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BESICOVITCH'S CIRCLE PAIRS, ANALYTIC CAPACITY, AND CAUCHY INTEGRALS FOR A CLASS OF UNRECTIFIABLE 1-SETS

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Abstract. We give an elementary proof of a theorem, which is originally due to Mattila on the vanishing of the analytic capacity of a certain natural subclass of the totally unrectifiable 1-sets in the plane, which includes self-similar sets (which are not line segments). We also give a version in $\mathbf{R}^n$, in relation to the Cauchy kernel. Our main tool is the circle pair idea which goes back to Besicovitch who used it in his study of the density properties of rectifiable, and totally unrectifiable 1-sets in the plane. Our method is also self-contained to some extent.

1. Introduction

In this paper we provide a different method of proof of a nice theorem of Mattila on the analytic capacity of a class of totally unrectifiable 1-sets in the plane. We also give an extension in the setting of $\mathbf{R}^n$. Our method is elementary, and exhibits a connection to some ideas that go back to Besicovitch. I would like to thank Pertti Mattila for very helpful remarks on this paper. The theorem under consideration is:

Theorem 1.1 [Ma2]. *Let E be a compact subset of $\mathbf{C}$ with $\mathscr{H}^1(E) < \infty$. If for $\mathscr{H}^1$ almost every $a \in E$ we have that the support of every $\nu \in \operatorname{Tan}(\mathscr{H}^1 \lfloor E, a)$ is not contained in a line, then $\gamma(E) = 0$.*

We recall a few facts and notations:

(1) E is a 1-set whenever $0 < \mathscr{H}^1(E) < \infty$.

(2) The lower circular density of E at a, denoted by $\Theta^1_*(E,a)$, is defined by

$$\Theta^1_*(E,a) \equiv \liminf_{r \to 0^+} \big((2r)^{-1} \mathscr{H}^1\big(E \cap B(a,r)\big)\big),$$

whereas the upper circular density of E at a, denoted by $\Theta^{*1}(E,a)$, is defined by

$$\Theta^{*1}(E,a) \equiv \limsup_{r \to 0^+} \big((2r)^{-1} \mathscr{H}^1\big(E \cap B(a,r)\big)\big),$$

where $B(a,r)$ is the closed ball centered at a, and having radius r.

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(3) If ν, μ are complex Radon measures, then $\nu \in \operatorname{Tan}(\mu, a)$ whenever there exist sequences c_i, r_i, of positive numbers, such that $r_i \to 0$, and $T_{a,r_i\#}\mu \to \nu$ weakly, as $i \to \infty$. Here $T_{a,r\#}\mu$ is the push forward of μ under the map $T_{a,r}(x) = (x-a)/r$.

(4) $\gamma(E) > 0$ (positive analytic capacity) whenever there exists a bounded analytic function $f(z)$ on $\mathbf{C}\backslash E$, which is not a constant, and $\gamma(E) = 0$ otherwise.

(5) E is called totally unrectifiable if $\mathscr{H}^1(E \cap K) = 0$ for every rectifiable curve K.

As of now, one has a complete characterization of the rectifiability properties of compact 1-sets in terms of their analytic capacity. Namely, $\gamma(E) = 0$ if and only if E is totally unrectifiable. The proof of this spanned decades of outstanding works. Recently, the last piece was completed by Guy David [D]. The reader is referred to the (short) list of references provided. However, for the class of sets under consideration, one has much simpler proofs. In [Ma1], a somewhat stronger version of Theorem 1.1 was given using a theorem of Besicovitch that is valid for subsets of $\mathbf{R}^2$. On the other hand the methods in [Ma2], and below, can be used to show corresponding results for 1-sets in $\mathbf{R}^n$ of the appropriate class (see Section 5).

2. A reformulation

In the spirit of providing an elementary treatment we will now state an equivalent version of the theorem which will be more suitable.

Recall that $\theta \in [0, \pi)$ is a *weak tangent direction* for E at a, if $\Theta^1_*(E, a) > 0$, and for all $\delta > 0$, $\liminf_{r\to 0}\big(r^{-1}\mathscr{H}^1(\{z \in E \cap B(a,r) : |\operatorname{Im} e^{-i\theta}(z-a)| \geq \delta|z-a|\}\big) = 0$. See [Ma1], [Fal]. We now state the following basic fact:

Lemma 2.1. *$\theta \in [0, \pi)$ is a weak tangent direction for E at a, if and only if there exists a positive Radon measure $\nu \in \operatorname{Tan}(\mathscr{H}^1\lfloor E, a)$, with $\operatorname{supp}(\nu)$ contained in a line through the origin with direction θ.*

For a proof one merely compares the definitions using some basic theorems on the existence of tangent measures. See [Ma2], [P]. We omit the details here. Theorem 1.1 can now be reformulated as:

Theorem 2.2. *Let E be a compact subset of $\mathbf{C}$ with $\mathscr{H}^1(E) < \infty$. If for almost all $a \in E$, $\Theta^1_*(E, a) > 0$, and no direction $\theta \in [0, \pi)$ is a weak tangent direction for E at a, then $\gamma(E) = 0$.*

We will give a proof of this theorem in Section 4 after we record some basic estimates which will be needed.

Remark 1. If in the definition of a weak tangent direction we replace the $\liminf$ by $\limsup$ we get the definition of a *tangent direction.* The class of sets in the above theorems is thus known a priori to be totally unrectifiable since the latter is characterized by the nonexistence of tangent almost everywhere (see e.g. [Fal]).

3. Basic estimates

In this section we recall some basic estimates that follow from our hypothesis, and basic results in geometric measure theory and analytic capacity:

(1) If E is a compact 1-set, $\gamma(E)>0$, then there exist $\phi\in L^\infty(E), A>0$, such that $\int_E\bigl(\phi(\zeta)/(\zeta-z)\bigr)\,d\mathscr{H}^1\zeta$ is a nonconstant analytic function on $\mathbf{C}\setminus E$, and the following estimate holds:

$$\sup_{\substack{r>0\\ z\in\mathbf{C}}}\left|\int_{E\setminus B(z,r)}\frac{\phi(\zeta)}{\zeta-z}\,d\mathscr{H}^1\zeta\right|<A. \tag{3.1}$$

See [Ma2] for an elegant treatment.

(2) By Lusin's theorem, and since ϕ cannot be identically zero, we can find $\alpha>0$, a compact 1-set $E_1\subset E$, such that, on E_1, $|\phi(\zeta)|>\alpha$, and $\phi(\zeta)$ is continuous. Thus, given $\varepsilon>0$, we can find $r_1>0$, such that

$$|\phi(\zeta)-\phi(\zeta')|<\varepsilon, \tag{3.2}$$

if $|\zeta-\zeta'|<r_1$, and $\zeta,\zeta'\in E_1$. We can also require that $|\phi\bigl(\zeta\bigr)|\le\|\phi\|_\infty$, whenever $\zeta\in E_1$.

(3) Given $D>1$, we can find a compact 1-set $E_2\subset E_1$, and $r_2>0$, such that

$$\mathscr{H}^1\bigl(E\cap B(a,r)\bigr)\le 2rD, \tag{3.3}$$

for $a\in E_2$, $r\le r_2$.

(4) Since at almost every $a\in E$, we have no weak tangent direction, and $\Theta^1_*(E,a)>0$, then (by compactness of the unit circle) we can find $\delta>0$, $\beta>0$, $r_3>0$, and a compact 1-set $E_3\subset E_2$, such that

$$\mathscr{H}^1\bigl(\{z\in E\cap B(a,r):|\operatorname{Im}e^{-i\theta}(z-a)|\ge\delta|z-a|\}\bigr)\ge\beta r \tag{3.4}$$

whenever $a\in E_3$, $r\le r_3$, $\theta\in[0,\pi)$.

(5) If F is a 1-set, then

$$\Theta^{*1}(F,a)\ge\tfrac12 \text{ for almost every } a\in F, \tag{3.5}$$

and

$$\Theta^{*1}(F,a)=0 \text{ for almost every } a\in F^c. \tag{3.6}$$

The above facts are rather standard reductions based on the definitions of $\mathscr{H}^1$, and its densities. See e.g. [Fal] or [Ma2].

4. Proof of Theorem 2.2

Let $F = E_3$ be as in Section 3, and let $r_0 = \frac{1}{2}\min\{r_1, r_2, r_3\}$. F is a totally unrectifiable 1-set, and is thus totally disconnected. We therefore have that F can be written as $F_1 \cup F_2$ with F_1, F_2 non-empty, compact, and disjoint, in infinitely many ways. Let F_1, F_2 be any choice of such sets, and let $w_1 \in F_1$, $w_2 \in F_2$, be such that $|w_1 - w_2| = s \equiv \operatorname{dist}(F_1, F_2)$. By focusing on a small piece of F, we may also assume that $s < r_0$. By a translation and a rotation, we can assume that w_1 is at the origin, and w_2 is positive . We can now prove:

Lemma 4.1. *If C is sufficiently large, ε (in (3.2)) sufficiently small, and $F' \equiv F \cap \big(B(w_1, s)\backslash B(w_1, s/C)\big)$, then (with A as in (3.1)),*

$$\left|\int_{F'} \frac{\phi(\zeta)}{\zeta - w_1}\, d\mathscr{H}^1\zeta\right| > 4A.$$

Proof. Choose $D \le 2$ in (3.3), $\theta = \frac{1}{2}\pi$ in (3.4), and $B \ge 8/\beta$. Let $F'_k = F' \cap \big(B(w_1, B^{-k}s)\backslash B(w_1, B^{-k-1}s)\big)$, observe that (using (3.2)),

$$\left|\operatorname{Re}\frac{\bar\phi(w_1)}{|\phi(w_1)|}\int_{F'}\frac{\phi(\zeta)}{\zeta - w_1}\, d\mathscr{H}^1\zeta\right| \ge |\phi(w_1)|\left|\int_{F'}\operatorname{Re}\frac{1}{\zeta - w_1}\, d\mathscr{H}^1\zeta\right| - \varepsilon\left|\int_{F'}\frac{1}{\zeta - w_1}\, d\mathscr{H}^1\zeta\right|. \tag{4.1}$$

Now observe that the Besicovitch circle pair $R(w_1, w_2) \equiv \operatorname{int}\big(B(w_1, s) \cap B(w_2, s)\big)$ is disjoint from F, and by an elementary computation (using (3.3)), we find that

$$\left|\int_{F',\, \operatorname{Re}\zeta \ge 0}\operatorname{Re}\frac{1}{\zeta - w_1}\, d\mathscr{H}^1\zeta\right| < M, \tag{4.2}$$

for some $M > 0$, which is independent of C. Thus we have that (for $n = [\ln C/\ln B]$)

$$\left|\int_{F'}\operatorname{Re}\frac{1}{\zeta - w_1}\, d\mathscr{H}^1\zeta\right| \ge \left|\sum_{k=0}^{n}\int_{F'_k,\, \operatorname{Re}\zeta < 0}\operatorname{Re}\frac{1}{\zeta - w_1}\, d\mathscr{H}^1\zeta\right| - M. \tag{4.3}$$

By (3.3), (3.4), and our choice of B, we find that

$$\mathscr{H}^1\big(\{\zeta \in F'_k : |\operatorname{Re}(\zeta - w_1)| \ge \delta|\zeta - w_1|\}\big) \ge \beta B^{-k}s - 4B^{-k-1}s \ge \tfrac{1}{2}\beta B^{-k}s, \tag{4.4}$$

so that the right hand side of (4.1) is at least $\frac{1}{2}(\alpha\delta\beta n) - 4\varepsilon(n+1) - M$. We can therefore satisfy the statement of the lemma by a proper choice of C, and ε.

Once C, and ε, have been chosen, they remain fixed. Lemma 4.1, and (3.1), together imply that

$$\left| \int_{E' \setminus F'} \frac{\phi(\zeta)}{\zeta - w_1} \, d\mathscr{H}^1 \zeta \right| > A, \tag{4.5}$$

where $E' = E \cap \big(B(w_1, s) \backslash B(w_1, s/C)\big)$. Since ϕ is bounded, there must exist $\eta > 0$, depending on $C, B, \|\phi\|_\infty$, but not on the pair w_1, w_2, so that

$$\mathscr{H}^1\big((E\backslash F) \cap B(w_1, s)\big) \geq \eta s. \tag{4.6}$$

At this point we can show that F, and hence E, cannot be totally unrectifiable. We will use a variation of the treatment in [Fal], closer in spirit to those in [Far1], [Far2].

Proposition 4.2. *Let E be a 1-set, and $F \subset E$, compact and $\mathscr{H}^1(F) > 0$. Suppose there exists $\eta > 0$, so that whenever F is represented as $F = F_1 \cup F_2$, with F_1, F_2 compact non empty and disjoint, and $w_1 \in F_1$, $w_2 \in F_2$ are such that $|w_1 - w_2| = s \equiv \operatorname{dist}(F_1, F_2)$, we have that $\mathscr{H}^1\big((E\backslash F) \cap B(w_1, s)\big) \geq \eta s$. Then there exists a continuum K such that $0 < \mathscr{H}^1(K \cap E) \leq \mathscr{H}^1(K) < \infty$.*

Proof. By (3.6), we can, for a given $\gamma > 0$, find $x_0 \in F$, $\rho > 0$ such that

$$\mathscr{H}^1\big((E\backslash F) \cap B(x_0, r)\big) \leq \gamma r \qquad \text{for } 0 < r \leq 2\rho. \tag{4.7}$$

By taking an appropriate subset of F, we can also require (by (3.3)) that

$$\mathscr{H}^1\big(E \cap B(x, r)\big) \leq 4r \qquad \text{for } 0 < r \leq 2\rho,\ x \in F, \tag{4.8}$$

and by (3.5),

$$\mathscr{H}^1\big(F \cap B(x_0, \rho)\big) \geq \tfrac{1}{2}\rho. \tag{4.9}$$

We can also assume that there is a point $y \in F \cap \partial B(x_0, \rho)$. Let $\mathscr{C}$ be the family of closed balls:

$$\begin{aligned} \mathscr{C} = \big\{ B(x, r) : x \in F \cap B(x_0, \rho),\ 0 < r < 2\rho, \\ \text{and } \mathscr{H}^1\big((E\backslash F) \cap B(x, r)\big) \geq \eta r \big\}. \end{aligned} \tag{4.10}$$

We now recall:

Lemma 4.3. *Let $\mathscr{C}$ be a collection of balls contained in a bounded subset of $\mathbf{R}^n$. Then we may find a finite or countably infinite disjoint collection $\{B_i\}$ such that*

$$\bigcup_{B \in \mathscr{C}} B \subset \bigcup_i B_i', \tag{4.11}$$

where B_i' is the ball concentric with B_i, and of five times the radius. Further, we may take the collection $\{B_i\}$ to be semidisjoint (i.e. $B_i' \not\subset B_j'$ if $i \neq j$).

See for example [Fal] for a proof. Now let

$$G = \big(F \cap B(x_0,\rho)\big) \cup \partial B(x_0,\rho) \cup \Big(\bigcup_i B'_i\Big), \tag{4.12}$$

and

$$K = \Big(G \backslash \bigcup_i B'_i\Big) \cup \bigcup_i \partial B'_i. \tag{4.13}$$

We need to show that K satisfies the conclusion of the proposition. First we check that G is a continuum (i.e. compact and connected).

(i) G is closed: Since F is closed, it is sufficient to observe that $\operatorname{diam}(B'_i) \to 0$ (by counting volume for instance). Hence a limit point of $\bigcup_i B'_i$ is either in some B'_j, or is a limit point of a sequence of centers of a subcollection of $\{B'_i\}$, and hence in F. In particular any limit point lies in G.

(ii) G is connected: Suppose $G = G_1 \cup G_2$, where G_1, G_2 are nonempty, disjoint, and compact. Without loss of generality we may assume $\partial B(x_0,\rho) \subset G_1$. We must then have $G_2 \subset \operatorname{int}\big(B(x_0,\rho)\big)$ (or else some $B'_i \subset G_1$, with center in $B(x_0,\rho)$ meets $\partial B(x_0,\rho)$, and hence connects G_1, G_2). Let

$$G'_1 = \big(\mathbf{R}^2 \backslash \operatorname{int}\big(B(x_0,\rho)\big)\big) \cup G_1. \tag{4.14}$$

Both G'_1, G_2 contain points from F (G'_1 contains y, and G_2 must contain either a point in F, or the center of any ball contained in it, which is also in F). Now $F_1 \equiv G'_1 \cap F$, $F_2 \equiv G_2 \cap F$ are compact, disjoint, and nonempty. Let $w_1 \in F_1$, $w_2 \in F_2$, be such that $|w_1 - w_2| = s \equiv \operatorname{dist}(F_1, F_2)$, and observe that $s < \rho$. By construction, the ball $B(w_1, s) \in \mathscr{C}$ and contains w_1, w_2, and is thus disconnected, which is absurd. Thus G is a continuum, and we can invoke:

Lemma 4.4. *Let G be a continuum in $\mathbf{R}^2$. Suppose $\{B_i\}$ is a countable semidisjoint collection of closed balls each contained in G and such that* $\operatorname{diam}(B_i) \ge d$ *for only finitely many i for any $d > 0$. Then if Γ_i is the boundary of B_i,*

$$K = \Big(G \backslash \bigcup_i B_i\Big) \cup \bigcup_i \Gamma_i \tag{4.15}$$

is a continuum.

For a proof see e.g. p. 42 in [Fal].

By applying this lemma to our collection $\{B'_i\}$, we conclude that K is a continuum. Now

$$\begin{aligned} \sum_i \operatorname{diam}(B'_i) &\le \frac{20}{\eta} \sum_i \mathscr{H}^1\big((E \backslash F) \cap B_i\big) \\ &\le \frac{20}{\eta} \mathscr{H}^1\big((E \backslash F) \cap B(x_0, 2\rho)\big) \le \frac{40\gamma\rho}{\eta} \end{aligned} \tag{4.16}$$

where we used (4.6). We also have

$$\mathscr{H}^1(K) \le \mathscr{H}^1(E) + 2\pi\rho + \pi \sum_i \operatorname{diam}(B_i') < \infty. \tag{4.17}$$

Thus K is a continuum of finite measure, and is hence rectifiable (in fact it is a rectifiable curve). See e.g. [Fal] for a proof of this fact, or p. 6 in [DS]. On the other hand

$$\begin{aligned}\mathscr{H}^1(K\cap E) &\ge \mathscr{H}^1\big(F\cap B(x_0,\rho)\big) - \sum_i \mathscr{H}^1(E\cap B_i') \\ &\ge \mathscr{H}^1\big(F\cap B(x_0,\rho)\big) - 2\sum_i \operatorname{diam}(B_i') \ge \frac{\rho}{2} - \frac{80\gamma\rho}{\eta}\end{aligned} \tag{4.18}$$

using (4.8), (4.9), and (4.16). If we then choose $\gamma \ll \eta$, we see that $\mathscr{H}^1(K\cap E) > 0$. This contradiction completes the proof of Theorem 2.2. ◻

5. Remarks and extensions

The circle pairs idea (of using (4.6) to construct a rectifiable set that has nontrivial intersection with E) is originally due to Besicovitch [B], who used it to show that if E is a totally unrectifiable 1-set in $\mathbf{R}^2$, then $\Theta^1_*(E,x) \le \frac{3}{4}$, for a.e. $x\in E$. In the above form (in the plane) it appears in [MR]. In $\mathbf{R}^n$ one can replace the collection $\mathscr{C}$ (see Section 4) by line segments connecting points like w_1, w_2. This was done in [Mo]. See also [PT] for a treatment in metric spaces.

One main application of Theorem 2.2 is for self-similar 1-sets in the sense of Hutchinson (see e.g. [Fal], [Ma1], [Ma2], [H]). The same method (with minimal changes) can also be used to prove the following:

Suppose $e\in S^{n-1}$, $x\in\mathbf{R}^n$, $\psi\in\big[0,\frac{1}{2}\pi\big)$, denote by $\Gamma(x,e,\psi)$ the double sided cone with vertex at x, axis in the direction of e, and opening angle ψ.

Theorem 5.1. *Let $E\subset\mathbf{R}^n$ be a 1-set such that for almost every $x\in E$, we have that for every $e\in S^{n-1}$ there exists $\delta>0$ so that*

$$\liminf_{r\to 0} \mathscr{H}^1\big(E\cap B(x,r)\cap\Gamma\big(x,e,\tfrac{1}{2}\pi-\delta\big)\big) > 0.$$

Suppose $\phi\in L^\infty(E)$, and let $F=\{x\in E:\phi(x)\ne 0\}$, then for almost every $x_0\in F$,

$$\limsup_{r\to\infty}\left|\int_{E\setminus B(x_0,r)} \frac{(x-x_0)}{|x-x_0|^2}\phi(x)\,d\mathscr{H}^1 x\right| = \infty.$$

Once again self similar 1-sets in $\mathbf{R}^n$ which are not line segments are immediate candidates for this theorem. While for $n>2$, such a set does not a priori satisfy the hypothesis if it happens to lie completely in some m-plane, we can reduce the problem to $\mathbf{R}^{m_0}$ where m_0 is the smallest dimension of such a plane and then we can use the theorem.

The analogue of the condition in Theorem 5.1 in the case of compact $(n-1)$-sets (in $\mathbf{R}^n$) was considered in [MP] to show that such sets are removable for Lipschitz harmonic functions. The general case of that question for compact totally unrectifiable 1-sets in the plane was settled in [DM]. Our theorem also gives an elementary proof of that theorem in the case of self similar 1-sets in the plane (more generally, for sets satisfying the hypothesis of Theorem 5.1).

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