then there exists $r_0 > 0$ such that if $w \in \partial \Omega$, $0 < r < r_0$, then we can find points $a_r(w) \in \Omega \cap \partial B(w, r)$ with $d(a_r(w), \partial \Omega) \geq c^{-1}r$ for a constant $c = c(M)$. In the following we let $a_r(w)$ denote one such point. Furthermore, if $w \in \partial \Omega$, $0 < r < r_0$, then we let $\Delta(w, r) = \partial \Omega \cap B(w, r)$. Finally we let $e_i$, $1 \leq i \leq n$, denote the point in $\mathbb{R}^n$ with one in the $i$th coordinate position and zeroes elsewhere and we let $\sigma$ denote surface measure, i.e., the $(n - 1)$-dimensional Hausdorff measure, on $\partial \Omega$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $w \in \partial \Omega$, $0 < r < r_0$. If $0 < b < 1$ and $x \in \Delta(w, 2r)$ then we let
\begin{equation}
\Gamma(x) = \Gamma_b(x) = \{y \in \Omega: d(y, \partial \Omega) > b|x - y|\} \cap B(w, 4r).
\end{equation}

Given a measurable function $k$ on $\bigcup_{x \in \Delta(w, 2r)} \Gamma(x)$ we define the non tangential maximal function $N(k): \Delta(w, 2r) \to \mathbb{R}$ for $k$ as
\begin{equation}
N(k)(x) = \sup_{y \in \Gamma(x)} |k(y)| \quad \text{whenever } x \in \Delta(w, 2r).
\end{equation}

We let $L^q(\Delta(w, 2r))$, $1 \leq q \leq \infty$, be the space of functions which are integrable, with respect to the surface measure, $\sigma$, to the power $q$ on $\Delta(w, 2r)$. Furthermore, given a measurable function $f$ on $\Delta(w, 2r)$ we say that $f$ is of bounded mean oscillation on $\Delta(w, r)$, $f \in BMO(\Delta(w, r))$, if there exists $A$, $0 < A < \infty$, such that
\begin{equation}
\int_{\Delta(x, s)} |f - f_\Delta|^2 \, d\sigma \leq A^2 \sigma(\Delta(x, s))
\end{equation}
whenever $x \in \Delta(w, r)$ and $0 < s \leq r$. Here $f_\Delta$ denotes the average of $f$ on $\Delta = \Delta(x, s)$ with respect to the surface measure $\sigma$. The least $A$ for which (1.7) holds is denoted by $||f||_{BMO(\Delta(w, r))}$. If $f$ is a vector valued function, $f = (f_1, \ldots, f_n)$, then $f_\Delta = (f_{1, \Delta}, \ldots, f_{n, \Delta})$ and the $BMO$-norm of $f$ is defined as in (1.7) with $|f - f_\Delta|^2 = \langle f - f_\Delta, f - f_\Delta \rangle$. Finally we say that $f$ is of vanishing mean oscillation on $\Delta(w, r)$, $f \in VMO(\Delta(w, r))$, provided for each $\varepsilon > 0$ there is a $\delta > 0$ such that
Then there exists \( \hat{u} \) in a similar manner to obtain these theorems in general. Concerning converse results starlike Lipschitz ring domains. Moreover, using Theorem 2 in [LN1] we can argue in addition, statement of Theorem 3 and extend contradiction we then use a blow-up argument. In particular, let \( V_{MO} \) (1.7) holds with \( A \) replaced by \( \varepsilon \) whenever \( 0 < s < \min(\delta, r) \) and \( x \in \Delta(w, r) \). For more on BMO we refer to [S1, chapter IV].

In this paper we first prove the following two theorems.

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain with constant \( M \). Given \( p, 1 < p < \infty, w \in \partial \Omega, 0 < r < r_0 \), suppose that \( u \) is a positive \( p \) harmonic function in \( \Omega \cap B(w, 4r) \), \( u \) is continuous in \( \Omega \cap B(w, 4r) \) and \( u = 0 \) on \( \Delta(w, 4r) \). Then

\[
\lim_{y \in \Gamma(x), y \to x} \nabla u(y) = \nabla u(x)
\]

for \( \sigma \) almost every \( x \in \Delta(w, 4r) \). Furthermore there exist \( q > p \) and a constant \( c, 1 \leq c < \infty \), which both only depend on \( p, n \) and \( M \) such that

1. \( N(|\nabla u|) \in L^q(\Delta(w, 2r)) \),
2. \[
\int_{\Delta(w, 2r)} |\nabla u|^q \, d\sigma \leq c r^{(n-1)(\frac{q}{p-1} - 1)} \left( \int_{\Delta(w, 2r)} |\nabla u|^{p-1} \, d\sigma \right)^{\frac{q}{p-1}},
\]
3. \[ \log |\nabla u| \in BMO(\Delta(w, r)), \quad \| \log |\nabla u| \|_{BMO(\Delta(w, r))} \leq c. \]

**Theorem 2.** Let \( \Omega, M, p, w, r \) and \( u \) be as in the statement of Theorem 1. If, in addition, \( \Omega \) is \( C^1 \) regular then

\[
\log |\nabla u| \in VMO(\Delta(w, r)).
\]

Theorem 1 and Theorem 2 are proved in [LN] for \( p \) capacitary functions in starlike Lipschitz ring domains. Moreover, using Theorem 2 in [LN1] we can argue in a similar manner to obtain these theorems in general. Concerning converse results we in this paper prove the following theorem.

**Theorem 3.** Let \( \Omega, M, p, w, r \) and \( u \) be as in the statement of Theorem 1. Then there exists \( \tilde{M} \), independent of \( u \), such that if \( M \leq \tilde{M} \) and \( \log |\nabla u| \in VMO(\Delta(w, r)) \), then the outer unit normal to \( \Delta(w, r) \) is in \( VMO(\Delta(w, r/2)) \).

We let \( n \) denote the outer unit normal to \( \partial \Omega \). To briefly discuss the proof of Theorem 3 we define

(1.8) \[
\eta = \lim_{\tilde{r} \to 0} \sup_{\tilde{w} \in \Delta(w, r/2)} \| n \|_{BMO(\Delta(\tilde{w}, \tilde{r}))}.
\]

To prove Theorem 3 it is enough to prove that \( \eta = 0 \). To do this we argue by contradiction and assume that (1.8) holds for some \( \eta > 0 \). This assumption implies that there exist a sequence of points \( \{ w_j \}, w_j \in \Delta(w, r/2) \), and a sequence of scales \( \{ r_j \}, r_j \to 0 \), such that \( \| n \|_{BMO(\Delta(w_j, r_j))} \to \eta \) as \( j \to \infty \). To establish a contradiction we then use a blow-up argument. In particular, let \( u \) be as in the statement of Theorem 3 and extend \( u \) to \( B(w, 4r) \) by putting \( u = 0 \) in \( B(w, 4r) \setminus \Omega \). For \( \{ w_j \}, \{ r_j \} \) as above we define \( \Omega_j = \{ r_j^{-1}(x - w_j): x \in \Omega \} \) and

(1.9) \[
u j(z) = \lambda_j u(w_j + r_j z) \text{ whenever } z \in \Omega_j.
\]
where \( \{\lambda_j\} \) is an appropriate sequence of real numbers defined in the bulk of the paper. We then show that subsequences of \( \{\Omega_j\} \), \( \{\partial\Omega_j\} \) converge to \( \Omega_\infty \), \( \partial\Omega_\infty \), in the Hausdorff distance sense, where \( \Omega_\infty \) is an unbounded Lipschitz domain with Lipschitz constant bounded by \( M \). Moreover, by our choice of the sequence \( \{\lambda_j\} \) it follows that a subsequence of \( \{u_j\} \) converges uniformly on compact subsets of \( \mathbb{R}^n \) to \( u_\infty \), a positive \( p \) harmonic function in \( \Omega_\infty \) vanishing continuously on \( \partial\Omega_\infty \). Defining 
\[ d\mu_j = |\nabla u_j|^{p-1} \, d\sigma_j, \]
where \( \sigma_j \) is surface measure on \( \partial\Omega_j \), it will also follow that a subsequence of \( \{\mu_j\} \) converges weakly as Radon measures to \( \mu_\infty \) and that

\[ \int_{\mathbb{R}^n} |\nabla u_\infty|^{p-2} \langle \nabla u_\infty, \nabla \phi \rangle \, dx = - \int_{\partial\Omega_\infty} \phi \, d\mu_\infty \]  
whenever \( \phi \in C^\infty_0(\mathbb{R}^n) \). Moreover, we prove that the limiting measure, \( \mu_\infty \), and the limiting function, \( u_\infty \), satisfy,

\[ \mu_\infty = \sigma_\infty \text{ on } \partial\Omega_\infty, \quad c^{-1} \leq |\nabla u_\infty(z)| \leq 1 \text{ whenever } z \in \Omega_\infty. \]  

In (1.11) \( \sigma_\infty \) is surface measure on \( \partial\Omega_\infty \) and \( c \) is a constant, \( 1 \leq c < \infty \), depending only on \( p, n \) and \( M \). Using (1.11) and results of Alt, Caffarelli and Friedman [ACF] we are then able to conclude that there exists \( \hat{M} \), independent of \( u_\infty \), such that if \( M \leq \hat{M} \) then (1.10) and (1.11) imply that \( \Omega_\infty \) is a halfplane. In particular, this will contradict the assumption that \( \eta \) defined in (1.8) is positive. Hence \( \eta = 0 \) and we are able to complete the proof of Theorem 3. Thus a substantial part of the proof of Theorem 3 is devoted to appropriate limiting arguments in order to conclude (1.10) and (1.11).

Of paramount importance to arguments in this paper is a result in [LN1] (listed as Theorem 2.7 in section 2), stating that the ratio of two positive \( p \) harmonic functions, \( 1 < p < \infty \), vanishing on a portion of the boundary of a Lipschitz domain \( \Omega \subset \mathbb{R}^n \) is Hölder continuous up to the boundary. This result implies (see Theorem 2.8 in section 2), that if \( \Omega, M, p, w, r \) and \( u \) are as in the statement of Theorem 1, then there exist \( c_3, 1 \leq c_3 < \infty, \lambda > 0 \), (both depending only on \( p, n, M \)) and \( \xi \in \partial B(0,1) \), independent of \( u \), such that if \( x \in \Omega \cap B(w, r/c_3) \), then

\[ \lambda^{-1} \frac{u(x)}{d(x, \partial\Omega)} \leq |\nabla u(x)| \leq \lambda \frac{u(x)}{d(x, \partial\Omega)}, \quad \text{ (ii) } \lambda^{-1} \frac{u(x)}{d(x, \partial\Omega)} \leq \langle \nabla u(x), \xi \rangle. \]  

If (1.12) (i) holds then we say that \( |\nabla u| \) satisfies a uniform non-degeneracy condition in \( \Omega \cap B(w, r/c_3) \) with constants depending only on \( p, n \) and \( M \). Moreover, using this non-degeneracy property of \( |\nabla u| \) it follows, by differentiation of (1.2), that if \( \zeta = \langle \nabla u, \xi \rangle \), for some \( \xi \in \mathbb{R}^n, |\xi| = 1 \), then \( \zeta \) satisfies, at \( x \) and in \( \Omega \cap B(w, r/(2c_3)) \), the partial differential equation \( \Lambda \zeta = 0 \), where

\[ L = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( b_{ij}(x) \frac{\partial}{\partial x_j} \right) \]

and

\[ b_{ij}(x) = |\nabla u|^{p-4}[(p-2)u_x u_x + \delta_{ij} |\nabla u|^2](x), \quad 1 \leq i, j \leq n. \]
In (1.14) $\delta_{ij}$ denotes the Kronecker $\delta$. Furthermore,

$$
(1.15) \quad \left( \frac{u(x)}{c d(x, \partial \Omega)} \right)^{p-2} |\xi|^2 \leq \sum_{i,j=1}^{n} b_{ij}(x) \xi_i \xi_j \leq \left( \frac{c u(x)}{d(x, \partial \Omega)} \right)^{p-2} |\xi|^2.
$$

To make the connection to the proof of Theorems 1–3 we first note that using (1.12)–(1.15) and we can use arguments from [LN] and apply classical theorems for elliptic PDE to get Theorems 1 and 2. The proof of Theorem 3 uses these results and the blow-up argument mentioned above and in the proof particular attention is paid to the proof of the refined upper bound for $|\nabla u_{\infty}|$ stated in (1.11).

The rest of the paper is organized as follows. In section 2 we state estimates for $p$ harmonic functions in Lipschitz domains and we discuss elliptic measure defined with respect to the operator $L$ defined in (1.13), (1.14). Most of this material is from [LN] and [LN1]. Section 3 is devoted to the proofs of Theorem 1 and Theorem 2. In section 4 we prove Theorem 3. In section 5 we discuss future work on free boundary problems beyond Lipschitz and $C^1$ domains.

Finally, we emphasize that this paper is not self-contained and that it relies heavily on work in [LN, LN1]. Thus the reader is advised to have these papers at hand.

2. Estimates for $p$ harmonic functions in Lipschitz domains

In this section we consider $p$ harmonic functions in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ having Lipschitz constant $M$. Recall that $\Delta(w, r) = \partial \Omega \cap B(w, r)$ whenever $w \in \partial \Omega$, $0 < r$. Throughout the paper $c$ will denote, unless otherwise stated, a positive constant $\geq 1$, not necessarily the same at each occurrence, which only depends on $p$, $n$ and $M$. In general, $c(a_1, \ldots, a_n)$ denotes a positive constant $\geq 1$, not necessarily the same at each occurrence, which depends on $p$, $n$, $M$ and $a_1, \ldots, a_n$. If $A \approx B$ then $A/B$ is bounded from above and below by constants which, unless otherwise stated, only depend on $p$, $n$ and $M$. Moreover, we let $\max_{B(z,s)} u$, $\min_{B(z,s)} u$ be the essential supremum and infimum of $u$ on $B(z,s)$ whenever $B(z,s) \subset \mathbb{R}^n$ and $u$ is defined on $B(z,s)$.

2.1. Basic estimates. For proofs and for references to the proofs of Lemma 2.1–2.5 stated below we refer to [LN].

**Lemma 2.1.** Given $p$, $1 < p < \infty$, let $u$ be a positive $p$ harmonic function in $B(w, 2r)$. Then

(i) \hspace{1cm} r^{p-n} \int_{B(w, r/2)} |\nabla u|^p dx \leq c(\max_{B(w,r)} u)^p,

(ii) \hspace{1cm} \max_{B(w,r)} u \leq c \min_{B(w,r)} u.

---

1For preprints we refer to www.ms.uky.edu/~john and www.math.umu.se/personal/nystrom_kaj.
Furthermore, there exists \( \alpha = \alpha(p, n, M) \in (0, 1) \) such that if \( x, y \in B(w, r) \) then
\[
(iii) \quad |u(x) - u(y)| \leq c \left( \frac{|x - y|}{r} \right)^\alpha \max_{B(w, 2r)} u.
\]

**Lemma 2.2.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain and suppose that \( p \) is given, \( 1 < p < \infty \). Let \( w \in \partial \Omega, \ 0 < r < r_0 \) and suppose that \( u \) is a positive \( p \) harmonic function in \( \Omega \cap B(w, 2r) \), continuous in \( \overline{\Omega} \cap B(w, 2r) \) and that \( u = 0 \) on \( \Delta(w, 2r) \). Then
\[
(i) \quad r^{p-n} \int_{\Omega \cap B(w, r/2)} |\nabla u|^p \, dx \leq c \left( \max_{\Omega \cap B(w, r)} u \right)^p.
\]
Furthermore, there exists \( \alpha = \alpha(p, n, M) \in (0, 1) \) such that if \( x, y \in \Omega \cap B(w, r) \) then
\[
(ii) \quad |u(x) - u(y)| \leq c \left( \frac{|x - y|}{r} \right)^\alpha \max_{\Omega \cap B(w, r)} u.
\]

**Lemma 2.3.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain and suppose that \( p \) is given, \( 1 < p < \infty \). Let \( w \in \partial \Omega, \ 0 < r < r_0 \) and suppose that \( u \) is a positive \( p \) harmonic function in \( \Omega \cap B(w, 2r) \), continuous in \( \overline{\Omega} \cap B(w, 2r) \) and that \( u = 0 \) on \( \Delta(w, 2r) \). There exists \( c = c(p, n, M) \geq 1 \) such that if \( \tilde{r} = r/c \), then
\[
\max_{\Omega \cap B(w, \tilde{r})} u \leq cu(\tilde{r}(w)).
\]

**Lemma 2.4.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain and suppose that \( p \) is given, \( 1 < p < \infty \). Let \( w \in \partial \Omega, \ 0 < r < r_0 \) and suppose that \( u \) is a positive \( p \) harmonic function in \( \Omega \cap B(w, 4r) \), continuous in \( \overline{\Omega} \cap B(w, 4r) \) and that \( u = 0 \) on \( \Delta(w, 4r) \). Extend \( u \) to \( B(w, 4r) \) by defining \( u \equiv 0 \) on \( B(w, 4r) \setminus \Omega \). Then \( u \) has a representative in \( W^{1,p}(B(w, 4r)) \) with Hölder continuous partial derivatives in \( \overline{\Omega} \cap B(w, 4r) \). In particular, there exists \( \sigma \in (0, 1] \), depending only on \( p, n \) such that if \( B(\tilde{w}, 4\tilde{r}) \subset \Omega \cap B(w, 4r) \) and \( x, y \in B(\tilde{w}, \tilde{r}/2) \), then
\[
c^{-1} |\nabla u(x) - \nabla u(y)| \leq (|x - y|/\tilde{r})^\sigma \max_{B(\tilde{w}, \tilde{r})} |\nabla u| \leq c\tilde{r}^{-1} (|x - y|/\tilde{r})^\sigma \max_{B(\tilde{w}, 2\tilde{r})} u.
\]

**Lemma 2.5.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain. Given \( p, 1 < p < \infty, \ w \in \partial \Omega, \ 0 < r < r_0 \), suppose that \( u \) is a positive \( p \) harmonic function in \( \Omega \cap B(w, 2r) \), continuous in \( \Omega \cap B(w, 2r) \) with \( u = 0 \) on \( \Delta(w, 2r) \). Extend \( u \) to \( B(w, 2r) \) by defining \( u \equiv 0 \) on \( B(w, 2r) \setminus \Omega \). Then there exists a unique finite positive Borel measure \( \mu \) on \( \mathbb{R}^n \), with support in \( \Delta(w, 2r) \), such that
\[
(i) \quad \int_{\mathbb{R}^n} |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx = - \int_{\mathbb{R}^n} \phi \, d\mu
\]
whenever \( \phi \in C_0^\infty(B(w, 2r)) \). Moreover, there exists \( c = c(p, n, M) \geq 1 \) such that if \( \tilde{r} = r/c \), then
\[
(ii) \quad c^{-1} \tilde{r}^{p-n} \mu(\Delta(w, \tilde{r})) \leq (u(a_r(w)))^{p-1} \leq c\tilde{r}^{p-n} \mu(\Delta(w, \tilde{r})).
\]
2.2. Refined estimates. In the following we state a number of results and estimates proved in [LN] and [LN1]. In particular, for the proof of Theorems 2.6–2.8 stated below we refer to [LN] and [LN1] and we note that Theorem 2.8 is referred to as Lemma 4.28 in [LN1] while Theorem 2.6 and Theorem 2.7 are two of the main results established in [LN] and [LN1] respectively.

**Theorem 2.6.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain with constant \( M \). Given \( p, 1 < p < \infty \), \( w \in \partial \Omega \), \( 0 < r < r_0 \), suppose that \( u \) and \( v \) are positive \( p \) harmonic functions in \( \Omega \cap B(w, 2r) \). Assume also that \( u \) and \( v \) are continuous in \( \bar{\Omega} \cap B(w, 2r) \), and \( u = 0 = v \) on \( \Delta(w, 2r) \). Under these assumptions there exists \( c_1, 1 \leq c_1 < \infty \), depending only on \( p \), \( n \) and \( M \), such that if \( \tilde{r} = r/c_1 \), \( u(a_{\tilde{r}}(w)) = v(a_{\tilde{r}}(w)) = 1 \), and \( y \in \Omega \cap B(w, \tilde{r}) \), then

\[
\frac{u(y)}{v(y)} \leq c_1.
\]

**Theorem 2.7.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain with constant \( M \). Given \( p, 1 < p < \infty \), \( w \in \partial \Omega \), \( 0 < r < r_0 \), suppose that \( u \) and \( v \) are positive \( p \) harmonic functions in \( \Omega \cap B(w, 2r) \). Assume also that \( u \) and \( v \) are continuous in \( \bar{\Omega} \cap B(w, 2r) \) and \( u = 0 = v \) on \( \Delta(w, 2r) \). Under these assumptions there exist \( c_2, 1 \leq c_2 < \infty \), and \( \alpha \in (0, 1) \), both depending only on \( p \), \( n \) and \( M \), such that if \( y_1, y_2 \in \Omega \cap B(w, r/c_2) \) then

\[
\left| \log \left( \frac{u(y_1)}{v(y_1)} \right) - \log \left( \frac{u(y_2)}{v(y_2)} \right) \right| \leq c_2 \left( \frac{|y_1 - y_2|}{r} \right)^\alpha.
\]

**Theorem 2.8.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain with constant \( M \). Let \( w \in \partial \Omega \), \( 0 < r < r_0 \), and suppose that (1.4) holds with \( x_i, r_i, \phi_i \) replaced by \( w, r, \phi \). Given \( p, 1 < p < \infty \), \( w \in \partial \Omega \), \( 0 < r < r_0 \), suppose that \( u \) is a positive \( p \) harmonic function in \( \Omega \cap B(w, 2r) \). Assume also that \( u \) is continuous in \( \bar{\Omega} \cap B(w, 2r) \) and \( u = 0 \) on \( \Delta(w, 2r) \). Then there exist \( c_3, 1 \leq c_3 < \infty \) and \( \lambda > 0 \), depending only on \( p \), \( n \) and \( M \), such that if \( y \in \Omega \cap B(w, r/c_3) \) then

\[
\lambda^{-1} \frac{u(y)}{d(y, \partial \Omega)} \leq \langle \nabla u(y), e_\alpha \rangle \leq |\nabla u(y)| \leq \lambda \frac{u(y)}{d(y, \partial \Omega)}.
\]

We note that Lemmas 2.9–2.12 below are stated and proved, for \( p \) capacitary functions in starlike Lipschitz ring domains, as Lemma 2.5 (iii), Lemma 2.39, Lemma 2.45 and Lemma 2.54 in [LN]. However armed with Theorem 2.8 the proofs of these lemmas can be extended to the more general situation of positive \( p \) harmonic functions vanishing on a portion of the boundary of a Lipschitz domain. Lemma 2.9 is only stated as it is used in the proof of Lemmas 2.10–2.12 as outlined in [LN], while Lemmas 2.10–2.12 are used in the proof of Theorems 1–3. We refer to [LN] for details (see also the discussion after Lemma 2.8 in [LN1]).

**Lemma 2.9.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain with constant \( M \). Given \( p, 1 < p < \infty \), \( w \in \partial \Omega \), \( 0 < r < r_0 \), suppose that \( u \) is a positive \( p \) harmonic
function in \( \Omega \cap B(w, 2r) \) and that \( u \) is continuous in \( \tilde{\Omega} \cap B(w, 2r) \) with \( u = 0 \) on \( \Delta(w, 2r) \). Then there exists a constant \( c = c(p, n, M) \), \( 1 \leq c < \infty \), such that

\[
\max_{B(x, 2s)} \sum_{i,j=1}^{n} |u_{y_i y_j}| \leq c \left( s^{-n} \int_{B(x, 3s/4)} \sum_{i,j=1}^{n} |u_{y_i y_j}|^2 \, dy \right)^{1/2} \leq c^2 u(x)/d(x, \partial \Omega)^{2}
\]

whenever \( x \in \Omega \cap B(w, r/c) \) and \( 0 < s \leq d(x, \partial \Omega) \).

**Lemma 2.10.** Let \( \tilde{\Omega}, M, p, w, r \) and \( u \) be as in the statement of Lemma 2.9. Let \( \mu \) be as in Lemma 2.5. Then there exists a constant \( c = c(p, n, M) \), \( 1 \leq c < \infty \), such that \( d\mu/d\sigma = k^{p-1} \) on \( \Delta(w, 2r/c) \) and

\[
\int_{\Delta(w, r/c)} k^p \, d\sigma \leq c r^{-n/p-1} \left( \int_{\Delta(w, r/c)} k^{p-1} \, d\sigma \right)^{p/(p-1)}.
\]

Recall that a bounded domain \( \Omega \subset \mathbb{R}^n \) is said to be starlike Lipschitz, with respect to \( \tilde{x} \in \Omega \), provided \( \partial \Omega = \{ \tilde{x} + R(\omega) \omega : \omega \in \partial B(0, 1) \} \) where \( R : \partial B(0, 1) \to \mathbb{R} \) is Lipschitz on \( \partial B(0, 1) \). We refer to \( \| \log R \|_{\partial B(0,1)} \) as the Lipschitz constant for \( \tilde{x} \). We observe that this constant is invariant under scalings about \( \tilde{x} \).

**Lemma 2.11.** Let \( \Omega, M, p, w, r \) and \( u \) be as in the statement of Lemma 2.9. Then there exist a constant \( c = c(p, n, M) \), \( 1 \leq c < \infty \), and a starlike Lipschitz domain \( \tilde{\Omega} \subset \Omega \cap B(w, 2r) \), with center at a point \( \tilde{w} \in \Omega \cap B(w, r) \), \( d(\tilde{w}, \partial \Omega) \geq c^{-1} r \), and with Lipschitz constant bounded by \( c \), such that

\[
c\sigma(\partial \tilde{\Omega} \cap \Delta(w, r)) \geq r^{n-1}.
\]

Moreover, the following inequality is valid for all \( x \in \tilde{\Omega} \),

\[
c^{-1} r^{-1} u(\tilde{w}) \leq |\nabla u(x)| \leq cr^{-1} u(\tilde{w}).
\]

**Lemma 2.12.** Let \( \tilde{\Omega}, M, p, w, r \) and \( u \) be as in the statement of Lemma 2.9. Let \( \tilde{\Omega} \) be constructed as in Lemma 2.11. Define, for \( y \in \tilde{\Omega} \), the measure

\[
d\tilde{\gamma}(y) = d(y, \partial \tilde{\Omega}) \max_{B(y, 1/2 d(y, \partial \tilde{\Omega}))} \{ |\nabla u|^{2p-6} \sum_{i,j=1}^{n} u_{x_i x_j}^2 \} \, dy.
\]

Then \( \tilde{\gamma} \) is a Carleson measure on \( \tilde{\Omega} \) and there exists a constant \( c = c(p, n, M) \), \( 1 \leq c < \infty \), such that if \( z \in \partial \tilde{\Omega} \) and \( 0 < s < r \), then

\[
\tilde{\gamma}(\tilde{\Omega} \cap B(z, s)) \leq cs^{-n-1} (u(\tilde{w})/r)^{2p-4}.
\]

Let \( u, \tilde{\Omega} \) be as in Lemma 2.12. We end this section by considering the divergence form operator \( L \) defined as in (1.13), (1.14), relative to \( u, \tilde{\Omega} \). In particular, we state a number of results for this operator which we will make use of in the following sections. Arguing as above (1.13) we first observe that

\[
L(\langle \nabla u, \xi \rangle) = 0 \text{ weakly in } \tilde{\Omega}
\]
whenever $\xi \in \partial B(0,1)$. Moreover, using Theorem 2.8, Lemma 2.11, and (1.15) we see that $L$ is uniformly elliptic in $\Omega$. Using this fact it follows from [CFMS] that if $z \in \partial \tilde{\Omega}$, $0 < s < r$, and if $v$ is a weak solution to $L$ in $\tilde{\Omega}$ which vanishes continuously on $\partial \tilde{\Omega} \cap B(z,s)$, then there exist $\tau$, $0 < \tau \leq 1$, and $c \geq 1$, both depending only on $p$, $n$, $M$, such that

$$
\max_{\tilde{\Omega} \cap B(z,t)} v \leq c \left(\frac{t}{s}\right)^\tau \max_{\tilde{\Omega} \cap B(z,s)} v, \text{ whenever } 0 < t \leq s.
$$

Moreover, using Lemma 2.12 we observe that if \( d \theta(y) = d(y, \partial \tilde{\Omega}) \max_{B(y, \frac{1}{2} d(y, \partial \tilde{\Omega})}} \left\{ \sum_{i,j=1}^{n} |\nabla b_{ij}|^2 \right\} dy, \) where \( \{b_{ij}\} \) is the matrix defining $L$ in (1.14), then $\theta$ is a Carleson measure on $\tilde{\Omega}$ and

$$
\theta(\tilde{\Omega} \cap B(z,s)) \leq cs^{n-1}(u(\tilde{w})/r)^{2p-4}
$$

whenever $z \in \partial \tilde{\Omega}$ and $0 < s < r$. Let $\tilde{\omega}(\cdot, \tilde{w})$ be elliptic measure defined with respect to $L$, $\tilde{\Omega}$, and $\tilde{w}$ (see [CFMS] for the definition of elliptic measure). We note that the above observation and the main theorem in [KP] imply the following lemma.

**Lemma 2.15.** Let $u, \tilde{\Omega}, \tilde{w}$ be as in Lemma 2.12 and let $L$ be defined as in (1.13), (1.14), relative to $u$, $\tilde{\Omega}$. Then $\tilde{\omega}(\cdot, \tilde{w})$ and the surface measure on $\partial \tilde{\Omega}$ (denoted $\tilde{\sigma}$) are mutually absolutely continuous. Moreover, $\tilde{\omega}(\cdot, \tilde{w})$ is an $A^\infty$ weight with respect to $\tilde{\sigma}$. Consequently, there exist $c \geq 1$ and $\gamma$, $0 < \gamma \leq 1$, depending only on $p$, $n$, $M$, such that

$$
\frac{\tilde{\omega}(E, \tilde{w})}{\tilde{\omega}(\partial \tilde{\Omega} \cap B(z,s), \tilde{w})} \leq c \left( \frac{\tilde{\sigma}(E)}{\tilde{\sigma}(\partial \tilde{\Omega} \cap B(z,s))} \right)^\gamma
$$

whenever $z \in \partial \tilde{\Omega}$, $0 < s < r$, and $E \subset \partial \tilde{\Omega} \cap B(z,s)$ is a Borel set.

For several other equivalent definitions of $A^\infty$ weights we refer to [CF] or [GR].

3. Proof of Theorem 1 and Theorem 2

In this section we prove Theorem 1 and Theorem 2. Hence we let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with constant $M$ and for given $p$, $1 < p < \infty$, $w \in \partial \Omega$, $0 < r < r_0$ we suppose that $u$ is a positive $p$ harmonic function in $\Omega \cap B(w,4r)$, continuous in $\tilde{\Omega} \cap B(w,4r)$ with $u = 0$ on $\Delta(w,4r)$.

3.1. Proof of Theorem 1. We first note that we can assume, without loss of generality, that

$$
\max_{\Omega \cap B(w,4r)} u = 1.
$$

We extend $u$ to $B(w,4r)$ by defining $u \equiv 0$ on $B(w,4r) \setminus \Omega$ and we let $\mu$ be the measure associated to $u$ as in the statement of Lemma 2.5. Using Lemma 2.10,
Lemma 2.5 (ii) and the Harnack inequality for $p$ harmonic functions we see that if $y \in \partial \Omega$, $s > 0$ and $B(y, 2cs) \subset B(w, 4r)$, then $d\mu/d\sigma = k^{p-1}$ on $\Delta(y, 2s)$ and

$$
\int_{\Delta(y,s)} k^p \, d\sigma \leq cs^{-\frac{n+1}{p-1}} \left( \int_{\Delta(y,s/2)} k^{p-1} \, d\sigma \right)^{p/(p-1)}.
$$

(3.2) and Lemma 2.5 (ii) imply (see [G], [CF]) that for some $q' > p$, depending only on $p$, $n$ and $M$, we have

$$
\int_{\Delta(w,3r)} k^q \, d\sigma \leq cr^{-\frac{(n-1)(q'+1-p)}{p-1}} \left( \int_{\Delta(w,3r)} k^{p-1} \, d\sigma \right)^{q'/(p-1)}.
$$

(3.3) Let $y \in \Delta(w,2r)$ and let $z \in \Gamma(y) \cap B(y, r/(4c_3))$, where $c_3$ is the constant appearing in the statement of Theorem 2.8 and $\Gamma(y)$, for $y \in \Delta(w, 2r)$, is defined in (1.5). Using Theorem 2.8, with $w$ replaced by $y$, $s = |z - y|$ and Lemma 2.5 (ii) we obtain

$$
|\nabla u(z)| \leq c \frac{u(z)}{s} \leq c^2 s^{-1} \left( s^{p-n} \mu(\Delta(y, s)) \right)^{1/(p-1)}
$$

(3.4)\[
= c^2 s^{1-n} \int_{\Delta(y,s)} k^{p-1} \, d\sigma \leq c^2 (M(k^{p-1})(y))^{1/(p-1)}.
\]

In (3.4),

$$
M(f)(y) = \sup_{0<s<r/4} s^{1-n} \int_{\Delta(y,s)} f \, d\sigma
$$

whenever $f$ is an integrable function on $\Delta(w, 3r)$. Next we define

$$
N_1(|\nabla u|)(y) = \sup_{\Gamma(y) \cap B(y,r/(4c_3))} |\nabla u| \text{ whenever } y \in \Delta(w, 2r).
$$

Using (3.3), (3.4) and the Hardy–Littlewood maximal theorem we see that if $q = (q' + p)/2$ then

$$
\int_{\Delta(w,2r)} N_1(|\nabla u|)^q \, d\sigma \leq c \int_{\Delta(w,2r)} M(k^{p-1})^{q/(p-1)} \, d\sigma
$$

(3.5)\[
\leq c^2 r^{-\frac{(n-1)(q+1-p)}{p-1}} \left( \int_{\Delta(w,2r)} k^{p-1} \, d\sigma \right)^{q/(p-1)}.
\]

Using Lemma 2.4 and (3.1) we also see that $|\nabla u(x)| \leq cr^{-1}$ whenever $x \in \Gamma(y) \setminus B(y, r/(4c_3))$ and $y \in \Delta(w, 2r)$. Thus $N(|\nabla u|) \leq N_1(|\nabla u|) + cr^{-1}$ on $\Delta(w, 2r)$. Therefore, using (3.5) as well as Lemma 2.5 (ii) and (3.1) once again we can conclude that statement (i) of Theorem 1 is true.
Next we prove by a contradiction argument that $\nabla u$ has non tangential limits for $\sigma$ almost every $y \in \Delta(w, 4r)$. To argue by contradiction we suppose

that there exists a set $F \subset \Delta(w, 4r)$, $\sigma(F) > 0$, such that if $y \in F$

then the limit of $\nabla u(z)$, as $z \to y$ with $z \in \Gamma(y)$, does not exist.

Assuming (3.6) we let $y \in F$ be a point of density for $F$ with respect to $\sigma$. Then

$$t^{1-n}\sigma(\Delta(y, t) \setminus F) \to 0 \text{ as } t \to 0,$$

so we can conclude that if $t > 0$ is small enough, then

$$\sigma(\partial \hat{\Omega} \cap \Delta(y, t) \cap F) \geq t^{n-1}$$

where $\hat{\Omega} \subset \Omega$ is the starlike Lipschitz domain defined in Lemma 2.11 with $w, \hat{w}, r$ replaced by $y, \hat{y}, t$. Using Lemma 2.11 we also see that $|\nabla u| \approx C$ in $\hat{\Omega}$ for some constant $C$. Let $\hat{L}$ be defined as in (1.13), (1.14), relative to $u, \hat{\Omega}$. Then, from (2.13), (1.15) and the fact $|\nabla u| \approx C$ in $\hat{\Omega}$, we have that $\hat{L}$ is uniformly elliptic on $\hat{\Omega}$ and $Lu_{x_k} = 0$ weakly in $\hat{\Omega}$. Moreover, since $u_{x_k}$ is bounded on $\hat{\Omega}$ for $1 \leq k \leq n$, we can therefore conclude, by well known arguments, see [CFMS], that $u_{x_k}$ has non tangential limits at almost every boundary point of $\hat{\Omega}$ with respect to elliptic measure, $\hat{\omega}(\cdot, \hat{y})$, associated with the operator $\hat{L}$, the domain $\hat{\Omega}$, and the point $\hat{y}$. Now from Lemma 2.15 we see that $\hat{\omega}(\cdot, \hat{y})$ and surface measure, $\hat{\sigma}$, on $\partial \hat{\Omega}$ are mutually absolutely continuous. Hence $u_{x_k}$ has non tangential limits at $\hat{\sigma}$ almost every boundary point. Since non tangential limits in $\hat{\Omega}$ agree with those in $\Omega$, for $\sigma$ almost every point in $F$, we deduce that this latter statement contradicts the assumption made in (3.6) that $\sigma(F) > 0$. Hence $\nabla u$ has non tangential limits for $\sigma$ almost every $y \in \Delta(w, 4r)$.

In the following we let $\nabla u(y), y \in \Delta(w, 2r)$, denote the non tangential limit of $\nabla u$ whenever this limit exists. To prove statement $(ii)$ of Theorem 1 we argue as follows. Let $y \in \Delta(w, 2r)$ and put $\tilde{r} = r/(4c_3)$ where $c_3$ is the constant appearing in the statement of Theorem 2.8. Using Theorem 2.8 we note, to start with, that $B(y, 2\tilde{r}) \cap \{u = t\}$, for $0 < t$ sufficiently small, can be represented as the graph of a Lipschitz function with Lipschitz constant bounded by $c = c(p, n, M)$, $1 \leq c < \infty$. In particular, $c$ can be chosen independently of $t$. In fact we can conclude, see [LN, Lemma 2.4] for the proof, that $u$ is infinitely differentiable and hence that $B(y, 2\tilde{r}) \cap \{u = t\}$ is a $C^\infty$ surface. Let $d\mu_t = |\nabla u|^{p-1}d\sigma_t$ where $\sigma_t$ is surface measure on $B(y, 2\tilde{r}) \cap \{u = t\}$. Using the definition of $\mu$ it is easily seen that $\mu_t$ converges weakly to $\mu$ as defined in Lemma 2.5 on $B(y, 2\tilde{r}) \cap \Omega$. Using the implicit function theorem, we can express $d\sigma_t$ and also $d\mu_t$ locally as measures on $\mathbb{R}^{p-1}$. Doing this, using non tangential convergence of $\nabla u$, Theorem 1 $(i)$, and dominated convergence we see first that

$$k(y) = |\nabla u|(y)$$

and

$$d\mu = |\nabla u|^{p-1}d\sigma.$$
Regularity and free boundary regularity for the \( p \) Laplacian in Lipschitz and \( C^1 \) domains

Theorem 1 (ii) by standard arguments, see [CF]. The proof of Theorem 1 is therefore complete.

3.2. Proof of Theorem 2. Let \( \Omega, M, p, w, r \) and \( u \) be as in the statement of Theorem 1. We prove that there exist \( 0 < \varepsilon_0 \) and \( \hat{r} = \hat{r}(\varepsilon) \), for \( \varepsilon \in (0, \varepsilon_0) \), such that whenever \( y \in \Delta(w, r) \) and \( 0 < s < \hat{r}(\varepsilon) \) then

\[
\int_{\Delta(y, s)} |\nabla u|^p \, d\sigma \leq (1 + \varepsilon) \left( \int_{\Delta(y, s)} |\nabla u|^{p-1} \, d\sigma \right)^{p/(p-1)}.
\]

Here

\[
\int_E f \, d\sigma = (\sigma(E))^{-1} \int_E f \, d\sigma
\]

whenever \( E \subset \partial \Omega \) is Borel measurable with finite positive \( \sigma \) measure and \( f \) is a \( \sigma \) integrable function on \( E \). Theorem 2 then follows, once (3.8) is established, from a lemma of Sarason, see [KT]. To prove (3.8) we argue by contradiction. Indeed, if (3.8) is false then

there exist two sequences \( \{y_m\}_1^\infty, \{s_m\}_1^\infty \) satisfying \( y_m \in \Delta(w, r) \)

and \( s_m \to 0 \) as \( m \to \infty \) such that (3.8) is false with

\( y, s \) replaced by \( y_m, s_m \) for \( m \in \mathbb{Z}_+ = \{1, 2, \ldots \} \).

To continue we first note that using the assumption that \( \Omega \) is \( C^1 \) regular it follows that \( \Delta(w, 2r) \) is Reifenberg flat with vanishing constant. That is, for given \( \hat{\varepsilon} > 0 \), small, there exists a \( \hat{r} = \hat{r}(\hat{\varepsilon}) < 10^{-\hat{\varepsilon}}r \), such that whenever \( y \in \Delta(w, 2r) \) and \( 0 < s \leq \hat{r} \), then

\[
\{ z + tn \in B(y, s), \ z \in P, \ t > \hat{\varepsilon}s \} \subset \Omega,
\]

\[
\{ z - tn \in B(y, s), \ z \in P, \ t > \hat{\varepsilon}s \} \subset \mathbb{R}^n \setminus \bar{\Omega}.
\]

In (3.10) \( P = P(y, s) \) is the tangent plane to \( \Delta(w, 2r) \) relative to \( y, s \), and \( n = n(y) \) is the inner unit normal to \( \partial \Omega \) at \( y \in \Delta(w, 2r) \). We let, for each \( m \in \mathbb{Z}_+ \), \( P(y_m) = P(y_m, s_m) \) denote the tangent plane to \( \Delta(w, 2r) \) relative to \( y_m, s_m \) where \( y_m, s_m \) are as in (3.9).

In the following we let \( A = e^{1/\varepsilon} \) and note that if we choose \( \varepsilon_0 \), and hence \( \varepsilon \), sufficiently small then \( A \) is large. Moreover, for fixed \( A > 10^6 \) we choose \( \hat{\varepsilon} = \hat{\varepsilon}(A) > 0 \) in (3.10) so small that if \( y'_m = y_m + As_m / 4 \), then the domain \( \Omega(y'_m) \), obtained by drawing all line segments from points in \( B(y'_m, As_m/4) \) to points in \( \Delta(y'_m, As_m) \), is starlike Lipschitz with respect to \( y'_m \). We assume, as we may, that \( s_m \leq \hat{r}(\hat{\varepsilon}) \) for \( m \in \mathbb{Z}_+ \) and we put \( D_m = \Omega(y'_m) \setminus \bar{B}(y'_m, As_m/8) \). From \( C^1 \) regularity of \( \Omega \) we also see that \( D_m \), for \( m \in \mathbb{Z}_+ \), has Lipschitz constant \( \leq c \) where \( c \) is an absolute constant. To continue we let \( u_m \) be the \( p \) capacitary function for \( D_m \) and we put \( u_m \equiv 0 \) on \( \mathbb{R}^n \setminus \bar{\Omega}(y'_m) \). From Theorem 2.7 with \( w, r, u_1, u_2 \) replaced by \( y_m, As_m/100, u, u_m \)
we deduce that if \( w_1, w_2 \in \Omega \cap B(y_m, 2s_m) \), then

\[
\left| \log \left( \frac{u_m(w_1)}{u(w_1)} \right) - \log \left( \frac{u_m(w_2)}{u(w_2)} \right) \right| \leq cA^{-\alpha}
\]

whenever \( m \) is large enough. The constants \( c, \alpha \) in (3.11) are the constants in Theorem 2.7 and these constants are independent of \( m \). If we let \( w_1, w_2 \to z_1, z_2 \in \Delta(y_m, 2s_m) \) in (3.11) and use Theorem 1, we get, for \( \sigma \) almost all \( z_1, z_2 \in \Delta(y_m, 2s_m) \), that

\[
\left| \log \left( \frac{\nabla u_m(z_1)}{\nabla u(z_1)} \right) - \log \left( \frac{\nabla u_m(z_2)}{\nabla u(z_2)} \right) \right| \leq cA^{-\alpha}.
\]

Therefore, taking exponentials in the inequality in (3.12) we see that, for \( A \) large enough,

\[
(1 - \tilde{c}A^{-\alpha}) \frac{\nabla u_m(z_1)}{\nabla u_m(z_2)} \leq \frac{\nabla u(z_1)}{\nabla u(z_2)} \leq (1 \pm cA^{-\alpha}) \frac{\nabla u_m(z_1)}{\nabla u_m(z_2)},
\]

whenever \( z_1, z_2 \in \Delta(y_m, 2s_m) \) and where \( \tilde{c} \) depends only on \( p, n \), and the Lipschitz constant for \( \Omega \). Using (3.13) we first obtain that

\[
\frac{\int_{\Delta(y_m, s_m)} |\nabla u_m|^p \, d\sigma}{\left( \int_{\Delta(y_m, s_m)} |\nabla u_m|^{p-1} \, d\sigma \right)^{p/(p-1)}} \geq (1 - cA^{-\alpha}) \frac{\int_{\Delta(y_m, s_m)} |\nabla u|^p \, d\sigma}{\left( \int_{\Delta(y_m, s_m)} |\nabla u|^{p-1} \, d\sigma \right)^{p/(p-1)}}.
\]

Secondly, using the assumption that (3.8) is false and (3.9), we from (3.14) obtain that

\[
\frac{\int_{\Delta(y_m, s_m)} |\nabla u_m|^p \, d\sigma}{\left( \int_{\Delta(y_m, s_m)} |\nabla u_m|^{p-1} \, d\sigma \right)^{p/(p-1)}} \geq (1 - cA^{-\alpha}) (1 + \varepsilon).
\]

Next for \( m \in \mathbb{Z}_+ \), let \( T_m \) be a conformal affine mapping of \( \mathbb{R}^n \) which maps the origin and \( e_n \) onto \( y_m \) and \( y'_m \) respectively and which maps \( W = \{ x \in \mathbb{R}^n : x_n = 0 \} \) onto \( P(y_m) \). \( T_m \) is the composition of a translation, rotation, dilation. Let \( D'_m, u'_m \) be such that \( T_m(D'_m) = D'_m \) and \( u_m(T_m x) = u'_m(x) \) whenever \( x \in D'_m \). Since the \( p \) Laplace equation is invariant under translations, rotations, and dilations, we see
that \( u'_m \) is the \( p \) capacitary function for \( D'_m \). Also, as

\[
\frac{\int_{\partial D'_m \cap B(0,1/A)} |\nabla u'_m|^p \, d\sigma'_m}{\int_{\partial D'_m \cap B(0,1/A)} (|\nabla u'_m|^{p-1})^{p/(p-1)} \, d\sigma'_m} = \frac{\int_{\partial D_m \cap B(0,1/A)} |\nabla u_m|^p \, d\sigma}{\int_{\partial D_m \cap B(0,1/A)} (|\nabla u_m|^{p-1})^{p/(p-1)} \, d\sigma},
\]

where \( \sigma'_m \) is the surface measure on \( \partial D'_m \), we see, using (3.15), that

\[
\int_{\partial D'_m \cap B(0,1/A)} |\nabla u'_m|^p \, d\sigma'_m \geq (1 - cA^{-\alpha})(1 + \varepsilon).
\]

(3.16)

Letting \( m \to \infty \) we see from Lemmas 2.1, 2.2 and 2.3 that \( u'_m \) converges uniformly on \( \mathbb{R}^n \) to \( u' \) where \( u' \) is the \( p \) capacitary function for the starlike Lipschitz ring domain, \( D' = \Omega' \setminus B(e_n,1/4) \). Also \( \Omega' \) is obtained by drawing all line segments connecting points in \( B(0,1) \cap W \) to points in \( B(e_n,1/4) \). We can now repeat, essentially verbatim, the argument in [LN, Lemma 5.28, (5.29)--(5.41)], to conclude that

\[
\limsup_{m \to \infty} \frac{\int_{\partial D'_m \cap B(0,1/A)} |\nabla u'_m|^p \, d\sigma'_m}{\int_{\partial D_m \cap B(0,1/A)} (|\nabla u'_m|^{p-1})^{p/(p-1)} \, d\sigma'_m} \leq \frac{\int_{W \cap B(0,1/A)} |\nabla u'|^p \, dx'}{\int_{W \cap B(0,1/A)} (|\nabla u'|^{p-1})^{p/(p-1)} \, dx'}. \tag{3.17}
\]

Here \( dx' \) denotes surface measure on \( W \). To complete the argument we show that (3.17) leads to a contradiction to our original assumption. Note that it follows from Schwarz reflection that \( u' \) has a \( p \) harmonic extension to \( B(0,1/8) \) with \( u' \equiv 0 \) on \( W \cap B(0,1/8) \). From barrier estimates we have \( c^{-1} \leq |\nabla u'| \leq c \) on \( B(0,1/16) \) where \( c \) depends only on \( p, n \), and from Lemma 2.4 we find that \( |\nabla u'| \) is Hölder continuous with exponent \( \theta = \theta(p,n) \) on \( W \cap B(0,1/16) \). In fact in this case we could take \( \theta = 1 \). Therefore, using these facts we first conclude that, for some \( c \),

\[
(1 - cA^{-\theta}) |\nabla u'(0)| \leq |\nabla u'(z)| \leq (1 + cA^{-\theta}) |\nabla u'(0)|
\]

whenever \( z \in B(0,1/A) \) and then from (3.16), (3.17) that

\[
(1 + cA^{-\theta}) \geq \frac{\int_{W \cap B(0,1/A)} |\nabla u'|^p \, dx'}{(\int_{W \cap B(0,1/A)} (|\nabla u'|^{p-1})^{p/(p-1)} \, dx')} \geq (1 - cA^{-\alpha})(1 + \varepsilon).
\]
As $A = e^{1/\varepsilon}$ the last inequality clearly can not hold if we choose $\varepsilon_0$, and hence $\varepsilon$, sufficiently small. From this contradiction we conclude that our original assumption was false, i.e., (3.9) can not hold. Hence (3.8) holds. This completes the proof of Theorem 2.

\[\square\]

4. Proof of Theorem 3

In this section we prove Theorem 3. Our argument is similar to the argument in [KT2], in that we argue by way of contradiction to get a sequence of blow-ups as in (1.8)–(1.10). We then use a theorem of [ACF] to show that a subsequence of this sequence converges to a linear function which turns out to be a contradiction. However, our argument is less voluminous and seems simpler to us than the one in [KT2]. The following lemma plays a key role in our blow-up argument.

4.1. A refined version of Lemma 2.11.

Lemma 4.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with constant $M$. Given $p, 1 < p < \infty$, $w \in \partial \Omega$, $0 < r < r_0$, suppose that $u$ is a positive $p$ harmonic function in $\Omega \cap B(w, 2r)$, $u$ is continuous in $\Omega \cap B(w, 2r)$ and $u = 0$ on $\Delta(w, 2r)$. Suppose also that $\log |\nabla u| \in VMO(\Delta(w, r))$. Given $\varepsilon > 0$ there exist $\tilde{r} = \tilde{r}(\varepsilon)$, $0 < \tilde{r} < r$, and $c = c(p, n, M)$, $1 \leq c < \infty$, such that the following is true whenever $0 < r' \leq \tilde{r}$. There exists a starlike Lipschitz domain $\tilde{\Omega} \subset \Omega \cap B(w, cr') \subset \Omega \cap B(w, r)$, with center at a point $\tilde{w} \in \Omega \cap B(w, cr')$, $d(\tilde{w}, \partial \Omega) \geq r'$, and with Lipschitz constant bounded by $c$, such that

\[(a) \quad \frac{\sigma(\partial \tilde{\Omega} \cap \Delta(w, r'))}{\sigma(\Delta(w, r'))} \geq 1 - \varepsilon,\]

\[(b) \quad (1 - \varepsilon)|b|^{p-1} \leq \frac{\mu(\Delta(y, s))}{\sigma(\Delta(y, s))} \leq (1 + \varepsilon)|b|^{p-1} \quad \text{whenever} \quad 0 < s < r', \quad y \in \partial \tilde{\Omega} \cap \Delta(w, r').\]

Here $\mu$ is the measure associated with $u$ as in Lemma 2.5 and $\log b$ is the average of $\log |\nabla u|$ on $\Delta(w, 4r')$. Moreover, for all $x \in \Omega$

\[c^{-1} \frac{u(\tilde{w})}{r'} \leq |\nabla u(x)| \leq c \frac{u(\tilde{w})}{r'}.\]

Proof. In the following we let $\tilde{\varepsilon} > 0$ and $r^* (\tilde{\varepsilon}) \ll r$ be small positive numbers. For the moment we allow $\tilde{\varepsilon}$ and $r^*$ to vary but we shall later fix these numbers to satisfy several conditions depending on $\varepsilon$. Using the assumption that $\log |\nabla u| \in VMO(\Delta(w, r))$ we see there exists $\tilde{r}$, $0 < \tilde{r} \leq r^*$, such that $\log |\nabla u| \in BMO(\Delta(w, 8\tilde{r}))$ with $BMO$ norm less than or equal to $\tilde{\varepsilon}^3$. Let $A$ denote the average of $f = \log |\nabla u|$ with respect to surface measure over $\Delta(w, 4\tilde{r})$. Using the definition of $BMO$, see (1.7), we have

\[\tilde{\varepsilon} \sigma(\{ x \in \Delta(w, 4\tilde{r}) : |f(x) - A| > \tilde{\varepsilon} \}) \leq (\sigma(\Delta(w, 4\tilde{r}))^{-1} \int_{\Delta(w, 4\tilde{r})} |f - A| d\sigma \leq c \tilde{\varepsilon}^3.\]
If \( b = e^A \), then from (4.2) we see that there exists a set \( E \subset \Delta(w, 4\hat{r}) \) such that \((1 - c\bar{\varepsilon})b \leq |\nabla u| \leq (1 + c\bar{\varepsilon})b\) on \( E \) and if \( F = \Delta(w, 4\hat{r}) \setminus E \) then \( \sigma(F) \leq c\bar{\varepsilon}^2 \sigma(\Delta(w, 4\hat{r})) \).

In (4.3), \( c \) is a universal constant. We introduce, for \( \sigma \) integrable functions \( h \) defined on \( \Delta(w, 5\hat{r}) \) and for \( x \in \Delta(w, 4\hat{r}) \), the maximal function

\[
M(h)(x) = \sup_{0 < s < \hat{r}} \frac{1}{\sigma(\Delta(x, s))} \int_{\Delta(x, s)} h \, d\sigma.
\]

Let \( G = \{ x \in \Delta(w, 4\hat{r}) : M(\chi_F)(x) \leq \bar{\varepsilon} \} \) where \( \chi_F \) is the indicator functions for the set \( F \) introduced in (4.3) and define \( K = \Delta(w, 4\hat{r}) \setminus G \). Using weak type estimates for the maximal function, see [S], it then follows that

\[
\sigma(K) \leq c\bar{\varepsilon} \sigma(\Delta(w, 4\hat{r})).
\]

Let \( y \in G \cap \Delta(w, \hat{r}), 0 < s \leq \hat{r} \). Then from Lemma 2.10, Theorem 1 and (3.7) we deduce

\[
\mu(\Delta(y, s)) = \int_{\Delta(y,s)} |\nabla u|^{p-1} \, d\sigma = \int_{E \cap \Delta(y,s)} |\nabla u|^{p-1} \, d\sigma + \int_{F \cap \Delta(y,s)} |\nabla u|^{p-1} \, d\sigma = T_1 + T_2.
\]

From the definitions of the sets \( E, F, G \), we see that

\[
(1 - c\bar{\varepsilon}) b^{p-1} \sigma(\Delta(y, s)) \leq T_1 \leq (1 + c\bar{\varepsilon}) b^{p-1} \sigma(\Delta(y, s)),
\]

for some \( c = c(p, n, M) \), provided \( \bar{\varepsilon} \) is sufficiently small. Also from Hölder’s inequality,

\[
(\sigma(\Delta(y, s)))^{-1} T_2 \leq \left( \frac{1}{\sigma(\Delta(y, s))} \int_{\Delta(y,s)} |\nabla u|^p \, d\sigma \right)^{(p-1)/p} \left( \frac{\sigma(F \cap \Delta(y, s))}{\sigma(\Delta(y, s))} \right)^{1/p}.
\]

Using \( y \in G \) and the reverse Hölder inequality for \( |\nabla u| \) in Theorem 1 we get from (4.7) that

\[
T_2 \leq c\bar{\varepsilon}^{3/p} \mu(\Delta(y, s)).
\]

Using (4.6) and (4.8) in (4.5), we obtain that

\[
(1 - c\bar{\varepsilon}^{3/p}) b^{p-1} \leq \frac{\mu(\Delta(y, s))}{\sigma(\Delta(y, s))} \leq (1 + c\bar{\varepsilon}^{3/p}) b^{p-1}.
\]

To construct \( \Omega \) we assume, as we may, that

\[
\Omega \cap B(w, 4r) = \{(x', x_n) : x_n > \phi(x') \} \cap B(w, 4r),
\]

\[
\partial \Omega \cap B(w, 4r) = \{(x', x_n) : x_n = \phi(x') \} \cap B(w, 4r),
\]

where \( \phi : \mathbb{R}^{n-1} \to \mathbb{R} \) is Lipschitz with \( \| \nabla \phi \|_{\infty} \leq M \). Let \( r' = \hat{r}/c \) and \( \hat{w} = w + \frac{1}{4} \hat{r} e_n \).

Let \( \hat{\Omega} \) be the domain obtained from drawing all open line segments from points in
$B(\hat{w}, r')$ to points in $\Delta(w, r') \cap G$. If $c$ is large enough and $\hat{r}$ small enough, it follows from Lipschitzness of $\Omega$ and elementary geometry that $\hat{\Omega} \subset \Omega$ is a starlike Lipschitz domain with center at $\hat{w}$ and Lipschitz constant $\hat{M} = \hat{M}(M)$. Now from (4.9) we see that if $\hat{\varepsilon} = (\varepsilon/c)^p$ and $\hat{r}(\varepsilon) = r^*\hat{\varepsilon}$, then (b) of Lemma 4.1 is valid. Also, (a) is an obvious consequence of (4.4) as $r' = \hat{r}/c$.

To prove the last display in Lemma 4.1 we first note from Theorem 2.8 that

\begin{equation}
\frac{c^{-1}}{d(x, \partial\Omega)} \leq |\nabla u(x)| \leq \frac{c}{d(x, \partial\Omega)}
\end{equation}

whenever $x \in \Omega \cap B(w, r/c)$. Second we note that if $x \in \hat{\Omega}$, there exists $y \in G$ with $d(x, \partial\Omega) \approx |x - y|$. If $s = |x - y|$, then from (4.11), the definition of the set $G$, Lemma 4.1 (b), Harnack’s inequality, and Lemma 2.5 we find that

\begin{equation}
\frac{\mu(\Delta(y, s))}{\sigma(\Delta(y, s))} \approx \left(\frac{u(x)}{d(x, \partial\Omega)}\right)^{p-1} \approx |\nabla u(x)|^{p-1}.
\end{equation}

From (4.12) and the fact that $\hat{w} \in \hat{\Omega}$ we obtain the last display in Lemma 4.1. The proof of Lemma 4.1 is now complete. \hfill \square

4.2. The blow-up argument. To begin the blow-up argument in the proof of Theorem 3 we first let

$$D(F_1, F_2) = \max\{\sup\{d(x, F_2) : x \in F_1\}, \sup\{d(y, F_1) : y \in F_2\}\}$$

be the Hausdorff distance between the sets $F_1, F_2 \subset \mathbb{R}^n$. Second, recall from section 1 that to prove Theorem 3 it suffices to obtain a contradiction to the assumption that

\begin{equation}
\eta = \lim_{r \to 0} \sup_{\hat{w} \in \Delta(w, r/2)} ||n||_{BMO(\Delta(\hat{w}, \hat{r}))} \neq 0
\end{equation}

where $n$ is the outer unit normal to $\Omega$. Moreover if (4.13) is false then there exist sequences, see the discussion after (1.8), $\{w_j\}, w_j \in \Delta(w, r/2)$, and $\{r_j\}, r_j \to 0$, such that

\begin{equation}
\eta = \lim_{j \to \infty} \left(\frac{1}{\sigma(\Delta(w_j, r_j))} \int_{\Delta(w_j, r_j)} |n - n_{\Delta(w_j, r_j)}|^2 d\sigma\right)^{1/2}
\end{equation}

where $n_{\Delta(w_j, r_j)}$ denotes the average of $n$ on $\Delta(w_j, r_j)$ with respect to $\sigma$. Let $\Omega \cap B(w, 4r)$ be as in (4.10) and let $u$ be as in Theorem 3. Extend $u$ to $B(w, 4r)$ by putting $u = 0$ in $B(w, 4r) \setminus \Omega$. Let $T_j(z) = w_j + r_j z$ and as in (1.9) we put, for $j = 1, 2, \ldots$,

\begin{equation}
\begin{align*}
\Omega_j &= T_j^{-1}(\Omega \cap B(w, 4r)) = \{r_j^{-1}(x - w_j) : x \in \Omega \cap B(w, 4r)\}, \\
u_j(z) &= \lambda_j u(T_j(z)) \text{ whenever } z \in T_j^{-1}(B(w, 4r)).
\end{align*}
\end{equation}

The sequence $\{\lambda_j\}$ used in (4.15) will be defined in (4.21) below. From translation and dilation invariance of the $p$ Laplace equation we see that $u_j$ is $p$ harmonic in $\Omega_j$
and continuous in $T_{j}^{-1}(B(w, 4r))$ with $u_{j} \equiv 0$ in $T_{j}^{-1}(B(w, 4r) \setminus \Omega)$. Also, we note, for $j = 1, 2, \ldots$, that

\begin{align}
\Omega_{j} &= \{(y', y_n): y_n > \psi_{j}(y')\} \cap T_{j}^{-1}(B(w, 4r)), \\
\partial \Omega_{j} &= \{(y', y_n): y_n = \psi_{j}(y')\} \cap T_{j}^{-1}(B(w, 4r)),
\end{align}

where if $w_{j} = (w_{j}'(w_{j})_{n})$, then

\begin{equation}
\psi_{j}(y') = r_{j}^{-1}[\phi(r_{j}y' + w_{j}') - (w_{j})_{n}] \text{ whenever } y' \in \mathbb{R}^{n}.
\end{equation}

Clearly, $\psi_{j}$ is Lipschitz with

\begin{equation}
\psi_{j}(0) = 0 \text{ and } ||\nabla \psi_{j}||_{\infty} = ||\nabla \phi||_{\infty} \leq M \text{ for } j = 1, 2, \ldots.
\end{equation}

Let $\mu, \mu_{j}$ be the measures associated with $u, u_{j}$ as in Lemma 2.5 and let $\sigma, \sigma_{j}$ be the surface measures on $\partial \Omega$ and $\partial \Omega_{j}$ respectively. From (4.16)–(4.18) and the definition of $\sigma_{j}$, we see that if $H_{j}$ is a Borel subset of $\partial \Omega_{j}$, then

\begin{equation}
\sigma_{j}(H_{j}) = r_{j}^{-1-n} \sigma(T_{j}(H_{j})), \quad \mu_{j}(H_{j}) = \lambda_{j}^{-1}r_{j}^{-p-n} \mu(T_{j}(H_{j})).
\end{equation}

We assume as we may that $2^{j}r_{j} \to 0$ as $j \to \infty$. We now apply Lemma 4.1 to $u$ with $w, r'$ replaced by $w_{j}, 2^{j}r_{j}$ and with $\varepsilon = 2^{-2j^{2}}$. Then for $j$ large enough there exists a starlike Lipschitz domain $\tilde{\Omega} = \tilde{\Omega}(j) \subset \Omega \cap B(w_{j}, 2^{j}r_{j})$, with Lipschitz constant $\tilde{M} = M(M)$ and center at $\tilde{w}_{j}$, such that $d(\tilde{w}_{j}, \partial \Omega) \approx 2^{j}r_{j}$ and such that

\begin{equation}
\frac{\sigma(\partial \Omega \cap \Delta(w_{j}, 2^{j}r_{j}))}{\sigma(\Delta(w_{j}, 2^{j}r_{j}))} \geq 1 - 2^{-2j^{2}},
\end{equation}

\begin{equation}
(1 - 2^{-2j^{2}})b_{j}^{p-1} \leq \frac{\mu_{j}(\Delta(y, s))}{\sigma(\Delta(y, s))} \leq (1 + 2^{-2j^{2}})b_{j}^{p-1} \text{ whenever } 0 < s < 2^{j}r_{j}
\end{equation}

and $y \in \partial \tilde{\Omega} \cap \Delta(w, 2^{j}r_{j})$,

\begin{equation}
\frac{1}{2^{j}r_{j}} \frac{\mu_{j}(\tilde{w}_{j})}{\sigma_{j}(\Delta(w, 2^{j}r_{j}))} \leq |\nabla u(x)| \leq \frac{c}{2^{j}r_{j}} \frac{\mu_{j}(\tilde{w}_{j})}{\sigma_{j}(\Delta(w, 2^{j}r_{j}))} \text{ whenever } x \in \tilde{\Omega}.
\end{equation}

In (4.20) (b'), $\log b_{j}$ denotes the average of $\log |\nabla u|$ on $\Delta(w_{j}, 2^{j+2}r_{j})$ with respect to $\sigma$. From (4.15), (4.19) and (4.20) we see that if

\begin{equation}
\lambda_{j} = (r_{j}b_{j})^{-1}, \quad \Omega_{j} = T_{j}^{-1}(\tilde{\Omega}(j)), \quad \zeta_{j} = T_{j}^{-1}(\tilde{w}_{j}),
\end{equation}

then $\Omega_{j} \subset \Omega \cap B(0, 2^{j})$ is a starlike Lipschitz domain with center at $\zeta_{j}$ and Lipschitz constant $\tilde{M} = M(M)$. Moreover, $d(\zeta_{j}, \partial \Omega_{j}) \approx 2^{j}$ and

\begin{equation}
\frac{\sigma_{j}(\partial \Omega_{j} \cap \partial \Omega_{j} \cap B(0, 2^{j}))}{\sigma_{j}(\partial \Omega_{j} \cap B(0, 2^{j}))} \geq 1 - 2^{-2j^{2}},
\end{equation}

\begin{equation}
(\beta) \quad (1 - 2^{-2j^{2}}) \frac{\mu_{j}(\partial \Omega_{j} \cap B(z, s))}{\sigma_{j}(\partial \Omega_{j} \cap B(z, s))} \leq (1 + 2^{-2j^{2}}) \text{ whenever } 0 < s < 2^{j}
\end{equation}

and $z \in \partial \Omega_{j} \cap \partial \Omega_{j}$,

\begin{equation}
(\gamma) \quad c^{-1} \leq |\nabla u_{j}(x)| \leq c \text{ whenever } x \in \Omega_{j}.
\end{equation}
In fact, (4.22) (α), (β) are straightforward consequences of (4.20) (a’), (b’) and (4.21). (4.22) (γ) follows from (4.20) (c’), (4.22) (β), and the fact that by Lemma 2.5,
\[
\frac{\mu_j(\partial \Omega_j \cap B(0, 2^j))}{\sigma_j(\partial \Omega_j \cap B(0, 2^j))} \approx \left( \frac{u_j(\zeta_j)}{2^j} \right)^{p-1}.
\]

Let \( \hat{\sigma}_j \) denote the surface measure on \( \partial \Omega_j \). We next show that the following holds for \( j \) large enough,
\[
(\hat{\alpha}) \quad \hat{\sigma}_j ((\partial \Omega_j \setminus \partial \Omega_j) \cap B(0, 2^{j/2})) \leq c2^{-j^2},
\]
\[
(\hat{\beta}) \quad D(\partial \Omega_j \cap B(0, 2^{j/2}), \partial \Omega_j \cap B(0, 2^{j/2})) \leq c2^{-j^2/(n-1)}.
\]

To prove (4.23) we observe from (4.22) (α) that for large \( j \),
\[
(4.24) \quad d(x, \partial \Omega_j) \leq 2^{-3j^2/(2(n-1))} \quad \text{whenever} \quad x \in \partial \Omega_j \cap B(0, 2^{j/2}).
\]

In fact, if the statement in (4.24) is false then there exists \( x \in \partial \Omega_j \cap B(0, 2^{j/2}) \) such that \( B(x, 2^{-3j^2/(2(n-1))}) \cap \partial \Omega_j = \emptyset \) and such that
\[
\frac{\sigma_j(\partial \Omega_j \cap B(0, 2^j))}{\sigma_j(\partial \Omega_j \cap B(0, 2^j))} \leq \left( 1 - c2^{-(j(n-1)+3j^2/2)} \right).
\]

As \( 1 - c2^{-(j(n-1)+3j^2/2)} < 1 - 2^{-j^2} \) if \( j \) is large enough the statement in the last display contradicts (4.22) (α) and hence (4.24) must hold. Moreover, if \( x \in (\partial \Omega_j \setminus \partial \Omega_j) \cap B(0, 2^{j/2}) \), then we can project \( x \) onto \( x^* \in \partial \Omega_j \) by way of radial projection from \( \zeta_j \). From the construction of \( O_j \) and (4.22) (α) we again see for large \( j \) that
\[
d(x, \partial \Omega_j) \approx d(x^*, \partial \Omega_j \cap \partial \Omega_j) \leq 2^{-3j^2/(2(n-1))}.
\]

Thus using the inequality in the last display and (4.24) we see that (4.23) (\( \hat{\alpha} \)) also follows from this inequality and a covering argument.

From (4.18) and a standard compactness argument we see there exists a subsequence \( \{ \psi_j' \} \) of \( \{ \psi_j \} \) with \( \psi_j' \to \phi_\infty \) uniformly on compact subsets of \( \mathbb{R}^{n-1} \) where \( \phi_\infty \) is Lipschitz and
\[
(\ast) \quad ||\nabla \phi_\infty||_\infty \leq M \quad \text{and} \quad \phi_\infty(0) = 0,
\]
\[
(\ast\ast) \quad \int_{\mathbb{R}^{n-1}} \frac{\partial \psi_j'}{\partial x_i} f \, dx' \to \int_{\mathbb{R}^{n-1}} \frac{\partial \phi_\infty}{\partial x_i} f \, dx' \quad \text{as} \quad j \to \infty \quad \text{for} \quad 1 \leq i \leq n
\]
and \( f \in C_0^\infty(\mathbb{R}^{n-1}) \).

Let \( \Omega'_j = \{ x \in \mathbb{R}^n : x_n > \psi_j'(x') \} \), \( \Omega_\infty = \{ x \in \mathbb{R}^n : x_n > \phi_\infty(x') \} \), and let \( n_j', \sigma_j' \) and \( n_{\infty}, \sigma_\infty \) denote, respectively, the outer unit normal and the surface measure to \( \partial \Omega_j' \) and \( \partial \Omega_\infty \). From (4.25) we find that
\[
(\ast) \quad D(\partial \Omega_j' \cap B(0, R), \partial \Omega_\infty \cap B(0, R)) \to 0 \quad \text{as} \quad j \to \infty \quad \text{for each} \quad R > 0,
\]
\[
(\ast\ast) \quad \int_{\mathbb{R}^{n-1}} \langle n_j, F \rangle \, ds' \to \int_{\mathbb{R}^{n-1}} \langle n, F \rangle \, ds_\infty \quad \text{as} \quad j \to \infty \quad \text{whenever} \quad F = (F_1, \ldots, F_n)
\]
with \( F_i \in C_0^\infty(\mathbb{R}^n) \) for \( 1 \leq i \leq n \).
In the last inequality we have used the fact that if \( y = (y', \psi_j(y')) \in \partial \Omega_j' \cap B(0, 2^j) \), then
\[
n_j'(y) \, d\sigma_j(y) = (\nabla \psi_j(y'), -1).
\]
(4.26) and measure theoretic type arguments imply
\[
\int_{\partial \Omega_{\infty}} f \, d\sigma \leq \liminf_{j \to \infty} \int_{\partial \Omega_j'} f \, d\sigma_j \quad \text{whenever } f \geq 0 \in C_0^\infty(\mathbb{R}^n).
\]
(4.27)

Let \( \{u'_j\}, \{\mu'_j\} \) be subsequences of \( \{u_j\}, \{\mu_j\} \), corresponding to \( (\Omega'_j) \). Then from Lemmas 2.1–2.5 applied to \( u'_j \) and (4.22) \((\beta)\) we deduce that \( u'_j \) is bounded, Hölder continuous, and locally in \( W^{1,p} \) on compact subsets of \( \mathbb{R}^n \) with norms of all functions bounded above by constants which are independent of \( j \). Also, if \( B(x, 2\rho) \subset \Omega_{\infty} \), then for large \( j \) we see from (4.23)(\( \beta \)) and Lemma 2.4 that \( \nabla u'_j \) is Hölder continuous and bounded on \( B(x, \rho) \) with constants independent of \( j \). Thus we assume, as we may, that \( \{u'_j\} \) converges uniformly and weakly in \( W^{1,p} \) on compact subsets of \( \mathbb{R}^n \) to \( u_{\infty} \) and that \( \{\nabla u'_j\} \) converges uniformly to \( \nabla u_{\infty} \) on compact subsets of \( \Omega_{\infty} \). Also, \( u_{\infty} \geq 0 \) is \( p \) harmonic in \( \Omega_{\infty} \) and continuous on \( \mathbb{R}^n \), with \( u_{\infty} \equiv 0 \) on \( \mathbb{R}^n \setminus \Omega_{\infty} \).

Furthermore, if \( \mu_{\infty} \) denotes the measure associated with \( u_{\infty} \) as in Lemma 2.5 and \( f \in C_0^\infty(\mathbb{R}^n) \), then
\[
- \int_{\mathbb{R}^n} f \, d\mu_{\infty} = \int_{\mathbb{R}^n} |\nabla u_{\infty}|^{p-2} \langle \nabla u_{\infty}, \nabla f \rangle \, dx
\]
(4.28)
\[
= \lim_{j \to \infty} \int_{\mathbb{R}^n} |\nabla u'_j|^{p-2} \langle \nabla u'_j, \nabla f \rangle \, dx
= - \lim_{j \to \infty} \int_{\mathbb{R}^n} f \, d\mu'_j.
\]

Thus \( \{\mu'_j\} \) converges weakly to \( \mu_{\infty} \).

Next we show that
\[
\sigma_{\infty} \leq \mu_{\infty}.
\]
(4.29)

To do this we first observe from Theorem 1 and (3.7) that \( d\mu'_j = |\nabla u'_j|^{p-1} \, d\sigma'_j \) on \( \partial \Omega_j' \). Using this inequality, (4.22) \((\beta)\), and differentiation theory we see that
\[
1 - 2^{-2j^2} \leq |\nabla u'_j| \leq 1 + 2^{-2j^2}
\]
(4.30)
\[\sigma'_j \text{ almost everywhere on } \partial \Omega_j' \cap \partial O_j' \cap B(0, 2^j), \]
where \( \{O_j'\} \) is the subsequence of \( \{O_j\} \) corresponding to \( \{\Omega'_j\} \). Let \( f \in C_0^\infty(\mathbb{R}^n) \) and \( f \geq 0 \). From (4.28), (4.27), (4.30), and (4.22) \((\alpha)\) we find that
\[
\int_{\partial \Omega_{\infty}} f \, d\sigma_{\infty} = \lim_{j \to \infty} \int_{\partial \Omega_j'} f |\nabla u'_j|^{p-1} \, d\sigma'_j \geq \liminf_{j \to \infty} \int_{\partial \Omega_j' \cap \partial \Omega_j'} f |\nabla u'_j|^{p-1} \, d\sigma'_j
\]
\[
\geq \liminf_{j \to \infty} (1 - 2^{-2j}) \int_{\partial \Omega_j' \cap \partial \Omega_j'} f \, d\sigma'_j = \liminf_{j \to \infty} \int_{\partial \Omega_j' \cap \partial \Omega_j'} f \, d\sigma'_j \geq \int_{\partial \Omega_{\infty}} f \, d\sigma_{\infty}.
\]
Thus (4.29) is true. We claim that
\[(4.31) \quad c^{-1} \leq |\nabla u_\infty| \leq 1 \text{ on } \Omega_\infty.\]
We note that once (4.31) is proved we get from Theorem 1 and (3.7) that
\[d\mu_\infty = |\nabla u_\infty|^{p-1} d\sigma_\infty \leq d\sigma_\infty.\]
From this inequality and (4.29) we conclude
\[(4.32) \quad \sigma_\infty = \mu_\infty.\]
To prove (4.31) let \(x \in \Omega_\infty\) and suppose that \(j\) is so large that \(|x| \leq 2^{j/4}\) and \(d(x, \partial O_j') \geq \frac{1}{2}d(x, \partial \Omega_\infty)\). The last assumption is permissible as we see from (4.23) and (4.26) (+). Let \(\xi \in \partial B(0,1)\) and for fixed \(j\) we set \(v = (\nabla u_j', \xi)\). Let \(\omega_j'(. , x)\) denote elliptic measure at \(x \in O_j'\) with respect to the operator \(L\) in (1.13), where \(u\) in (1.14) is replaced by \(u_j'\). From (1.15) and (4.22) (\(\gamma\)) we see that
\[(4.33) \quad |v| \leq c \text{ and } Lv \equiv 0 \text{ weakly in } O_j'.\]
Let \(\tilde{\sigma}_j'\) be surface measure on \(\partial O_j'\). Using Lemma 2.15 and Harnack’s inequality for the operator \(L\) we see that \(\tilde{\sigma}_j'\) and \(\omega_j'(x, \cdot)\) are mutually absolutely continuous. Hence, arguing as in [CFMS] we get that \(v\) has non-tangential limits \(\tilde{\sigma}_j'\) almost everywhere on \(\partial O_j'\). Moreover, \(v\) can be interpreted as the ‘Poisson integral’ of its boundary values. Using these facts, (4.33), (4.22) (\(\alpha\)) and the maximum principle for the operator \(L\), we deduce that
\[(4.34) \quad |v(x)| \leq (1 + 2^{-2j^2})T_1(x) + c(T_2(x) + T_3(x))\]
where
\[T_1(x) = \omega_j'(\partial O_j' \cap \partial O_j' \cap B(0, 2^{j/2}), x)\]
\[T_2(x) = \omega_j'((\partial O_j' \setminus \partial O_j') \cap B(0, 2^{j/2}), x)\]
\[T_3(x) = \omega_j'(\partial O_j' \setminus B(0, 2^{j/2}), x).\]
Next we estimate \(T_1(x)\), \(T_2(x)\) and \(T_3(x)\) for \(|x| \leq 2^{j/4}\). In particular, using (2.14) we see that if \(|x| \leq 2^{j/4}\) then
\[(4.35) \quad T_3(x) \leq c 2^{-j^4/4}\]
where \(c \geq 1, 0 < \tau \leq 1\), depend only on \(p, n, M\). Also from Lemma 2.15 and (4.23) (\(\tilde{\alpha}\)) we obtain
\[(4.36) \quad T_2(\zeta_j') \leq c \left( \frac{\sigma_j'((\partial O_j' \setminus \partial O_j') \cap B(0, 2^{j/2}))}{\sigma_j'(\partial O_j' \cap B(0, 2^{j/2}))} \right)^{\gamma} \leq c 2^{-\gamma j^2/2}\]
for \(j\) large enough. Here \(\zeta_j'\) is the center of \(O_j'\). Moreover, using Harnack’s inequality for the operator \(L\) and the fact that \(d(\zeta_j', \partial O_j') \approx 2^j\) we see there exist \(c \geq 1\) and \(\kappa \geq 1\), depending only on \(p, n, M\), such that
\[T_2(x) \leq cT_2(\zeta_j') (2^j/d(x, \partial \Omega_\infty))^{\kappa}\]
provided \( j \) is large enough. In view of this inequality and (4.36) we can conclude that
\[
T_2(x) \leq 2^{-\gamma \delta^2/4} d(x, \partial \Omega_\infty)^{-\kappa}
\]
for large \( j \). Using (4.34), the fact that \( T_1 \leq 1 \), (4.37) as well as (4.35) we find, by taking limits, that
\[
|\langle \nabla u_\infty, \xi \rangle|(x) = \lim_{j \to \infty} |\langle \nabla u_j', \xi \rangle|(x) \leq 1.
\]
Since \( x \in \Omega_\infty \) and \( \xi \in \partial B(0,1) \) are arbitrary, we conclude that the right-hand inequality in (4.31) is true. The left-hand inequality in (4.31) follows from (4.22) \((\gamma)\) and the fact that \( \{\nabla u_j'\} \) converges to \( \nabla u_\infty \) uniformly on compact subsets of \( \Omega_\infty \).

4.3. The final proof. For those well versed in [ACF] we can now rapidly obtain a contradiction to (4.14) and thus prove Theorem 3. Indeed from (4.31), (4.32), (4.25) \((\ast)\), and [ACF] it follows, for \( \hat{M} \) small enough, that if \( M \leq \hat{M} \) then
\[
u_j = \langle x, \nu \rangle \quad \text{and} \quad \Omega_\infty = \{ x \in \mathbb{R}^n: \langle x, \nu \rangle > 0 \} \quad \text{for some} \quad \nu \in \partial B(0,1).
\]
Using (4.38) and (4.26) \((+++)\) we see that
\[
\lim_{j \to \infty} \int_{\partial \Omega_j' \cap B(0,1)} \langle n_j', \nu \rangle \, d\sigma_j' = -\sigma_\infty(\partial \Omega_\infty \cap B(0,1)).
\]
Also from (4.31), (4.22), and the fact that \( d\sigma_\infty = d\mu_\infty \), see (4.32), we obtain for \( f \geq 0 \) and \( f \in C_0^\infty(\mathbb{R}^n) \), as in the argument leading to (4.29),
\[
\int f \, d\sigma_\infty = \lim_{j \to \infty} \int_{\partial \Omega_j'} f |\nabla u_j'|^{p-1} \, d\sigma_j' \geq \limsup_{j \to \infty} \int_{\partial \Omega_j' \cap \partial \Omega_j'} f |\nabla u_j'|^{p-1} \, d\sigma_j'
\]
\[
\geq \limsup_{j \to \infty} (1 - 2^{-\gamma \delta^2}) \int f \, d\sigma_j' = \limsup_{j \to \infty} \int f \, d\sigma_j'.
\]
Combining (4.40) and (4.27) we see that
\[
\sigma_j' \to \sigma_\infty \text{ weakly as } j \to \infty.
\]
Finally, let \( a_j' \) denote the average of \( n_j' \) on \( \partial \Omega_j' \cap B(0,1) \) with respect to \( \sigma_j' \). From (4.41) and (4.26) \((+++)\) we deduce that \( a_j \to \nu \) as \( j \to \infty \). Using this fact, (4.41), (4.39), the fact that (4.14) is scale invariant, and the triangle inequality, we get
\[
0 < \eta = \lim_{j \to \infty} \left( \frac{1}{\sigma_j'(\partial \Omega_j' \cap B(0,1))} \int_{\partial \Omega_j' \cap B(0,1)} |n_j' - a_j'|^2 \, d\sigma_j' \right)^{1/2}
\]
\[
\leq \limsup_{j \to \infty} \left( \frac{1}{\sigma_j'(\partial \Omega_j' \cap B(0,1))} \int_{\partial \Omega_j' \cap B(0,1)} |n_j' + \nu|^2 \, d\sigma_j' \right)^{1/2} + \lim_{j \to \infty} |a_j + \nu|
\]

\[
\leq \limsup_{j \to \infty} \left( \frac{1}{\sigma_j'(\partial \Omega_j' \cap B(0,1))} \int_{\partial \Omega_j' \cap B(0,1)} |n_j' + \nu|^2 \, d\sigma_j' \right)^{1/2} + |\lim_{j \to \infty} a_j + \nu|
\]

\[
\leq \limsup_{j \to \infty} \left( \frac{1}{\sigma_j'(\partial \Omega_j' \cap B(0,1))} \int_{\partial \Omega_j' \cap B(0,1)} |n_j' + \nu|^2 \, d\sigma_j' \right)^{1/2} + |\lim_{j \to \infty} a_j + \nu|
\]

\[
\leq \limsup_{j \to \infty} \left( \frac{1}{\sigma_j'(\partial \Omega_j' \cap B(0,1))} \int_{\partial \Omega_j' \cap B(0,1)} |n_j' + \nu|^2 \, d\sigma_j' \right)^{1/2} + |\lim_{j \to \infty} a_j + \nu|
\]

\[
\leq \limsup_{j \to \infty} \left( \frac{1}{\sigma_j'(\partial \Omega_j' \cap B(0,1))} \int_{\partial \Omega_j' \cap B(0,1)} |n_j' + \nu|^2 \, d\sigma_j' \right)^{1/2} + |\lim_{j \to \infty} a_j + \nu|
\]

\[
\leq \limsup_{j \to \infty} \left( \frac{1}{\sigma_j'(\partial \Omega_j' \cap B(0,1))} \int_{\partial \Omega_j' \cap B(0,1)} |n_j' + \nu|^2 \, d\sigma_j' \right)^{1/2} + |\lim_{j \to \infty} a_j + \nu|
\]

\[
\leq \limsup_{j \to \infty} \left( \frac{1}{\sigma_j'(\partial \Omega_j' \cap B(0,1))} \int_{\partial \Omega_j' \cap B(0,1)} |n_j' + \nu|^2 \, d\sigma_j' \right)^{1/2} + |\lim_{j \to \infty} a_j + \nu|
\]

\[
\leq \limsup_{j \to \infty} \left( \frac{1}{\sigma_j'(\partial \Omega_j' \cap B(0,1))} \int_{\partial \Omega_j' \cap B(0,1)} |n_j' + \nu|^2 \, d\sigma_j' \right)^{1/2} + |\lim_{j \to \infty} a_j + \nu|
\]
\[
= \limsup_{j \to \infty} \left( \frac{1}{\sigma_j' (\partial \Omega_j' \cap B(0,1))} \int_{\partial \Omega_j' \cap B(0,1)} 2(1 + \langle n_j', \nu \rangle) \, d\sigma_j' \right)^{1/2} = 0.
\]

We have therefore reached a contradiction and thus Theorem 3 is true. \hfill \Box

For the reader not so well versed in [ACF] we outline the proof of (4.38). First we remark that from (4.31) it follows (see [LN, Lemma 2.4]) that \( u_\infty \) is infinitely differentiable in \( \Omega_\infty \). Using this fact and (4.31) once again it is easily checked that the argument in sections 5 and 6 of [ACF] applies to \( u_\infty \). To briefly outline these sections in our situation we need a definition.

**Definition 4.43.** Let \( 0 \leq \sigma_+, \sigma_- \leq 1, \xi \in \partial B(0,1) \) and \( \lambda \in (0,1) \). For fixed \( p, 1 < p < \infty \), we say that \( u \) belongs to the class \( F(\sigma_+, \sigma_-, R, \xi, \lambda) \), \( 0 < R \), if the following conditions are fulfilled,

(i) \( u(x) \geq \langle x, \xi \rangle - \sigma_+ R \) whenever \( x \in B(0,R) \) and \( \langle x, \xi \rangle \geq \sigma_+ R \),

(ii) \( u(x) = 0 \) whenever \( x \in B(0,R) \) and \( \langle x, \xi \rangle \leq \sigma_- R \),

(iii) \( \lambda \leq |\nabla u(x)| \leq 1 \) whenever \( x \in \Omega_\infty \cap B(0,R) \),

(iv) \( u \geq 0 \) is \( p \) harmonic in \( \{ u > 0 \} \cap B(0,R) \) and continuous in \( B(0,R) \).

From (4.31), (4.32), one can deduce, as in the proof Theorem 5.1 and Lemma 5.6 in [ACF] (see also Lemma 7.2 and Lemma 7.9 in [AC]), that the following two lemmas hold.

**Lemma 4.44.** There exist constants \( 0 < \sigma_1 \) and \( 0 < c_1 \) such that if \( 0 < \sigma \leq \sigma_1 \) and if \( u_\infty \in F(1, \sigma, R, \xi, \lambda) \) then \( u_\infty \in F(c_1 \sigma, 2\sigma, R/2, \xi, \lambda) \).

**Lemma 4.45.** Given \( \theta \in (0,1) \) there exist constants \( 0 < \sigma_2 = \sigma_2(\theta) \) and \( \beta = \beta(\theta) \in (0,1) \) such that if \( 0 < \sigma \leq \sigma_2 \) and if \( u_\infty \in F(\sigma, \sigma, R, \xi, \lambda) \) then \( u_\infty \in F(1, \theta \sigma, \beta R, \xi, \lambda) \) for some \( \xi \in \partial B(0,1) \) with \( |\xi - \xi| \leq c \sigma \).

In the following we let \( \hat{\theta} \in (0,1/2) \) be a constant to be chosen. Let \( \delta = \sigma_2(\hat{\theta}) \) where \( \sigma_2 \) is as in Lemma 4.45. Note from (4.25)(*) and (4.31), that there exists \( M = M(\hat{\delta}) \) such that \( |\xi_0 - \epsilon_n|, M \leq M, \) and \( \lambda = c^{-1}, c \) as in (4.31), then \( u_\infty \in F(\delta, \hat{\theta} \delta, \beta(\hat{\theta}) R, \xi_0, \lambda) \) for any \( R > 0 \). We can now apply Lemma 4.45 to conclude that \( u_\infty \in F(1, \hat{\theta} \delta, \beta(\hat{\theta}) R, \xi_1, \lambda) \) where \( |\xi_0 - \xi_1| \leq c \delta \). Subsequently using Lemma 4.44 we also see that \( u_\infty \in F(c_1 \hat{\theta} \delta, 2\hat{\delta} \beta, \beta(\hat{\theta}) R/2, \xi_1, \lambda) \). We let \( \theta = \max\{ c_1 \hat{\theta}, 2\hat{\theta} \} \) and choose \( \hat{\theta} \in (0,1/2) \) so small that \( \theta < 1 \). We also let \( \beta = \beta(\hat{\theta})/2 \). Based on this we can conclude that if \( u_\infty \in F(\delta, \delta, R, \xi_0, \lambda) \) then \( u_\infty \in F(\theta \delta, \theta \delta, \beta R, \xi_1, \lambda) \) and \( |\xi_0 - \xi_1| \leq c \delta \). By iteration we see that,

\[
(4.46) \quad u_\infty \in F(\theta^m \delta, \theta^m \delta, \beta^m R, \xi_m, \lambda) \quad \text{and} \quad |\xi_m - \xi_{m-1}| \leq c \theta^m \delta \quad \text{for} \quad m = 1, 2, \ldots.
\]

If we let \( R = m \beta^{-m} \) for a fixed positive integer \( m \), then we note from (4.46) that if \( x \in \partial \Omega_\infty \cap B(0,m) \), then

\[
(4.47) \quad |\langle x, \xi_m \rangle| \leq c \theta^m \delta.
\]
Letting $m \to \infty$ in (4.47) we see that (4.38) is valid, where $\nu$ is the limit of a certain subsequence of $\{\xi_m\}$.

5. Closing remarks

As noted in section 1, in a future paper, we shall prove Theorems 1–3 in the setting of Reifenberg flat chord arc domains and thus carry out the full program in [KT], [KT1], [KT2] when $1 < p < \infty$, $p \neq 2$. We also plan to study and remove the smallness assumption in Theorem 3 on $M$ by generalizing the results in [C] for harmonic functions (see also [C1], [C2], [J]) to $p$-harmonic functions. We also note that one can state interesting codimension problems similar to Theorems 1–3 for certain values of $p$. For example if $\gamma \subset B(0,1/2) \subset \mathbb{R}^3$ is a curve and $p > 2$, then there exists a unique $p$-harmonic function $u$ in $B(0,1) \setminus \gamma$ which is continuous in $\bar{B}(0,1)$ with boundary values $u = 0$ on $\gamma$ and $u = 1$ on $\partial B(0,1)$. Moreover, there exists a unique measure $\mu$ with support $\subset \gamma$. If $\gamma$ is Lipschitz, is it true that $\mu$ is absolutely continuous with respect to Hausdorff one measure $(H^1)$ on $\gamma$? If so, we next assume $\gamma$ is $C^1$, and put $k = d\mu/d\sigma$. Is it true that $\log k \in VMO(\gamma)$, where $VMO$ is the space of functions of vanishing mean oscillation? If $\mu = H^1$ measure on $\gamma$, is it true that $\gamma$ is a line segment or a circular arc? That is, to what extent do the theorems of Caffarelli and coauthors generalize to the codimension $> 1$ case.

As for related problems, we note that in [LV], see also [LV1], Lewis and Vogel study over-determined boundary conditions for positive solutions to the $p$-Laplace equation in a bounded domain $\Omega$. They prove that conditions akin to (4.32) imply uniqueness in certain free boundary problems. In particular, in [LV] the following free boundary problem is considered. Given a compact convex set $F \subset \mathbb{R}^n$, $a > 0$, and $1 < p < \infty$, find a function $u$, defined in a domain $\Omega = \Omega(a,p) \subset \mathbb{R}^n$, such that $\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0$ weakly in $\Omega \setminus F$, $u(x) \to 1$ whenever $x \to y \in F$, $u(x) \to 0$ whenever $x \to y \in \partial \Omega$ and such that $\mu = a^{p-1}H^{n-1}$ on $\partial \Omega$. Here $H^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure on $\partial \Omega$ and $\mu$ is the unique finite positive Borel measure associated with $u$ as in Lemma 2.5. If in addition, $\mu$ is upper Ahlfors regular, then the above authors show that this over-determined boundary value problem has a unique solution. An important part of their argument is to show that $\limsup_{x \to \partial \Omega} |\nabla u(x)| \leq a$. If $\partial \Omega$ is Lipschitz we note that this inequality is an easy consequence of Theorem 1 and (3.7). However, in [LV] it is only assumed that $\Omega$ is bounded, so a different argument, based on finiteness of a certain square function, is used.

References


Received 1 October 2007